562 Qualifying Exam–Spring 2014

1. Let M be a differentiable manifold of dimension n. Let p be a point on M.
(a). Show that there exist small open sets $U \subset V \subset M$ and a smooth function h on M satisfying $h \cong 1$ on U, $h \cong 0$ on $M - V$ and $0 \leq h \leq 1$ everywhere on M.
(b). Does there exists a real analytic function h satisfying the same properties on some U and V as above? Explain.

2. Is the surface S defined by

$$\begin{aligned} x^2 &= y^2 \\ z^2 &= xw \end{aligned}$$

in R^4 a differentiable manifold? Explain.

3. (a). Given vector fields $X_1 = \frac{\partial}{\partial w} + z\frac{\partial}{\partial z}$ and $X_2 = \frac{\partial}{\partial z} + w\frac{\partial}{\partial y}$ on R^4. Can we find a two dimensional manifold M in R^4 such that X_1 and X_2 are tangential to M everywhere on M? Explain.
(b). Can we find a two dimensional manifold N such that X_1 is tangential to M everywhere on N? Explain.

4. Let M be the unit ball in R^4, with ∂M the unit sphere of dimension 3 in R^4. Let

$$\omega = w^3 dx \wedge dy \wedge dz + x^3 dy \wedge dz \wedge dw + y^3 dz \wedge dw \wedge dx + z^3 dw \wedge dx \wedge dy.$$

Find $\int_{\partial M} \omega$.

5. For each of the following manifolds, determine if there exists some smooth nowhere zero vector fields. Give an example when the answer is affirmative and explain with reason if the answer is negative.
(a). The unit sphere S^3 in R^4.
(b). The four torus $T^4 = R^4/Z^4$
(c). The unit sphere S^4 in R^5.

6. Let $T^n = R^n/Z^n$ be a torus of dimension n. Show from definition of the de Rham cohomology group $H^i(T^n)$ that dimension of the first de Rham cohomology group $\dim(H^1(T^n)) \geq n$. Based on the above fact, give a lower bound of $\dim(H^i(T^n))$ for all other values of $0 \leq i \leq n$. Explain your work.

7. Let α be a closed differential n form with compact support on R^n. Prove that there exists a n form β with support on the unit ball and a $(n-1)$ form γ with compact support on R^n such that

$$\alpha = \beta + d\gamma.$$