PUID: ________________________________

Instructions:
1. The point value of each exercise occurs to the left of the problem.
2. No books or notes or calculators are allowed.

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1. (12 pts) Let $V$ be a finite-dimensional vector space over a field $F$ and let $W_1$ and $W_2$ be subspaces of $V$. Prove that
\[ \dim W_1 + \dim W_2 = \dim (W_1 \cap W_2) + \dim (W_1 + W_2). \]

2. (6 pts) Let $V$ be a finite-dimensional vector space over the field $F$ and let $W$ be a subspace of $V$. If $f$ is a linear functional on $W$, prove that there is a linear functional $g$ on $V$ such that $g(\alpha) = f(\alpha)$ for each vector $\alpha$ in the subspace $W$. 
3. Let $V$ and $W$ be finite-dimensional vector spaces over a field $F$, and let $T: V \to W$ be a linear transformation.
   
   (a) (2 pts) Define the rank of $T$.
   
   (b) (2 pts) Define the nullity of $T$.
   
   (c) (10 pts) State and prove a theorem involving the rank of $T$ and the nullity of $T$.

4. (8 pts) Let $F$ be an infinite field and let $g \in F[x]$ be a monic polynomial of degree $n > 0$.
   
   (a) Describe the ideals in $F[x]$ that contain $g$.
   
   (b) Are there finitely many or infinitely many ideals in $F[x]$ that contain $g$?
5. (12 pts) Let $F$ be a field, let $S$ be a set, and let $F(S,F)$ be the set of all functions from $S$ to $F$.

(a) As in Chapter 2 of Hoffman and Kunze, define vector addition and scalar multiplication on the set $F(S,F)$ so that $F(S,F)$ is a vector space over the field $F$.

(b) If $S$ is a finite set with $n$ elements what is the dimension of the vector space $F(S,F)$? Justify your answer.

6. (6 pts) State true or false and justify your answer: If $V$ is a finite-dimensional vector space and $W_1$ and $W_2$ are subspaces of $V$ such that $V = W_1 \oplus W_2$, then for any subspace $W$ of $V$ we have $W = (W \cap W_1) \oplus (W \cap W_2)$. 
7. Define the following terms as in Hoffman and Kunze.

(a) (4 pts) \( \mathfrak{A} \) is a linear algebra over the field \( F \).

(b) (4 pts) The vector space \( V \) of polynomial functions over a field \( F \).

(c) (4 pts) The vector space \( F[x] \) of polynomials over a field \( F \).

8. (6 pts) For what fields \( F \) is the vector space of polynomial functions over \( F \) isomorphic to the vector space of polynomials over \( F \)? Justify your answer.
9. Let $D$ be a principal ideal domain and let $M$ be a finitely generated $D$-module.

(a) (3 pts.) What does it mean for a subset $S = \{z_1, \ldots, z_n\}$ of $M$ to be a generating set for $M$?

(b) (3 pts.) What does it mean for a subset $S = \{z_1, \ldots, z_n\}$ of $M$ to be a basis for $M$.

(c) (6 pts.) What does it mean for a matrix $A \in D^{m \times n}$ to be a relation matrix for $M$? How is a relation matrix for $M$ constructed?

(d) (6 pts.) State true or false and justify your answer with either a proof or a counterexample: Every nonzero finitely generated $D$-module has a basis.
10. (20 pts) Let $T : V \to V$ be a linear operator on an $n$-dimensional vector space $V$, and let $\mathcal{F}$ be the vector space of linear operators $U : V \to V$ that commute with $T$.

(a) Prove that $\dim \mathcal{F} \geq n$.

(b) Prove that $T$ has a cyclic vector if and only if every $U \in \mathcal{F}$ is a polynomial in $T$. 
11. (20 pts) Let \( V \) be an abelian group generated by elements \( a, b, c \). Assume the following relations hold: \( 2a = 4b, 2b = 4c, 2c = 4a \), and these three relations generate all the relations on \( a, b, c \).

(a) Write down a relation matrix for \( V \).

(b) Find generators \( x, y, z \) for \( V \) such that \( V = \langle x \rangle \oplus \langle y \rangle \oplus \langle z \rangle \) is the direct sum of cyclic subgroups generated by \( x, y, z \), and express your generators \( x, y, z \) in terms of \( a, b, c \).

(c) What is the order of \( V \)?

(d) What is the order of the element \( a \)?
12. (10 pts) Let $V$ be a finite-dimensional vector space over an infinite field $F$. Prove that $V$ is not the union of finitely many proper subspaces.

13. (10 pts) Let $V$ be a finite-dimensional vector space over an infinite field $F$ and let $\alpha_1, \ldots, \alpha_m$ be finitely many nonzero vectors in $V$. Prove that there exists a linear functional $f$ on $V$ such that $f(\alpha_i) \neq 0$ for each $i$ with $1 \leq i \leq m$. 
14. (18 pts) Let $T : V \to V$ be a linear operator on an $n$-dimensional vector space over a field $F$. Let $c_1, \ldots, c_k$ be distinct elements in $F$ and let $p = (x - c_1)^{r_1} \cdots (x - c_k)^{r_k}$ be the minimal polynomial of $T$. Let $W_i = \{ v \in V \mid (T - c_i I)^{r_i}(v) = 0 \}$.

(a) Describe linear operators $E_i : V \to V$, $i = 1, \ldots, k$, such that $E_i(V) = W_i$, $E_i^2 = E_i$ for each $i$, $E_i E_j = 0$ if $i \neq j$, and $E_1 + \cdots + E_k = I$ is the identity operator on $V$.

(b) Describe how to obtain linear operators $D$ and $N$ such that $T = D + N$, where $D$ is diagonalizable, $N$ is nilpotent and $D$ and $N$ are polynomials in $T$.

(c) If $T = D' + N'$, where $D'$ is diagonalizable and $N'$ is nilpotent and $D'N' = N'D'$, prove that $D = D'$ and $N = N'$. 
15. (18 pts) Let notation be as in the previous problem and let \( f = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k} \) be the characteristic polynomial for \( T \). Thus \( n = d_1 + \cdots + d_k \) and \( 1 \leq r_i \leq d_i \) for each \( i \).

(a) If \( r_i + 1 = d_i \) for each \( i \in \{1, \ldots, k\} \), describe the Jordan form for \( T \).

(b) If \( r_i + 2 = d_i \) for each \( i \in \{1, \ldots, k\} \), how many different Jordan forms are possible for \( T \)?

(c) If \( r_i + 3 = d_i \) for each \( i \in \{1, \ldots, k\} \), how many different Jordan forms are possible for \( T \)?
16. (10 pts.) Let $V$ be an $n$-dimensional vector space over a field $F$.

(a) True or false: Every monic polynomial in $F[x]$ of degree $n$ is the characteristic polynomial of some linear operator on $V$. Justify your answer.

(b) True or false: Every monic polynomial in $F[x]$ of degree $n$ is the minimal polynomial of some linear operator on $V$. Justify your answer.