Math 55400 **Qualifying Exam** **January 6, 2017** **W. Heinzer**

PUID: ________________________________

Instructions:

1. The point value of each exercise occurs to the left of the problem.

2. No books or notes or calculators are allowed.

Page	**Points Possible**	**Points**
2	24	
3	20	
4	18	
5	20	
6	16	
7	20	
8	20	
9	20	
10	20	
11	22	
Total	200	

1. (24 pts) Let $T : V \to V$ be a linear operator on an n-dimensional vector space over a field F. Let $c_1, \ldots, c_k$ be distinct elements in F and let $p(x) = (x - c_1)^{r_1} \cdots (x - c_k)^{r_k}$ be the minimal polynomial of T. Let $W_i = \{v \in V \mid (T - c_i I)^{r_i}(v) = 0\}$.

(a) Describe linear operators $E_i : V \to V$, $i = 1, \ldots, k$, such that $E_i(V) = W_i$, $E_i^2 = E_i$ for each i, $E_i E_j = 0$ if $i \neq j$, and $E_1 + \cdots + E_k = I$ is the identity operator on V.

(b) Describe how to obtain linear operators D and N such that $T = D + N$, where D is diagonalizable, N is nilpotent and D and N are polynomials in T.

(c) If $T = D' + N'$, where D' is diagonalizable and N' is nilpotent and $D'N' = N'D'$, prove that $D = D'$ and $N = N'$.

2. (20 pts) Let notation be as in the previous problem and let $f(x) = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}$ be the characteristic polynomial for T. Thus $n = d_1 + \cdots + d_k$ and $1 \leq r_i \leq d_i$ for each i.

(a) Describe the possible Jordan forms for T.

(b) What are necessary and sufficient conditions in order that $\operatorname{rank} T = n$?

(c) If $\operatorname{rank} T < n$, prove or disprove that $\operatorname{rank} T - \operatorname{rank} T^2 \;\geq\; \operatorname{rank} T^2 - \operatorname{rank} T^3$.

3. (18 pts) Let notation be as in the previous problem.

(a) If $r_i + 1 = d_i$ for each $i \in \{1, \ldots, k\}$, how many different Jordan forms are possible?

(b) If $r_i + 2 = d_i$ for each $i \in \{1, \ldots, k\}$, how many different Jordan forms are possible?

(c) If $r_i + 3 = d_i$ for each $i \in \{1, \ldots, k\}$, how many different Jordan forms are possible?

4. Let M be a module over the integral domain D. A submodule N of M is *pure* in M if the following holds: given $y \in N$ and $a \in D$ such that there exists $x \in M$ with $ax = y$, then there exists $z \in N$ with $az = y$.

(a) (10 pts) Let N be a submodule of M and for $x \in M$, let $\overline{x} = x + N$ denote the coset representing the image of x in the quotient module M/N. If N is a pure submodule of M, and $\operatorname{ann} \overline{x} = \{a \in D \mid a\overline{x} = 0\}$ is the principal ideal (d) of D, prove that there exists $x' \in M$ such that $x + N = x' + N$ and $\operatorname{ann} x' = \{a \in D \mid ax' = 0\}$ is the principal ideal (d).

(b) (10 pts) If $M = \langle \alpha \rangle$ is a cyclic $\mathbb{Z}$-module of order 12, list the submodules of M and indicate which of the submodules of M are pure in M.

5. (16 pts) Let M be a finitely generated module over the polynomial ring $F[x]$, where F is a field, and let N be a pure submodule of M. Prove that there exists a submodule L of M such that $N + L = M$ and $N \cap L = 0$.

6. (20 pts) Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V and let $R = T(V)$ denote the range of T.

(a) Prove that R has a complementary T-invariant subspace if and only if R is independent of the null space N of T, i.e., $R \cap N = 0$.

(b) If R and N are independent, prove that N is the unique T-invariant subspace of V that is complementary to R.

7. (20 pts) Let p be a prime integer and let $F = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements. Let V be a vector space over F and $T : V \to V$ a linear operator. Assume that T has characteristic polynomial x^4 and minimal polynomial x^3.

(a) Express V as a direct sum of cyclic $F[x]$-modules.

(b) How many cyclic 3-dimensional T-invariant subspaces does V have?

(c) How many cyclic 3-dimensional T-invariant subspaces of V are direct summands of V?

(d) How many cyclic 2-dimensional T-invariant subspaces does V have?

(e) How many cyclic 2-dimensional T-invariant subspaces of V are direct summands of V?

(f) How many 1-dimensional T-invariant subspaces does V have?

(g) How many 1-dimensional T-invariant subspaces of V are direct summands of V?

8. (20 pts) Let V be a finite dimensional inner product space over $\mathbb{C}$ and let $T : V \to V$ be a linear operator.

(a) (2 pts) Define the adjoint T^* of T.

(b) (6 pts) If $T = T^*$, prove that every characteristic value of T is a real number.

(c) (6 pts) Assume that $T = T^*$ and that c and d are distinct characteristic values of T. If α and β in V are such that $T\alpha = c\alpha$ and $T\beta = d\beta$, prove that α and β are orthogonal.

(d) (6 pts) State true or false and justify: If $A \in \mathbb{R}^{5\times 5}$ is symmetric, then A is diagonalizable.

9. (20 pts) Consider the abelian group $V = \mathbb{Z}/(5^4) \oplus \mathbb{Z}/(5^3) \oplus \mathbb{Z}$.

(a) Write down a relation matrix for V as a $\mathbb{Z}$-module.

(b) Let W be the cyclic subgroup of V generated by the image of the element $(5^2, 5, 5)$ in $\mathbb{Z}/(5^4) \oplus \mathbb{Z}/(5^3) \oplus \mathbb{Z}$. Write down a relation matrix for W.

(c) Write down a relation matrix for the quotient module V/W.

(d) What is the cardinality of the quotient module V/W?

10. (12 pts) Prove or disprove: if V is a vector space over a field F and $T : V \to V$ is a linear operator such that every subspace of V is invariant under T, then T is a scalar multiple of the identity operator.

11. Let F be a field and let $g(x) \in F[x]$ be a monic polynomial.

(a) (5 pts) Describe the $F[x]$-submodules of $V = F[x]/(g(x))$.

(b) (5 pts) If $g(x) = x^3(x-1)$, diagram the lattice of $F[x]$-submodules of $V = F[x]/(g(x))$.