Math 562: Summer 2015 Qualifying Exam (McReynolds)

PUID Number:______________________________________

Work **four out of five** of the following problems. The time limit is two hours. Please explicitly indicate which four problems you want graded.

Problem 1. [15 points]

Let $\mathrm{M}(n,\mathbf{R})$ be the set of all n by n matrices (this is a manifold diffeomorphic to $\mathbf{R}^{n^2}$). Let $\mathrm{M}_k(n,\mathbf{R})$ denote the subset of all rank k matrices. Prove that $\mathrm{M}_k(n,\mathbf{R})$ is a submanifold and find its dimension.

Problem 2. [15 points]

Let $f\colon \mathbf{R}^5 \to \mathbf{R}^3$ be a smooth map. Prove that there exists a sphere $S \subset \mathbf{R}^3$ centered about the origin such that $f^{-1}(S)$ is a smooth submanifold of $\mathbf{R}^5$.

Problem 3. [15 points]

Let X,Y be compact, oriented n–manifolds without boundary and assume that Y is connected. Prove that if $f\colon X \to Y$ is a smooth function, then

$$\deg(f) = I(\mathrm{Graph}(f), X \times \{y\})$$

for any $y \in Y$.

Problem 4. [15 points]

Let $S^2 \subset \mathbf{R}^3$ be the standard 2–sphere and $i\colon S^2 \to \mathbf{R}^3$ the inclusion map. Define

$$\omega = (x^2+x+y)dy \wedge dz.$$

(a) Calculate

$$\int_{S^2} \omega.$$

State which orientation you are using.

(b) Prove or disprove: there exists a closed form $\alpha \in \Omega^2(\mathbf{R}^3)$ such that $i^*(\alpha) = i^*(\omega)$.

Problem 5. [15 points]

Let M,N be compact, oriented manifolds of dimension m,n, respectively. Orient $M \times N$ with the product orientation and let $P_M, P_N\colon M \times N \longrightarrow M,N$ be the projection maps onto M,N, respectively. Prove that if $\omega \in \Omega^m(M)$ and $\eta \in \Omega^n(N)$ that

$$\int_{M\times N} P_M^*(\omega) \wedge P_N^*(\eta) = \int_M \omega \cdot \int_N \eta.$$