Solve all six problems. Justify your answers. Each problem is worth 10 points.

1. Let $M$ be a manifold of dimension $n$ and let $a$ and $b$ be real numbers with $a < b$. Let $f: M \to (a,b)$ and $g: (a,b) \to M$ be $C^\infty$ functions. If $f(g(t)) = t$ for every $t \in (a,b)$, does it follow that $n = 1$? If $g(f(x)) = x$ for every $x \in M$ does it follow that $n = 1$? In each case, give a proof or a counterexample.

2. For each $a \in \mathbb{R}$, let let $f_a: \mathbb{R} \to \mathbb{R}^2$ be given by
   
   $f_a(t) = (\sin(a \arctan t), \sin(2a \arctan t))$.

   (a) For which values of $a$ is $f_a$ an immersion?

   (b) For which values of $a$ is $f_a$ injective?

   (c) For which values of $a$ is $f_a$ an embedding?

3. Let $M$ and $N$ be manifolds, and let $\Gamma$ be a subset of $N$. Let $\mathcal{I}$ be the set of injective immersions $f: M \to N$ such that $f(M) = \Gamma$. For $f$ and $g$ in $\mathcal{I}$, define the relation $f \sim g$ to mean that there is a diffeomorphism $h$ of $M$ such that $f \circ h = g$.

   (a) Prove that $\sim$ is an equivalence relation.

   (b) Prove that if $f \in \mathcal{I}$ and $g \in \mathcal{I}$ are embeddings, then $f \sim g$.

   (c) In the case that $M = \mathbb{R}$ and $N = \mathbb{R}^2$; give an example of $\Gamma \subset N$ and $f \in \mathcal{I}$ and $g \in \mathcal{I}$ such that $f \not\sim g$.

4. Let $N$ be a closed regular submanifold of a manifold $M$. Let $f \in C^\infty(N)$.

   (a) Show there is a function $g \in C^\infty(M)$ such that $f(p) = g(p)$ for all $p \in N$.

   (b) Suppose additionally that $f$ is bounded. Show that there is a function $h \in C^\infty(M)$ such that $f(p) = h(p)$ for all $p \in N$ and such that $|h(p)| \leq \sup |f|$ for all $p \in M \setminus N$.

   (c) Under what necessary and sufficient additional condition on $f$ can we replace the ‘$\leq$’ in part (b) by ‘<’?

5. Let $A$ be an antisymmetric $3 \times 3$ matrix, and consider the vector field on $\mathbb{R}^3$ given by $V(x) = Ax$.

   (a) Prove that $V(x)$ is tangent to all spheres in $\mathbb{R}^3$ centered at the origin.

   (b) For any $w \in \mathbb{R}^3$ such that $Aw = 0$, prove that $V(x)$ is tangent to all spheres in $\mathbb{R}^3$ centered at $w$.

   (c) Let $\theta(t,p)$ be the one-parameter group on the unit sphere $S^2$ corresponding to $V(x)$. Prove that for every $p \in S^2$, there is a circle $C_p$ containing the orbit $\{\theta(t,p): t \in \mathbb{R}\}$. 
6. Let $f: (0, \infty) \to \mathbb{R}$ be a $C^\infty$ function and define a differential 1-form on $\mathbb{R}^n \setminus \{0\}$ by

$$\omega = f(|x|^2) \sum_{j=1}^{n} x^j dx^j.$$

(a) Let $G: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ be given by $G(y) = \exp(-|y|^2)y$. Compute $G^* \omega$.

(b) Compute $d\omega$.

(c) If $f$ is the natural logarithm function, do there exist functions $h \in C^\infty(\mathbb{R}^n \setminus \{0\})$ such that $dh = \omega$? If yes, give an explicit formula for all of them. If no, prove it.