Qualifying Exam | Math 562
January 2017 | Professor Donnelly

1. Give an example to show that a closed connected submanifold of a connected manifold does not necessarily carry the relative topology.

2. Show that $G\ell(n, C)$ is connected.

3. Let $X = F\partial_x + G\partial_y$ be a C^∞ vector field defined on all of R^2 and suppose that there is a constant K such that $|F|+|G| \leq K$. Show that X is complete. Is this a necessary condition for completeness?

4. Find a vector field $\vec{V}$ such that curl $\vec{V} = y\vec{\imath} + z\vec{\jmath} + x\vec{k}$.

5. Prove that the set of all 3×3 matrices of the form $\begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix}$ is a Lie group.

6. Show that $\exp\colon \mathfrak{g} \to G$ is $1-1$ and onto, where G is the Lie group of Problem 5.

7. Let M be a smooth manifold and B a closed subset of M. Show that there is a continuous function $\psi\colon M \to R$ that is smooth and positive on $M - B$ and zero on B.

8. If $\pi\colon M \to N$ is a submersion and X is a vector field on N, show that there is a smooth vector field on M that is π-related to X. Is it unique?

9. Show that the tangent bundle of S^3 is diffeomorphic to $S^3 \times R^3$.

10. Find the area inside the loop of Descartes' folium, $0 \leq t < \infty$,

$$x = \frac{t}{1+t^3} \quad , \quad y = \frac{t^2}{1+t^3}$$