Qualifying Examination
January 2009
Math 562 – Professor Donnelly

1. Show that $x^3z^2 - z^3xy = 0$ is solvable for z as a function of (x, y) near $(1, 1, 1)$ but not near the origin. Compute z_x and z_y at $(1, 1)$.

2. Let M be a connected manifold and $p, q \in M$. Show that there exists a diffeomorphism ϕ with $\phi(p) = q$.

3. Is RP^n orientable?

4. Show that the only solutions on R^3 of both $(\partial_x + x\partial_y)f = 0$ and $(\partial_y + y\partial_z)f = 0$ are constant.

5. What is the geometrical significance of the differential form
$\phi = r^{-n} \sum_{i=1}^{n} (-1)^i x_i dx_1 \wedge dx_2, \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$, where $r^2 = \sum_{i=1}^{n} x_i^2$?

6. Let $X \in sl(2, R)$. If $\det X > 0$, show that $e^X = \cos(\det X)^{1/2}\, I + \sin(\det X)^{1/2}\, X/(\det X)^{1/2}$. What are the corresponding formulas when $\det X \leq 0$?

7. Does there exist a function on the two torus having exactly three critical points?

8. Show that the space of derivations $D : C^1(p) \to R$ is infinite dimensional.

9. Let $\gamma(t)$ be a 1–parameter subgroup of a Lie group. Suppose that $\gamma(t)$ intersects itself. Prove that $\gamma(t) = \gamma(t + L)$, for some L and all t.

10. Show that the system of equations

$$3x + y - z + u^2 = 0$$
$$x - y + 2z + u = 0$$
$$2x + 2y - 3z + 2u = 0$$

defines an embedded submanifold in R^4. Which of the coordinates x, y, z, u can serve as parameters?