QUALIFYING EXAMINATION

AUGUST 2004

MATH 571 - Prof. McClure

Each problem is worth 14 points and you get two points for free.

1. Let X be a topological space, let D be a connected subset of X, and let $\{E_\alpha\}$ be a collection of connected subsets of X.

   Prove that if $D \cap E_\alpha \neq \emptyset$ for all α, then $D \cup (\bigcup_\alpha E_\alpha)$ is connected.

2. Let X be a topological space with an equivalence relation $\sim$. Suppose that the quotient space $X/\sim$ is Hausdorff.

   Prove that the set

   $$S = \{(x, y) \in X \times X \mid x \sim y\}$$

   is a closed subset of $X \times X$.

3. For any space X, let us say that two points are "inseparable" if there is no separation $X = U \cup V$ into disjoint open sets such that $x \in U$ and $y \in V$.

   Write $x \sim y$ if x and y are inseparable. Then $\sim$ is an equivalence relation (you don't have to prove this).

   Now suppose that X is locally connected (this means that for every point x and every open neighborhood U of x, there is a connected open neighborhood V of x contained in U).

   Prove that each equivalence class of the relation $\sim$ is connected.

4. Let X be a compact metric space and let $\mathcal{U}$ be a covering of X by open sets.

   Prove that there is an $\epsilon > 0$ such that, for each set $S \subset X$ with diameter $< \epsilon$, there is a $U \in \mathcal{U}$ with $S \subset U$. (This fact is known as the "Lebesgue number lemma.")

5. Recall that a space X is *locally compact* if, for each $x \in X$, there is an open set U containing x whose closure is compact.

   Let X be a locally compact Hausdorff space, let Y be any space, and let the function space $\mathcal{C}(X, Y)$ have the compact-open topology.

   Prove that the map

   $$e : X \times \mathcal{C}(X, Y) \to Y$$

   defined by the equation

   $$e(x, f) = f(x)$$

   is continuous.

6. Let X and Y be topological spaces and let $x \in X$, $y \in Y$.

   Prove that there is a 1-1 correspondence between

   $$\pi_1(X \times Y, (x, y))$$

   and

   $$\pi_1(X, x) \times \pi_1(Y, y).$$

   (You do **not** have to show that the 1-1 correspondence is compatible with the group structures.)

7. Let $p : Y \to X$ be a covering map, let $y \in Y$, and let $x = p(y)$.

   Let σ be a loop beginning and ending at x and let $[\sigma]$ be the corresponding element of $\pi_1(X, x)$.

   Let $\tilde{\sigma}$ be the unique lifting of σ to a path starting at y.

   Prove that if $[\sigma] \in p_*(Y, y)$ then $\tilde{\sigma}$ ends at y.