562 Qualifying Exam–2001 Spring

1(a). Let $X_1 = z\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, X_2 = y\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$ be vector fields in R^3. Do they span the tangent space of a two dimensional surface at $(1, -1, 1)$? Explain.
(b). Given a dimension one distribution of vector fields on a differentiable manifold, is it always integrable? Why?
2 (a). Explain the notions of fundamental group $\pi_1(M)$ and de Rham coholomogy group $H^1(M)$ of a differentiable manifolds.
(b). Explain from definition that $\pi_1(R^2) = 0$ and $H^1(R^2) = 0$.

3. Consider a two torus T^2 in R^3 parametrized by

$$(x, y, z) = (r \sin u, (R + r\cos u)\sin v, (R + r\cos u)\cos v),$$

where $R > r > 0$ and $0 \le u, v < 2\pi$.
(a). Let $z : T^2 \to R$ be the projection into the third coordinate. Explain why the level set $z^{-1}(c)$ on T^2 is a regular submanifold for $c \neq R - r, -R + r$.
(b). Explain why the level set $z^{-1}(R - r)$ on T^2 is the image of an immersed submanifold. A simple geometric picture may help.
(c). Explicitly find two linear independent closed $1-$form on T^2 and a closed $2-$form on T^2. Explain why T^2 is orientable.
(d). Let $f : T^2 \to T^2$ be the map sending (u, v) to $(u + \pi, 2\pi - v)$ (modulo 2π). Show that f and the identity map i on T^2 form a group G of two elements. Is the quotient space T^2/G an orientable surface? Why?
(e). Explain, by looking into a picture of the manifold, the regions of u, v on which the Gaussian curvature is negative, positive or zero.

4. Let M be a orientable two dimensional Riemannian manifold. Let Ω be a non-trivial volume form on M.
(a). Explain Ω must be a closed two form on M.
(b). Is Ω an exact two-form? Explain.

5. Let C be a curve in R^4 parametrized by arc length. Prove that there exists an orthonormal frame $\{X_1, X_2, X_3, X_4\}$ of vectors along C such that

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{pmatrix} = \begin{pmatrix} 0 & a_1 & 0 & 0 \\ -a_1 & 0 & a_2 & 0 \\ 0 & -a_2 & 0 & a_3 \\ 0 & 0 & -a_3 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix},$$

where a_i are some functions along the curve C, and the differentiation is taken with respect to the parametrization of C.