MATH 571 - McClure QUALIFYING EXAM August 1998

1. Let X_α be an infinite family of topological spaces.

   (a) (6 points) Define the product topology on
   $$\prod_\alpha X_\alpha$$

   (b) (8 points) For each α, let A_α be a subspace of X_α. Prove that
   $$\overline{\prod_\alpha A_\alpha} = \prod_\alpha \overline{A_\alpha}.$$

2. (14 points) Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. Let G_f (called the *graph* of f) be the subspace $\{\, (x, f(x)) \,|\, x \in X \,\}$ of $X \times Y$. Prove that if Y is Hausdorff then G_f is closed.

3. (14 points) Let X be a topological space and let $f, g : X \to \mathbb{R}$ be continuous. Define $h : X \to \mathbb{R}$ by
   $$h(x) = \min\{f(x), g(x)\}$$
   Use the pasting lemma to prove that h is continuous. (You will not get full credit for any other method.)

4. (14 points) Define an equivalence relation on the interval $[-1, 1]$ by
   $$x \sim y \Leftrightarrow x = y \text{ or } x = -y$$
   (you may *assume* that this is an equivalence relation, you do not have to prove it). Let X be the quotient space determined by this equivalence relation. Prove that X is homeomorphic to the interval $[0, 1]$.

5. (14 points) Prove from the definitions that a compact Hausdorff space is normal. (You may use the fact that a closed subset of a compact space is compact.)

6. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. Let $x_0 \in X$ and let $y_0 = f(x_0)$.

   (a) (6 points) Give the definition of the function $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, including the proof that it is well-defined.

   (b) (10 points) Prove that if f is a covering map then f_* is one-to-one.

7. (14 points) Let D^2 be the unit disk $\{\, x^2 + y^2 \leq 1 \,\}$ and let S^1 be the unit circle $\{\, x^2 + y^2 = 1 \,\}$. Prove that S^1 is not a retract of D^2 (that is, prove that there is no continuous function $f : D^2 \to S^1$ whose restriction to S^1 is the identity function). You may use anything in Munkres for this.