Math 562: August 2013 Qualifying Exam (McReynolds)

PUID Number:____

Work **four out of five** of the following problems. The time limit is two hours. Please explicitly indicate which four problems you want graded as otherwise this decision will be made for you with no guarantee it will be the optimal outcome.

Problem 1. [15 points]
Let $f\colon M \to N$ be a submersion of manifolds without boundary.

(a) Prove that if $U \subset M$ is open then $f(U) \subset N$ is open.

(b) Prove that if M is compact and N is connected, then f is onto.

(c) Prove that there is no submersion of a compact manifold without boundary to $\mathbf{R}^n$.

Problem 2. [15 points]
Identify $\mathbf{R}^4$ with 2 by 2 real matrices $\mathrm{M}(2,\mathbf{R})$ via

$$(x,y,z,w) \longmapsto \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

(a) Prove that $\det^{-1}(a)$ is a submanifold for all $a > 0$ where

$$\det\colon \mathrm{M}(2,\mathbf{R}) \to \mathbf{R}$$

is the determinant map.

(b) Prove that $\det^{-1}(1)$ intersects the planes $x+y+z+w=r$ transversely for almost every $r \in \mathbf{R}$.

Problem 3. [15 points]
Let $f\colon S^n \to S^n$ be a smooth map. Prove that if $\deg(f) \neq (-1)^n$, then f has a fixed point.

Problem 4. [15 points]
Prove that there exists $(n+1)$ vector fields $v_1,\dots,v_{n+1}$ on $S^1 \times S^n$ that are linearly independent at every $x \in S^1 \times S^n$. Use this to prove that $T(S^1 \times S^n)$ is diffeomorphic to $S^1 \times S^n \times \mathbf{R}^{n+1}$.

Problem 5. [15 points]
Let $f\colon \mathbf{R}^2 \to \mathbf{R}^3$ be given by

$$f(u,v) = (\sin u \cos v, \sin u \sin v, \cos u).$$

(a) With coordinates x, y, z on $\mathbf{R}^3$, compute

$$f^*(dx), f^*(dy), f^*(dx).$$

(b) Let $U = (0, \pi) \times (0, 2\pi)$ and

$$\omega = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy.$$

Compute

$$\int_U f^*(\omega).$$

Please indicate what orientation you are using.