Math 55400 **Qualifying Exam** **January 3, 2023** **W. Heinzer**

PUID: ______________________________

Instructions:

1. The point value of each exercise occurs to the left of the problem.
2. No books or notes or calculators are allowed.

Page	**Points Possible**	**Points**
2	18	
3	22	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
11	20	
Total	200	

1. (18 pts) Let V be a finite-dimensional vector space over a field F and let W_1 and W_2 be subspaces of V. Prove that

$$\dim W_1 \quad + \quad \dim W_2 \quad = \quad \dim(W_1 \cap W_2) \quad + \quad \dim(W_1 + W_2).$$

2. Let V and W be finite-dimensional vector spaces over a field F, and let $T : V \to W$ be a linear transformation.

(a) (2 pts) Define the rank of T.

(b) (2 pts) Define the nullity of T.

(c) (10 pts) Prove that $\dim V = \operatorname{rank} T + \operatorname{nullity} T$

3. (8 pts) How many possible Jordan forms are there for a matrix $A \in \mathbb{C}^{3\times 3}$ that has the property that $A^3 = I$, the identity matrix? Justify your answer.

4. (12 pts) Let F be a field, let S be a set, and let $\mathcal{F}(S, F)$ be the set of all functions from S to F.

(a) As in Chapter 2 of Hoffman and Kunze, define vector addition and scalar multiplication on the set $\mathcal{F}(S, F)$ so that $\mathcal{F}(S, F)$ is a vector space over the field F.

(b) If S is a finite set with n elements what is the dimension of the vector space $\mathcal{F}(S, F)$? Justify your answer.

5. (8 pts) State true or false and justify your answer: If V is a finite-dimensional vector space and W_1 and W_2 are subspaces of V such that $V = W_1 \oplus W_2$, then for any subspace W of V we have $W = (W \cap W_1) \oplus (W \cap W_2)$.

6. Let D be a principal ideal domain and let M be a finitely generated D-module.

(a) (3 pts.) What does it mean for a subset $S = \{z_1, \dots, z_n\}$ of M to be a *generating set* for M?

(b) (3 pts.) What does it mean for a subset $S = \{z_1, \dots, z_n\}$ of M to be a *basis* for M.

(c) (7 pts.) What does it mean for a matrix $A \in D^{m \times n}$ to be a *relations matrix* for M? How is a relations matrix for M constructed?

(d) (7 pts.) State true or false and justify your answer: Every nonzero finitely generated D-module has a basis.

7. (20 pts) Let $T : V \to V$ be a linear operator on an n-dimensional vector space V, and let $\mathcal{F}$ be the vector space of linear operators $U : V \to V$ that commute with T.

(a) Prove that $\dim \mathcal{F} \geq n$.

(b) Prove that T has a cyclic vector if and only if every $U \in \mathcal{F}$ is a polynomial in T.

8. (20 pts) Let V be an abelian group generated by elements a, b, c. Assume the following relations hold: $2a = 4b, 2b = 4c, 2c = 4a$, and these three relations generate all the relations on a, b, c.

(a) Write down a relations matrix for V.

(b) Find generators x, y, z for V such that $V = \langle x \rangle \oplus \langle y \rangle \oplus \langle z \rangle$ is the direct sum of cyclic subgroups generated by x, y, z.

(c) What is the order of V?

(d) What is the order of the element a?

(e) List all the possible orders of elements in V.

9. (10 pts) Let V be a finite-dimensional vector space over an infinite field F. Prove that V is not the union of finitely many proper subspaces.

10. (10 pts) Let V be a finite-dimensional vector space over an infinite field F and let $\alpha_1, \ldots, \alpha_m$ be finitely many nonzero vectors in V. Prove that there exists a linear functional f on V such that $f(\alpha_i) \neq 0$ for each i with $1 \leq i \leq m$.

11. (20 pts) Let $T : V \to V$ be a linear operator on an n-dimensional vector space over a field F. Let $c_1, \ldots, c_k$ be distinct elements in F and let $p = (x - c_1)^{r_1} \cdots (x - c_k)^{r_k}$ be the minimal polynomial of T. Let $W_i = \{v \in V \mid (T - c_i I)^{r_i}(v) = 0\}$.

(a) Describe linear operators $E_i : V \to V$, $i = 1, \ldots, k$, such that $E_i(V) = W_i$, $E_i^2 = E_i$ for each i, $E_i E_j = 0$ if $i \neq j$, and $E_1 + \cdots + E_k = I$ is the identity operator on V.

(b) Describe how to obtain linear operators D and N such that $T = D + N$, where D is diagonalizable, N is nilpotent and D and N are polynomials in T.

(c) If $T = D' + N'$, where D' is diagonalizable and N' is nilpotent and $D'N' = N'D'$, prove that $D = D'$ and $N = N'$.

12. (20 pts) Let $\mathbb{Z}^{(3)} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and let $V = \mathbb{Z}/(5^4) \oplus \mathbb{Z}/(5^3) \oplus \mathbb{Z}$.

(a) Write down a relations matrix for V as a $\mathbb{Z}$-module.

(b) Let $x = (5^2, 5, 5) \in \mathbb{Z}^{(3)}$ and let w denote the image of x in V. Write down a relation matrix for the cyclic group $W = \langle w \rangle$.

(c) Write down a relations matrix for the quotient module V/W.

(d) What is the cardinality of the quotient module V/W?

13. (20 pts.) Let V be an n-dimensional vector space over a field F.

(a) True or false: Every monic polynomial in $F[x]$ of degree n is the characteristic polynomial of some linear operator on V. Justify your answer.

(b) True or false: Every monic polynomial in $F[x]$ of degree n is the minimal polynomial of some linear operator on V. Justify your answer.