Math 562: January 2013 Qualifying Exam

PUID Number:______________________________________

Work **four out of five** of the following problems. The time limit is two hours. Please explicitly indicate which four problems you want graded as otherwise this decision will be made for you with no guarantee it will be the optimal outcome.

Problem 1. [15 points]

Let X_1, X_2, Y be smooth manifolds and V_1, V_2, W be finite dimensional vector spaces.

(a) Let $L_j \colon V_j \longrightarrow W$ be linear maps for $j = 1,2$. Prove that if L_1 is onto, then

$$\mathrm{Image}(L_1) \oplus \{0\} + \{0\} \oplus \mathrm{Image}(L_2) + \Delta_W = W \times W$$

where Δ_W is the diagonal in $W \times W$.

(b) Let $f_j \colon X_j \longrightarrow Y$ be smooth maps for $j = 1,2$. Prove that if f_1 is a submersion, then

$$Z = \{(x_1, x_2) \in X_1 \times X_2 \ : \ f_1(x_1) = f_2(x_2)\}$$

is a submanifold.

Problem 2. [15 points]

Let S^1 be the standard unit circle in $\mathbf{R}^2$.

(a) Let $f \colon S^1 \longrightarrow \mathbf{R}^3$ be a smooth function such that df_x is nonzero for all $x \in S^1$. Prove that there exists a 2–plane V in $\mathbf{R}^3$ with $P_V \colon \mathbf{R}^3 \longrightarrow V$ being the orthogonal projection map such that $V \circ f \colon S^1 \longrightarrow V$ has $d(P_V \circ f)_x$ nonzero for all $x \in S^1$.

(b) Give an example of f in (a) that is also injective but $(P_V \circ f)$ is not injective for all choices P_V.

(c) Let $F \colon S^1 \longrightarrow \mathbf{R}^4$ be smooth and injective. Prove that there exists a 3–space W in $\mathbf{R}^4$ with $P_W \colon \mathbf{R}^4 \longrightarrow W$ being orthogonal projection such that $(P_W \circ F)$ is injective.

Problem 3. [15 points]

Let S^2 be the unit 2–sphere in $\mathbf{R}^3$ and define

$$F \colon S^2 \longrightarrow S^2$$

by

$$F(x,y,z) = (-z, -x, -y).$$

(a) Compute the Lefschetz number of F.

(b) Compute the oriented degree of F.

Problem 4. [15 points]
Let M be a smooth compact n–manifold without boundary. Prove or disprove the following:

(a) M admits a vector field with only finitely many zeroes.

(b) M admits a vector field with no zeroes.

Problem 5. [15 points]
Let S^2 be the standard 2–sphere in $\mathbf{R}^3$ and

$$\omega = (3x^2 \cos y + e^{xy})dx \wedge dy + 17x^3 dx \wedge dz + (x + yz^5 + \sin z)dy \wedge dz.$$

Compute

$$\int_{S^2} \omega.$$

Please indicate what orientation you are using.