# 554 QUALIFYING EXAM, AUG 9, 2022

Attempt all questions. Time 2 hrs.

1. (5 + 10 pts)
   (a) Let $V$ be a vector space over a field $k$, and $S \subset V$. What does it mean to say that $S$ is linearly independent ?
   (b) Let $u \in \mathrm{End}(V)$. Suppose there exists $\alpha \in V$, and $m > 0$, such that
   $$u^m(\alpha) = \mathbf{0}, u^{m-1}(\alpha) \neq \mathbf{0}.$$
   Prove that $S = \{\alpha, u(\alpha), \ldots, u^{m-1}(\alpha)\}$ is linearly independent.
2. (5 + 10 pts) Let $V$ be a finite dimensional vector space over a field $k$ and $u \in \mathrm{End}(V)$.
   (a) Define $\mathrm{rank}(u)$.
   (b) Prove that
   $$2 \cdot \mathrm{rank}(u^2) \leq \mathrm{rank}(u) + \mathrm{rank}(u^3).$$
3. (5 + 10 pts) Let $V$ be a finite dimensional complex inner product space and $u \in \mathrm{End}(V)$.
   (a) What does it mean to say that $u$ is normal?
   (b) Prove that $u$ is normal if and only if there exists an orthonormal basis of $V$ consisting of eigenvectors of $u$.
4. (10 + 10 pts) Let $V$ be a finite-dimensional vector space over a field $k$. Let $u, v \in \mathrm{End}(V)$. Prove or disprove (with an example) the following statements.
   (a) Every eigenvector of $u \circ v$ is also an eigenvector of $v \circ u$.
   (b) Every eigenvalue of $u \circ v$ is an eigenvalue of $v \circ u$.
5. (5 + 10 pts) Let $V$ be an $n$-dimensional complex inner product space.
   (a) Define unitary transformations of $V$.
   (b) Suppose that $u, v \in \mathrm{End}(V)$ are unitary transformations. Prove that
   $$|\det(u + v)| \leq 2^n.$$
6. (5 + 5 + 10 pts)
   (a) State but do not prove the additive Jordan decomposition theorem for finite dimensional complex vector spaces.
   (b) Suppose that $V$ is a finite dimensional complex vector space and $u \in \mathrm{End}(V)$. Let $\mathrm{ad}(u) : \mathrm{End}(V) \to \mathrm{End}(V)$ be the map defined by
   $$\mathrm{ad}(u)(v) = u \circ v - v \circ u$$
   for each $v \in \mathrm{End}(V)$.
      (i) Prove that ad is a linear map $V \to \mathrm{End}(\mathrm{End}(V))$ (and so in particular $\mathrm{ad}(u) \in \mathrm{End}(\mathrm{End}(V))$ for each $u \in \mathrm{End}(V)$).
      (ii) Let $u \in \mathrm{End}(V)$ and suppose that $u = u_s + u_n$ is the additive Jordan decomposition of $u$. Prove that $\mathrm{ad}(u_s)+\mathrm{ad}(u_n)$ is the additive Jordan decomposition of $\mathrm{ad}(u)$.