Attempt all questions. Time 2 hrs.

1. (10 pts) Let V be a complex vector space and \( L \in \text{End}(V) \). Let \( \alpha \in V \) such that \( L^m \alpha = 0 \), and \( L^{m-1} \alpha \neq 0 \), for some positive integer \( m \). Prove that \( \alpha, L\alpha, \ldots, L^{m-1}\alpha \) are linearly independent.

2. (10 pts) Let \( V \) be a finite dimensional vector space over a field \( k \), and \( W \) a subspace of \( V \). Let \( L \in \text{End}(V) \) such that \( \text{Im}(L) \subset W \). Let \( L' \in \text{End}(W) \) denote the restriction of \( L \) to \( W \). Prove that 
\[
\det(Id_V + \lambda L) = \det(Id_W + \lambda L')
\]
as elements of \( k[\lambda] \).

3. (10 pts) Let \( E, F, G, H \) be four finite dimensional vector spaces over a field \( k \) and \( u : E \to F, v : F \to G, w : G \to H \) be linear transformations. Prove that 
\[
\text{rk}(v \circ u) + \text{rk}(w \circ v) \leq \text{rk}(v) + \text{rk}(w \circ v \circ u),
\]
where \( \text{rk}(\cdot) \) denotes the rank.

4. (5 + 5 pts) Let \( L, L' \) be endomorphisms of a finite dimensional vector space \( V \) over a field \( k \). Prove or disprove (by providing a counter-example) the following statements.
   (a) Every eigenvalue of \( L \circ L' \) is also an eigenvalue of \( L' \circ L \).
   (b) Every eigenvector of \( L \circ L' \) is also an eigenvector of \( L' \circ L \).

5. (5 + 5 + 5 pts) Let \( A \) be a complex \( n \times n \) matrix all of whose entries are equal to 1.
   (a) Find the characteristic polynomial of \( A \).
   (b) Is \( A \) diagonalizable? Prove or disprove.
   (c) Find the Jordan canonical form of \( A \).

6. (5 + 5 + 5 pts) Let \( V \) be a finite dimensional complex Hermitian space and \( u \in \text{End}(V) \).
   (a) Define the adjoint of \( u \) and prove that it exists.
   (b) Prove that if \( u \) is self-adjoint then the eigenvalues of \( u \) are all real.
   (c) Prove that if \( u \) is self-adjoint then \( u \) is diagonalizable.

7. (5 + 5 pts) Let \( V \) be the vector space of complex \( n \times n \) matrices, \( A \in V \), and \( C(A) \subset V \) the set of \( n \times n \) complex matrices which commutes with \( A \).
   (a) Prove that \( C(A) \) is a subspace of \( V \).
   (b) Prove that \( \dim C(A) \geq n \).

8. (5 + 10 + 5 pts) Let \( V \) be a finite dimensional complex Hermitian vector space.
   (a) What does it mean to say that \( U \in \text{End}(V) \) is a unitary transformation?
   (b) Suppose that \( U \in \text{End}(V) \) is unitary. Prove that \( U \) is diagonalizable, and if \( \lambda \) is an eigenvalue of \( U \), then \( |\lambda| = 1 \).
   (c) Let \( L \in \text{End}(V) \) such that \( \text{Id}_V + L, \text{Id}_V + L^2, \text{Id}_V + L^3 \) are all unitary. Prove that \( L = 0 \).