QUALIFYING EXAMINATION

AUGUST 2005

MATH 571 - Prof. McClure

Each problem is worth 14 points and you get two points for free.

1. Suppose that we are given an indexing set A, and for each $\alpha \in A$ a topological space X_α.

   Suppose also that for each $\alpha \in A$ we are given a point $b_\alpha \in X_\alpha$.

   Let $Y = \prod_\alpha X_\alpha$, with the product topology. Let $\pi_\alpha : Y \to X_\alpha$ be the projection.

   Prove that the set

   $$S = \{y \in Y \mid \pi_\alpha y = b_\alpha \text{ except for finitely many } \alpha\}$$

   is dense in Y (that is, its closure is Y).

2. Let X be a Hausdorff space and let A be a compact subset of X. **Prove** from the definitions that A is closed.

3. Let $p : X \to Y$ be a quotient map.

   Let us say that a subset S of X is *saturated* if it has the form $p^{-1}(T)$ for some subset T of Y.

   Suppose that for every $y \in Y$ and every open neighborhood U of $p^{-1}(y)$ there is a saturated open set V with $p^{-1}(y) \subset V \subset U$.

   Prove that p takes closed sets to closed sets.

4. Let X be a topological space and let $f : X \to X$ be a homeomorphism for which $f \circ f$ is the identity map.

   Suppose also that each $x \in X$ has an open neighborhood V_x for which $V_x \cap f(V_x)$ is empty.

   Define an equivalence relation $\sim$ on X by: $x \sim y$ if and only if $x = y$ or $f(x) = y$. (You do **not** have to prove that this is an equivalence relation; this is the only place where the assumption that $f \circ f$ is the identity is used).

   (a) (5 points) **Prove** that the quotient map $q : X \to X/\sim$ takes open sets to open sets.

   (b) (9 points) **Prove** that q is a covering map. (You may use part (a) even if you didn't prove it.)

5. Let S^1 be the circle

   $$\{(x_1, x_2) \mid x_1^2 + x_2^2 = 1\}$$

   in $\mathbb{R}^2$. Let $\mathbf{0}$ be the origin in $\mathbb{R}^2$.

   Prove from the definitions that S^1 is a deformation retract of $\mathbb{R}^2 - \mathbf{0}$.

6. Let X be a topological space and let $x_0 \in X$.

   Let U and V be open sets containing x_0, and suppose that the hypotheses of the Seifert-van Kampen theorem are satisfied (that is,

   $$U \cup V = X,$$

   and U, V, $U \cap V$ are path-connected).

   Let $i_1 : U \cap V \to U$, $i_2 : U \cap V \to V$, $j_1 : U \to X$ and $j_2 : V \to X$ be the inclusion maps.

   Suppose that $(i_1)_* : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ is an isomorphism.

   Prove, using the Seifert-van Kampen theorem, that there is an homomorphism

   $$\Phi : \pi_1(X, x_0) \to \pi_1(V, x_0)$$

   for which $\Phi \circ (j_2)_*$ is the identity map of $\pi_1(V, x_0)$.

7. Prove that every continuous map $f : P^2 \to S^1$ is homotopic to a constant map (where P^2 is the projective plane and S^1 is the circle). (Hint: use facts about covering spaces.)