1. (10 pts) Let $U, V, W$ be finite dimensional subspaces of a real vector space. Prove that 
\[ \dim(U) + \dim(V) + \dim(W) - \dim(U + V + W) \geq \max(\dim(U \cap V), \dim(V \cap W), \dim(W \cap U)) \]

2. (5 + 5 + 5 pts) Let $M_{2\times2}$ be the vector space of all real $2 \times 2$ matrices. Let 
\[ A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}. \]

Let $L : M_{2\times2} \to M_{2\times2}$ be the map defined by 
\[ L(X) = AXB. \]

(a) Prove that $L$ is a linear transformation.
(b) Calculate the determinant of $L$.
(c) Calculate the trace of $L$.

3. (5 + 10 pts) 
(a) Let $A, B$ be $n \times n$ real matrices, such that $A$ is invertible. Prove that 
\[ \text{rank}(AB) = \text{rank}(B). \]
(b) Let $A$ and $B$ be $n \times n$ real matrices such that $A^2 = A, B^2 = B$, and $I_n - (A + B)$ is invertible. Prove that 
\[ \text{rank}(A) = \text{rank}(B). \]

4. (5 + 5 + 10 pts) 
(a) When are two $n \times n$ complex matrices similar ?
(b) Let $A$ be an $n \times n$ complex matrix with characteristic polynomial $(\lambda - 1)^n$. Prove that $A$ is invertible and that $A$ is similar to $A^{-1}$.
(c) Let $A$ be an $n \times n$ complex matrix. Prove that $A$ and $A^T$ are similar matrices.

5. (5 + 10 pts) Let $V$ be a finite dimensional complex inner product space and $f \in \text{End}(V)$.
(a) What does it mean to say that $f$ is self-adjoint ?
(b) If $f$ is self-adjoint prove that all eigenvalues of $f$ are real.

6. (5 + 5 + 5 pts) Let $V$ be a finite dimensional complex vector space and $f \in \text{End}(V)$.
(a) What does it mean to say that $f$ is diagonalizable ?
(b) Define the minimal polynomial of $f$.
(c) Suppose that $f^k = 1_V$ for some positive integer $k$. Prove that $f$ is diagonalizable.

7. (10 pts) Let $V = \mathbb{R}^3$ with the standard inner product and $(a, b, c)^T$ a vector of length 1. Let $W$ be the subspace defined by $aX_1 + bX_2 + cX_3 = 0$. Find the matrix (with respect to the standard basis) which represents the orthogonal projection, $p : V \to V$, of $V$ on to $W$. 