554 QUALIFYING EXAM. SPRING 2025

Attempt all questions. Time 2 hrs.

1. (10 pts) Let V be a n-dimensional complex vector space and $u \in \mathrm{End}(V)$ such that $u^2 = \mathrm{Id}_V$. Prove that
$$\mathrm{rank}(\mathrm{Id}_V + u) + \mathrm{rank}(\mathrm{Id}_V - u) = n.$$
2. (5 + 5 + 5 pts)
   (a) Prove that a non-zero nilpotent square matrix with complex entries is not diagonalizable.
   (b) Prove that if A is a complex $n \times n$ matrix such that $A^k = \mathrm{Id}_n$ for some $k > 0$, then A is diagonalizable.
   (c) Find the possible Jordan canonical forms of an $n \times n$ complex idempotent matrix A (i.e. having the property that $A^2 = A$).
3. (5 + 5 pts) Let V be a complex inner product space and $\alpha_1, \ldots, \alpha_m \in V$. Prove that $\alpha_1, \ldots, \alpha_m$ are linearly dependent if and only if the determinant of the gram matrix
$$G(\alpha_1, \ldots, \alpha_m) = ((\alpha_i, \alpha_j))_{1 \leq i,j \leq m}$$
is equal to 0.
4. (10 + 10 pts) Let V be a finite dimensional complex vector space, and let F be a set of diagonalizable endomorphisms of V. Prove that the set F is simultaneously diagonalizable if and only if for each $u, v \in F$, $u \circ v = v \circ u$.
5. (10 + 5 + 5 pts) Let A be a complex $n \times n$ matrix, $n \geq 2$, all of whose entries are equal to 1.
   (a) Find the characteristic and minimal polynomials of A ?
   (b) Is A diagonalizable ? Justify your answer.
   (c) Find the Jordan and the rational canonical forms of A.
6. (5 + 10 + 10 pts) Let V be a finite dimensional complex inner product space.
   (a) What does it mean to say that $u \in \mathrm{End}(V)$ is a unitary transformation ?
   (b) Suppose that $u \in \mathrm{End}(V)$ is unitary. Prove that U is diagonalizable, and if λ is an eigenvalue of u, then $|\lambda| = 1$.
   (c) Let $v \in \mathrm{End}(V)$ such that $\mathrm{Id}_V + v, \mathrm{Id}_V + v^2, \mathrm{Id}_V + v^3$ are all unitary. Prove that $v = \mathbf{0}$.