554 QUALIFYING EXAM, JAN 6, 2022

Attempt all questions. Time 2 hrs.

1. (5 + 10 pts) Let A be an $n \times n$ complex matrix.
   (a) Define the adjugate, $\mathrm{adj}(A)$, of A.
   (b) Suppose eigenvalues of A are $\lambda_1, \ldots, \lambda_n$. Express the eigenvalues of $\mathrm{adj}(A)$ in terms of $\lambda_1, \ldots, \lambda_n$.
2. (5+5+5+5 pts) Let A, B be complex $n \times n$ matrices. Prove or disprove each of the following statements.
   (a) If A and B are diagonalizable, then so is $A + B$.
   (b) If A and B are diagonalizable, then so is AB.
   (c) If $A^2 = A$, then A is diagonalizable.
   (d) If A is invertible, and A^2 is diagonalizable, then A is diagonalizable.
3. (10+5+5 pts) Let V be a finite dimensional complex inner product space and $T \in \mathrm{End}(V)$.
   (a) Prove that there exists $P, U \in \mathrm{End}(V)$ such that P is positive semi-definite, U is unitary and such that $T = PU$.
   (b) Comment on the uniqueness of P, U.
   (c) Prove that in the decomposition in part (a), $PU = UP$ if and only if T is normal.
4. (5 + 10 pts) pts
   (a) When is a complex square matrix unitary ?
   (b) Prove that any complex square matrix A can be written as a product
   $$A = UDV$$
   where D is a diagonal matrix and U, V are unitary matrices.
5. (20 pts) Let A, B be $n \times n$ diagonalizable $n \times n$ complex matrices. Prove that A, B are simultaneously diagonalizable if and only if $AB = BA$.
6. (10 pts) Let V be the vector space of complex 2×2 matrices and $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in V$. Let $T \in \mathrm{End}(V)$ be defined by
   $$T(X) = XA - AX.$$
   Find the Jordan canonical form for the endomorphism T.