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QUALIFYING EXAM MA 571 FALL 1995

1. Let X be the union in R^2 of the two line segments I and J where
   $I = \{(t,0)| -1 \le t \le 1\}$, (a VERTICAL interval) and
   $J = \{(0,t)| -1 \le t \le 1\}$ (a HORIZONTAL interval).
   Let O be the point of X common to both I and J.
   Prove that any continuous injective map of X to itself maps O to O.

2. Let X be a metric space. Let $B(x;r) = \{y|d(x,y) < r\}$, the open ball about x of radius r.
   A) Give an example where $X = B(x_0;1)$ but no finite number of 1/2 balls covers X,
   B) Suppose Y is a dense subset of X and is totally bounded. Prove X is too.

3. A) Let the dimensions p, q and r each be at least 2. Then the one-point union of the spheres $S^p \vee S^q \vee S^r$ is simply connected.
   B) Construct a universal covering space for $P^2 \vee S^2$, where P^2 is the projective space and S^2 the sphere.
   C) From your answer to 3B), find the fundamental group of $P^2 \vee S^2$.

4. Let K be the topologist's comb: K is the subset of R^2 which is the union of the vertical closed segments
   $t \times I = \{t\} \times \{(t,u)|\ 0 \le u \le 1\}$ for $t = 1, 1/2, 1/3, .., 1/n, ..$ and $t = 0$;
   PLUS the horizontal segment $\{(z,0)|\ 0 \le z \le 1\}$.
   Let A be the comb after we replace the VERTICAL segment $0 \times I$ with $0 \times Q$;
   let B be the comb after we replace the VERTICAL segment $0 \times I$ with $0 \times S$,
   where:
   Q is the rational numbers in $[0,1]$; and
   S is the IRRATIONAL numbers in $[0,1]$.
   PROVE A and B are NOT homeomorphic.

5. A) Let X be a compact Hausdorff space. If $C_1 \supseteq C_2 \supseteq C_3 \supseteq ...$ is a decreasing sequence of closed connected subsets of X. PROVE that $C_* = \bigcap C_n$, the intersection of ALL the sets, is connected too.
   B) Give an example in which X is NOT compact but all else is as above, and in which the result in (A) fails.

6. Let A be a T_2 space. For any space X define $F(X)$ to be the set of all continuous maps $f : X \longrightarrow A$.
   Define a map $\Phi_X : X \longrightarrow A^{F(X)}$ by defining the value on the point x of X to be the point with f-coordinate $\pi_f(x) = f(x)$ for each index $f \in F(X)$.

   A) Prove Φ is continuous.

   B) Prove Φ is injective if for each $x \ne y$ there is a continuous map $f : X \longrightarrow A$ for which $f(x) \ne f(y)$.

   C) Prove Φ is an embedding if for each point x and open set U containing x, there are maps $f_1, ..., f_n$ in $F_A(X)$ and open sets $O_1, ..., O_n$ in A for which $x \in f_1^{-1}(O_1) \cap ... \cap f_n^{-1}(O_n) \subseteq U$.