554 QUALIFYING EXAM, JAN 2018

Attempt all questions. Time 2 hrs.

1. (5+10+5 pts) Let E be a finite dimensional complex vector space and $u \in \mathrm{End}(E)$.
   (a) Prove that if $\mathrm{Tr}(u^i) = 0$ for each $i > 0$, then u is nilpotent.
   (b) Suppose that
   $$u = [u_1, v_1] + \cdots + [u_m, v_m],$$
   (where for any $f, g \in \mathrm{End}(E)$ we denote $[f, g] = f \circ g - g \circ f$), and u commutes with each u_i for $1 \leq i \leq m$. Prove that u is nilpotent.
   (c) Suppose that $\mathrm{Tr}(u \circ v) = 0$ for all $v \in \mathrm{End}(E)$ satisfying $\mathrm{Tr}(v) = 0$. Prove that $u = \lambda \cdot \mathrm{Id}_E$ for some $\lambda \in k$.
2. (10+10 pts) Let E be a finite dimensional complex inner product vector space and $u, v \in \mathrm{End}(E)$.
   (a) Prove that if u is normal, then $u^* = p(u)$ for some polynomial $p \in \mathbb{C}[X]$.
   (b) Suppose that $u, v \in \mathrm{End}(E)$, such that u, v are normal and $u \circ v = v \circ u$. Prove that $u \circ v$ is normal.
3. (5+15 pts) Let E be a finite dimensional k-vector space and $u \in \mathrm{End}(E)$. Consider $\mathrm{End}(E)$ as a k-vector space, and denote by $\mathrm{ad}(u)$ the element of $\mathrm{End}(\mathrm{End}(E))$ defined by $\mathrm{ad}(u)(v) = u \circ v - v \circ u$.
   (a) State the additive Jordan decomposition theorem for endomorphisms of finite dimensional complex vector spaces.
   (b) Prove that
   $$\mathrm{ad}(u)_s = \mathrm{ad}(u_s), \mathrm{ad}(u)_n = \mathrm{ad}(u_n).$$
   (using the notation for the additive Jordan decomposition).
4. (5+5+10 pts) Let E be a finite dimensional complex vector space, and $u \in \mathrm{End}(E)$.
   (a) Define the algebraic and geometric multiplicity of an eigenvalue λ of u.
   (b) What are the algebraic and geometric multiplicities of the various eigenvalues of the endomorphism whose matrix with respect to a certain basis is given by
   $$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$
   (c) Compute the rational canonical form of the matrix given in Part (4b).
5. (10+10 pts)
   (a) Let M be a 3×3 matrix with complex entries. If M^3 is the identity matrix, what are the possibilities for the Jordan canonical form of M ?

(b) Let M be a 3×3 matrix with integer entries and $\det(M) = -1$. Assume that every eigenvalue of M is rational. What are the possibilities for the minimal polynomial and Jordan canonical form of M?