QUALIFYING EXAMINATION

August 2025

MA 554

SCORES	
Problem	Points
1	
2	
3	
4	
5	
6	
7	
Total=	**/100**

Problem 1. (8 points) Let r, s, n be positive integers. Determine the number of distinct $\mathbb{Z}$-linear maps $\mathbb{Z}^r \longrightarrow (\mathbb{Z}/\mathbb{Z}n)^s$.(Justify your answer.)

Problem 2. (12 points) Let A be an $n \times n$ matrix with entries in a field K. Prove that $\det A = 0$ if and only if there exists an $n \times (n-1)$ matrix B and an $(n-1) \times n$ matrix C, both with entries in K, so that $A = BC$.

Problem 3. (16 points) Let R be a principal ideal domain, M an R-module, and φ, ψ nonzero R-linear maps $M \longrightarrow R$. Prove that $\ker\varphi = \ker\psi$ if and only if $a \cdot \varphi = b \cdot \psi$ for some nonzero elements a, b of R.

Problem 4. (19 points) Let $R = K[T]$ be the polynomial ring in one variable over a field K, let A be an $n \times n$ matrix with entries in R, and let $\varphi : R^n \longrightarrow R^n$ be the R-linear map with $\varphi(x) := A \cdot x$, where $x \in R^n$ is considered as a column vector. Write $M = \operatorname{coker} \varphi := R^n / \operatorname{im} \varphi$, which is an R-module and, in particular, a K-vector space via the inclusion $K \subset R$. Prove that:

(a) $\det A \neq 0$ if and only if M is a torsion R-module;

(b) if $\det A \neq 0$, then $\dim_K M = \deg(\det A)$, the degree of the polynomial $\det A \in R$.

Problem 5. (18 points) Let V be a finite dimensional vector space over a field K, let $\varphi \in \mathrm{End}_K(V)$ (i.e., φ is a K-linear map $V \longrightarrow V$), and let $q_\varphi \in K[T]$ be the minimal polynomial of φ (i.e., the monic polynomial of minimal degree with $q_\varphi(\varphi) = 0$). Prove that

(a) φ is diagonalizable if and only if q_φ is the product of distinct monic polynomials of degree one (i.e., polynomials of the form $T - \lambda$, $\lambda \in K$);

(b) if φ is diagonalizable and W is a φ-invariant subspace of V, then the restriction $\varphi_{|W}$ is also diagonalizable.

Problem 6. (14 points) Up to similarity determine all 5×5 matrices A with entries in $\mathbb{Q}$ that have characteristic polynomial $\chi_A = T^5 + 2\,T^4 + T^3 - 5\,T^2 - 10\,T - 5$. (Notice that $\chi_A = (T^3 - 5)(T^2 + 2\,T + 1)$; you may use the fact that $\sqrt[3]{5}$ is irrational.)

Problem 7. (13 points) Let A be a symmetric $n \times n$ matrix with entries in $\mathbb{R}$. Prove that for every positive integer k there exists a symmetric $n \times n$ matrix B with entries in $\mathbb{C}$ so that $A = B^k$.