1. (5 + 10 pts) Let $A$ be an $n \times n$ complex matrix.
   (a) Define the adjugate, $\text{adj}(A)$, of $A$.
   (b) Suppose eigenvalues of $A$ are $\lambda_1, \ldots, \lambda_n$. Express the eigenvalues of $\text{adj}(A)$ in terms of $\lambda_1, \ldots, \lambda_n$.

2. (5+5+5+5 pts) Let $A, B$ be complex $n \times n$ matrices. Prove or disprove each of the following statements.
   (a) If $A$ and $B$ are diagonalizable, then so is $A + B$.
   (b) If $A$ and $B$ are diagonalizable, then so is $AB$.
   (c) If $A^2 = A$, then $A$ is diagonalizable.
   (d) If $A$ is invertible, and $A^2$ is diagonalizable, then $A$ is diagonalizable.

3. (10+5+5 pts) Let $V$ be a finite dimensional complex inner product space and $T \in \text{End}(V)$.
   (a) Prove that there exists $P, U \in \text{End}(V)$ such that $P$ is positive semi-definite, $U$ is unitary and such that $T = PU$.
   (b) Comment on the uniqueness of $P, U$.
   (c) Prove that in the decomposition in part (a), $PU = UP$ if and only if $T$ is normal.

4. (5 + 10 pts) pts
   (a) When is a complex square matrix unitary ?
   (b) Prove that any complex square matrix $A$ can be written as a product

$$A = UDV$$

where $D$ is a diagonal matrix and $U, V$ are unitary matrices.

5. (20 pts) Let $A, B$ be $n \times n$ diagonalizable $n \times n$ complex matrices. Prove that $A, B$ are simultaneously diagonalizable if and only if $AB = BA$.

6. (10 pts) Let $V$ be the vector space of complex $2 \times 2$ matrices and $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in V$. Let $T \in \text{End}(V)$ be defined by

$$T(X) = XA - AX.$$ 

Find the Jordan canonical form for the endomorphism $T$. 