QUALIFYING EXAMINATION

JANUARY 2004

MATH 562 - Professor Donnelly

Each question is worth ten points.

1. Prove that a d-dimensional manifold X, for which there exists an immersion $f : X \to R^{d+1}$, is orientable if and only if there is a smooth nowhere vanishing normal vector field along (X, f).

2. Define $\omega = \dfrac{-y\,dx}{x^2+y^2} + \dfrac{x\,dy}{x^2+y^2}$. Calculate $\displaystyle\int_\gamma \omega$, where γ is the curve $x^8 + y^8 = 1$, oriented counterclockwise.

3. Let $f(x, y, z) = x^2y + e^x + z$. Show that there exists a differentiable function $g(y, z)$, defined near $(y, z) = (1, -1)$, so that $g(1, -1) = 0$ and $f(g(y, z), y, z) = 0$.

4. Prove that the set of all 3×3 matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

   is a Lie group and that the exponential mapping in G maps T_eG in a one-one manner globally onto G.

5. Let M be a compact manifold and $f : M \to R$, a C^1 function. Show that there exist at least two points where $df = 0$. Give an example with exactly two points.

6. Suppose $p \le d$ and let $\omega_1, \omega_2, \ldots, \omega_p$ be linearly independent 1-forms on M^d such that for some $\theta_1, \theta_2, \ldots, \theta_p$, $\displaystyle\sum_{i=1}^{p} \theta_i \wedge \omega_i = 0$. Show $\theta = \displaystyle\sum_{j=1}^{p} A_{ij}\omega_j$ for C^∞ functions A_{ij}, satisfying $A_{ij} = A_{ji}$.

7. Show that $S^k \times S^\ell$ can be embedded in $R^{k+\ell+1}$.

8. Is the open ball B^n in R^n diffeomorphic to R^n?

9. Define $\zeta = \dfrac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{r^3}$ in $R^3 - 0$.
   a. Show $d\zeta = 0$
   b. Is ζ exact in $R^3 - 0$?
   c. Is ζ exact in the complement of each line through 0?

10. Prove that the unit tangent bundle of S^2 is diffeomorphic to $SO(3)$.