554 QUALIFYING EXAM, AUG 6 2018

Attempt all questions. Time 2 hrs.

1. (20 pts) Suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a basis of the vector space V. Does it follow that $\vec{w}_k = \sum_{i \neq k} \vec{v}_i$ is also a basis of V?

2. (5 + 15 pts) Let E be the vector space of 3×3 real matrices. Let $A \in E$, and let $L_A : E \to E$ be the map defined by $B \mapsto AB$.
   (a) Prove that L_A is an endomorphism of the vector space E.
   (b) Suppose that $\det(A) = 32$ and the minimal polynomial of A equals $(T-2)(T-4)$. What is the trace of L_A ?

3. (20 pts) Find an orthogonal matrix which diagonalizes the given symmetric matrix
$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}.$$

4. (10 + 5 + 5 pts) Let A be a complex $n \times n$ matrix, $n \geq 2$, all of whose entries are equal to 1.
   (a) Find the characteristic and minimal polynomials of A ?
   (b) Is A diagonalizable ? Justify your answer.
   (c) Find the Jordan canonical form of A.

5. (20 pts) Suppose that T is a self-adjoint operator on an inner product space. If $T^k \vec{v} = \vec{0}$, for some $k \geq 2$, show that $T\vec{v} = \vec{0}$.

6. (20 pts) Let A be a 3×5, and B a 5×3 complex matrix such that

$$AB = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Find the Jordan canonical form of the matrix BA.

7. (20 pts) If $A = \frac{1}{\sqrt{10}} \begin{bmatrix} 10 & 6 \\ 0 & 8 \end{bmatrix}$, then decompose $A = QS$, where Q is orthogonal and S is symmetric with positive eigenvalues.

8. (6+6+8 pts) Let A, B be complex $n \times n$ matrices. Prove or disprove each of the following statements.
   (a) If A and B are diagonalizable, then so is $A + B$.
   (b) If A and B are diagonalizable, then so is AB.
   (c) If $A^2 = A$, then A is diagonalizable.