QUALIFYING EXAMINATION
JANUARY 2005
MATH 562 - Prof. Catlin

1. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be linearly independent vector fields defined in neighborhoods about $p \in M$ and $q \in N$, respectively, where M and N are manifolds of dimension n. Let $\eta_1, \ldots, \eta_n$ and $\omega_1, \ldots, \omega_n$ be the the corresponding dual frames of T^*M and T^*N. Let $f : M \to N$ be a smooth map satisfying $f(p) = q$.

   (a) Calculate the matrix of $f^* : T_q^*N \to T_p^*M$ in terms of the matrix of T_*f at p.

   (b) If $g = \sum_{k,l=1}^n g_{k,l}\omega_k \otimes \omega_l$ is a Riemannian metric defined in a neighborhood of $q \in N$, and if $G = \sum G_{i,j}\eta_i \otimes \eta_j$ is defined near p by $G = f^*g$, then calculate the matrix $[G_{i,j}]$ in terms of the matrices of g and f_*.

2. Let $O(N)$ denote the set of $n \times n$ real matrices A, such that ${}^tA\colon A = I$ where ${}^tA$ denotes the transpose of A. Show that A is a compact manifold.

3. Show that the de Rham cohomology group $H^n(M)$ has positive dimension if M is compact and orientable.

4. Using differential forms, show that real projective space $\mathbf{P}^n$ is not orientable if n is even.

5. Let $f(x, y, z)$ be a smooth positive function in $W = \mathbf{R}^n/\{0\}$ and define $F : W \to W$ by $F(x, y, z) = f(x, y, z)(x, y, z)$. Let

$$T = \{F(x, y, z); (x, y, z) \in S^2 \text{ and } x, y, z > 0\},$$

where $S^2 = \{(x, y, z); |x|^2 + |y|^2 + |z|^2 = 1\}$. If μ is the standard orientation on S^2, let T be given the orientation $F_*\mu$. Calculate

$$\int_T \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(|x|^2 + |y|^2 + |z|^2)^{3/2}}.$$