554 QUALIFYING EXAM, AUG 11, 2023

Attempt all questions. Time 2 hrs.

1. (5 + 10 pts)
   (a) When are two $n \times n$ complex matrices similar ?
   (b) Can an $n \times n$ complex matrix A be similar to the matrix $A + \mathrm{Id}_n$?
2. (5 + 10 pts)
   (a) Let V be a finite dimensional complex vector space and $u \in \mathrm{End}(V)$. What does it mean to say that u is diagonalizable ?
   (b) Let V be the vector space of complex 2×2 matrices and $u : V \to V$ be the map that takes each matrix to its transpose. Prove that u is a diagonalizable endomorphism of V.
3. (5 + 10 pts) Let V be a finite dimensional complex inner product space and $u \in \mathrm{End}(V)$.
   (a) What does it mean to say that u is self-adjoint ?
   (b) If u is self-adjoint prove that all eigenvalues of u are real.
4. (5+10 pts) Let V be a finite dimensional complex inner product vector space and $u, v \in \mathrm{End}(V)$.
   (a) What does it mean to say that u is normal ?
   (b) Prove that if u is normal, then $u^* = p(u)$ for some polynomial $p \in \mathbb{C}[X]$.
5. (10 pts) Let A be a 3×5, and B a 5×3 complex matrix such that
$$AB = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$
   Find the Jordan canonical form of the matrix BA.
6. (10 pts) Let V be a finite dimensional complex vector space, and let F be a set of diagonalizable endomorphisms of V. Prove that the set F is simultaneously diagonalizable if and only if for each $u, v \in F$, $u \circ v = v \circ u$.
7. (5 + 10 + 5 pts) Let V be a finite dimensional complex inner product space.
   (a) What does it mean to say that $u \in \mathrm{End}(V)$ is a unitary transformation ?
   (b) Suppose that $u \in \mathrm{End}(V)$ is unitary. Prove that U is diagonalizable, and if λ is an eigenvalue of u, then $|\lambda| = 1$.
   (c) Let $v \in \mathrm{End}(V)$ such that $\mathrm{Id}_V + v, \mathrm{Id}_V + v^2, \mathrm{Id}_V + v^3$ are all unitary. Prove that $v = \mathbf{0}$.