Qualifying Exam — MA 56200 — Jan 2010

Name: ______________________________

Each problem is worth 6 points.

(1) (a) Prove that

$$M := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$$

is a smooth manifold. Determine its dimension.

(b) Does M have boundary? If yes, determine ∂M.

(2) Let M be a manifold without boundary.

(a) Give one definition of $T_x M$, $x \in M$.

(b) Let $f : M \longrightarrow \mathbb{R}$ be a smooth function. Define $Df(x) : T_x M \longrightarrow \mathbb{R}$.

(c) Let $m \in M$ be a maximum of f, that is, $f(x) \leq f(m)$ for all $x \in M$.
Prove that

$$Df(m) : T_m M \longrightarrow \mathbb{R}$$

vanishes.

(3) We denote by $M(n)$ the vector space of all $n \times n$ matrices. Let $O(n)$ be the orthogonal group, that is

$$O(n) := \{A \in M(n) \mid AA^t = \mathbb{1}\} .$$

(a) Prove that $O(n)$ is a manifold. Determine its dimension.

(b) Give an explicit description of $T_{\mathbb{1}} O(n)$.

(4) We denote by $\mathbb{D} := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 100\}$ the closed disk of radius 10. Let $v : \mathbb{D}^2 \longrightarrow \mathbb{R}^2$ be the vector field given by

$$v(x, y) = (p(x)e^x, q(y)e^y)$$

where $p(x) = x^3 - x$ and $q(y) = y^3 + 3y^2 + 2y$.

Compute the index $\mathrm{ind}(v)$.

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(5) Let $X = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}$ and $Y = \frac{\partial}{\partial y} - z\frac{\partial}{\partial x}$ be vector fields on $\mathbb{R}^3$ with respect to the standard coordinates $(x, y, z) \in \mathbb{R}^3$.

Compute $[X, Y]$.

(6) Let G be a Lie group. We denote by L_g resp. R_g left resp. right multiplication with $g \in G$.

(a) Give the definition of a left invariant vector field X and prove that $(R_g)_*X$ is also left invariant.

(b) Use part (a) to prove that the map $Ad(g) : \mathfrak{g} \longrightarrow \mathfrak{g}$ defined by

$$Ad(g)X := (R_{g^{-1}})_*X$$

is well-defined. That is, prove that $X \in \mathfrak{g}$ implies $Ad(g)X \in \mathfrak{g}$.

(c) State the definition of the exponential map $\exp : \mathfrak{g} \longrightarrow G$ in terms of the flow of a left invariant vector field. Prove

$$\big(Ad(g)X\big)(e) = \frac{d}{dt}\bigg|_{t=0} \big(g \cdot \exp(tX) \cdot g^{-1}\big) \ .$$

(7) We consider $\mathbb{R}^6$ with coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$ and define the 1-form

$$\lambda = \sum_{i=1}^{3} x_i \, dy_i$$

(a) Compute $\omega := d\lambda$.

(b) Compute $\omega \wedge \omega \wedge \omega$. Simplify all expressions!

(c) Show that $\omega \wedge \omega$ is exact, that is, find a 3-form Λ with $\omega \wedge \omega = d\Lambda$.