554 QUALIFYING EXAM, AUG, 2020

Attempt all questions. Time 2 hrs.

1. (10 pts) Let U, V, W be finite dimensional subspaces of a real vector space. Prove that $\dim(U)+\dim(V)+\dim(W)-\dim(U+V+W) \geq \max(\dim(U\cap V), \dim(V\cap W), \dim(W\cap U))$.
2. (5 + 5 + 5 pts) Let $M_{2\times 2}$ be the vector space of all real 2×2 matrices. Let
$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}.$$
Let $L : M_{2\times 2} \to M_{2\times 2}$ be the map defined by
$$L(X) = AXB.$$
   (a) Prove that L is a linear transformation.
   (b) Calculate the determinant of L.
   (c) Calculate the trace of L.
3. (5 +10 pts)
   (a) Let A, B be $n \times n$ real matrices, such that A is invertible. Prove that
$$\text{rank}(AB) = \text{rank}(B).$$
   (b) Let A and B be $n \times n$ real matrices such that $A^2 = A, B^2 = B$, and $I_n - (A + B)$ is invertible. Prove that
$$\text{rank}(A) = \text{rank}(B).$$
4. (5 + 5 + 10 pts)
   (a) When are two $n \times n$ complex matrices similar ?
   (b) Let A be an $n \times n$ complex matrix with characteristic polynomial $(\lambda - 1)^n$. Prove that A is invertible and that A is similar to A^{-1}.
   (c) Let A be an $n \times n$ complex matrix. Prove that A and A^T are similar matrices.
5. (5 + 10 pts) Let V be a finite dimensional complex inner product space and $f \in \text{End}(V)$.
   (a) What does it mean to say that f is self-adjoint ?
   (b) If f is self-adjoint prove that all eigenvalues of f are real.
6. (5 + 5 + 5 pts) Let V be a finite dimensional complex vector space and $f \in \text{End}(V)$.
   (a) What does it mean to say that f is diagonalizable ?
   (b) Define the minimal polynomial of f.
   (c) Suppose that $f^k = 1_V$ for some positive integer k. Prove that f is diagonalizable.
7. (10 pts) Let $V = \mathbb{R}^3$ with the standard inner product and $(a, b, c)^T$ a vector of length 1. Let W be the subspace defined by $aX_1 + bX_2 + cX_3 = 0$. Find the matrix (with respect to the standard basis) which represents the orthogonal projection, $p : V \to V$, of V on to W.