554 QUALIFYING EXAM, AUG 12, 2021

Attempt all questions. Time 2 hrs.

1. (10 pts) Let V be a complex vector space and $L \in \mathrm{End}(V)$. Let $\alpha \in V$ such that $L^m\alpha = \mathbf{0}$, and $L^{m-1}\alpha \neq \mathbf{0}$, for some positive integer m. Prove that $\alpha, L\alpha, \ldots, L^{m-1}\alpha$ are linearly independent.
2. (10 pts) Let V be a finite dimensional vector space over a field k, and W a subspace of V. Let $L \in \mathrm{End}(V)$ such that $\mathrm{Im}(L) \subset W$. Let $L' \in \mathrm{End}(W)$ denote the restriction of L to W. Prove that
$$\det(\mathrm{Id}_V + \lambda L) = \det(\mathrm{Id}_W + \lambda L')$$
as elements of $k[\lambda]$.
3. (10 pts) Let E, F, G, H be four finite dimensional vector spaces over a field k and $u : E \to F, v : F \to G, w : G \to H$ be linear transformations. Prove that
$$\mathrm{rk}(v \circ u) + \mathrm{rk}(w \circ v) \leq \mathrm{rk}(v) + \mathrm{rk}(w \circ v \circ u),$$
where $\mathrm{rk}(\cdot)$ denotes the rank.
4. (5 + 5 pts) Let L, L' be endomorphisms of a finite dimensional vector space V over a field k. Prove or disprove (by providing a counter-example) the following statements.
   (a) Every eigenvalue of $L \circ L'$ is also an eigenvalue of $L' \circ L$.
   (b) Every eigenvector of $L \circ L'$ is also an eigenvector of $L' \circ L$.
5. (5 + 5 + 5 pts) Let A be a complex $n \times n$ matrix all of whose entries are equal to 1.
   (a) Find the characteristic polynomial of A.
   (b) Is A diagonalizable ? Prove or disprove.
   (c) Find the Jordan canonical form of A.
6. (5 + 5 + 5 pts) Let V be a finite dimensional complex Hermitian space and $u \in \mathrm{End}(V)$.
   (a) Define the adjoint of u and prove that it exists.
   (b) Prove that if u is self-adjoint then the eigenvalues of u are all real.
   (c) Prove that if u is self-adjoint then u is diagonizable.
7. (5 + 5 pts) Let V be the vector space of complex $n \times n$ matrices, $A \in V$, and $C(A) \subset V$ the set of $n \times n$ complex matrices which commutes with A.
   (a) Prove that $C(A)$ is a subspace of V.
   (b) Prove that $\dim C(A) \geq n$.
8. (5 + 10 + 5 pts) Let V be a finite dimensional complex Hermitian vector space.
   (a) What does it mean to say that $U \in \mathrm{End}(V)$ is a unitary transformation ?
   (b) Suppose that $U \in \mathrm{End}(V)$ is unitary. Prove that U is diagonalizable, and if λ is an eigenvalue of U, then $|\lambda| = 1$.
   (c) Let $L \in \mathrm{End}(V)$ such that $\mathrm{Id}_V + L, \mathrm{Id}_V + L^2, \mathrm{Id}_V + L^3$ are all unitary. Prove that $L = \mathbf{0}$.