MA 571 Qualifying Exam   August 2006   Prof. McClure

Each problem is worth 14 points and you get two points for free.

Unless otherwise stated, you may use anything in Munkres's book—but be careful to make it clear what fact you are using.

1. Let X be a connected space and let $f : X \to Y$ be a function which is continuous and onto. **Prove** that Y is connected. (This is a theorem in Munkres—prove it from the definitions).

2. Let X be the Cartesian product

$$\prod_{i=1}^{\infty} \mathbb{R}$$

with the **box topology** (recall that a basis for this topology consists of all sets of the form $\prod_{i=1}^{\infty} U_i$, where each U_i is open in $\mathbb{R}$).

Let

$$f : \mathbb{R} \to X$$

be the function which takes t to $(t, t, t, \dots)$

Prove that f is not continuous.

3. Let Y be a topological space.

Let X be a set and let $f : X \to Y$ be a function. Give X the topology in which the open sets are the empty set and the sets $f^{-1}(V)$ with V open in Y (you do **not** have to verify that this is a topology).

Let $a \in X$ and let B be a closed set in X not containing a.

Prove that $f(a)$ is not in the closure of $f(B)$.

For the next two problems, let P be the Cartesian product

$$\prod_{i=1}^{\infty} \{0, 1\}$$

with the usual Cartesian product topology. (**Note** that $\{0, 1\}$ is a set with two points, it is not an interval.)

4. **Prove** that every function from the Cantor set C to P which is 1-1, onto and continuous is a homeomorphism. (**Note:** the definition of the Cantor set is recalled at the end of this exam).

5. (a) Give a clear and specific description of a function from the Cantor set C to P which is 1-1 and onto. You do **not** have to prove that your function is 1-1 and onto.

   (b) **Prove** that the function you described in part (a) is continuous. (If it isn't continuous, go back and choose a different function that is).

6. Let X and Y be topological spaces, let $x_0 \in X$, $y_0 \in Y$, and let $f : X \to Y$ be a continuous function which takes x_0 to y_0.

   Is the following statement true? If f is 1-1 then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is 1-1. Prove or give a counterexample (and if you give a counterexample justify it). You may use anything in Munkres's book.

7. Let S^2 be the 2-sphere, that is, the following subspace of $\mathbb{R}^3$:

$$\{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \}.$$

   Let x_0 be the point $(0, 0, 1)$ of S^2.

   Use the Seifert-van Kampen theorem to **prove** that $\pi_1(S^2, x_0)$ is the trivial group. You may use either of the two versions of the Seifert-van Kampen theorem given in Munkres's book. You will **no**t get credit for any other method.

Definition of the Cantor set C. Let A_1 be the closed interval $[0, 1]$ in $\mathbb{R}$. Let A_2 be the set obtained from A_1 by deleting its "middle third" $(\frac{1}{3}, \frac{2}{3})$. Let A_3 be the set obtained from A_2 by deleting its "middle thirds" $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. Continue in this way to define a sequence of sets A_n: each A_n will be a finite disjoint union of closed intervals, and A_{n+1} is obtained by removing the open middle third of each of these closed intervals. The *Cantor set* is the intersection

$$C = \bigcap_{n=1}^{\infty} A_n.$$