MA 562 Qualify Exam August 2018

Notations:

- $\mathbb{R}$: set of real numbers; $\mathbb{C}$: set of complex numbers.
- $S^n := \{x \in \mathbb{R}^{n+1}; x_1^2 + \cdots + x_{n+1}^2 = 1\}$: unit sphere in $\mathbb{R}^{n+1}$.
- $\mathbb{R}P^n = S^n/\{\text{anti-podal map}\}$: n-dimensional projective space.

Problems (each problem has 20 points)

1. Consider the two form $F = \frac{x dy\wedge dz + y dz\wedge dx + z dx\wedge dy}{(x^2+4y^2+9z^2)^{3/2}}$ on $\mathbb{R}^3 - \{(0,0,0)\}$. Prove that F is a closed form. Is F an exact form?

2. Prove that $M = \{(z_1, z_2) \in \mathbb{C}^2; z_1^2 + z_2^2 = 1\} = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2; x_1^2 - y_1^2 + x_2^2 - y_2^2 = 1, x_1 y_1 + x_2 y_2 = 0\}$ is a real 2-dimensional smooth manifold that is diffeomorphic to $S^1 \times \mathbb{R}$.

3. Prove that the tangent bundle of S^3 is trivial. In other words, there are 3 smooth vector fields on S^3 that are linearly independent at any point. Is the tangent bundle of $\mathbb{R}P^3$ trivial?

4. Let M, N be two n-dimensional manifolds.
   (a) Prove that if $f : M \to N$ is an immersion, then f is also a submersion.
   (b) Prove that if $f : M \to N$ is a submersion, then f is an open map, i.e. it maps open sets on open sets. Use this to show that if M is compact and N is connected, then $f(M) = N$.
   (c) Prove that there is no immersion from S^n into $\mathbb{R}^n$.

5. Let $M = \mathbb{R} \times \mathbb{R}_+ = \{(x, y) \in \mathbb{R}^2; y > 0\}$ be the upper half plane. Calculate the local one parameter group of diffeomorphisms $\theta : W \to M$ generated by the vector field $V = \frac{\partial}{\partial x} + xy^2 \frac{\partial}{\partial y}$ on M. Find the domain $W = \{(t, x, y); \alpha(x, y) < t < \beta(x, y)\} \subseteq \mathbb{R} \times M$ of θ.

6. Consider the distribution Δ on $\mathbb{R}^3$ defined as the kernel of a nowhere zero 1-form α. In other words, $\Delta_p = \mathrm{Ker}(\alpha_p) \subset T_p\mathbb{R}^3$ for any $p \in \mathbb{R}^3$. Prove that the following conditions are equivalent:
   (a) Δ is involutive: $[X, Y] \subset \Delta$ for any smooth vector fields $X, Y \subset \Delta$.
   (b) In a neighborhood of any $p \in \mathbb{R}^3$, there exists a 1-form φ such that $d\alpha = \alpha \wedge \varphi$.
   (c) $d\alpha \wedge \alpha = 0$.