MA 571 Qualifying Exam. August 2007. Professor McClure.

$\boxed{\text{Each problem is worth 14 points and you get two points for free.}}$

Unless otherwise stated, you may use anything in Munkres's book—but be careful to make it clear what fact you are using.

When you use a set theoretic fact that isn't obvious, be careful to give a clear explanation.

1. Let X and Y be topological spaces and let $f : X \to Y$ be a function with the property that

$$f(\overline{A}) \subset \overline{f(A)}$$

for all subsets A of X.

Prove that f is continuous.

2. Let X be a compact space and suppose we are given a nested sequence of subsets

$$C_1 \supset C_2 \supset \cdots$$

with all C_i closed. Let U be an open set containing $\cap C_i$.

Prove that there is an i_0 with $C_{i_0} \subset U$.

3. Prove the Tube Lemma: let X and Y be topological spaces with Y compact, let $x_0 \in X$, and let N be an open set of $X \times Y$ containing $\{x_0\} \times Y$, **then** there is an open set W of X containing x_0 with $W \times Y \subset N$.

4. Let $f : X \to Y$ be a function that takes closed sets to closed sets. Let $y \in Y$ and let U be an open set containing $f^{-1}(y)$.

Prove that there is an open set V containing y such that $f^{-1}(V)$ is contained in U.

5. Let $p : \mathbb{R} \to S^1$ be the usual covering map (specifically, $p(t) = (\cos 2\pi t, \sin 2\pi t)$). Let $b_0 \in S^1$ be the point $(1, 0)$. Recall that the standard map

$$\phi : \pi_1(S^1, b_0) \to \mathbb{Z}$$

is defined by $\phi([f]) = \tilde{f}(1)$, where $\tilde{f}$ is a lifting of f with $\tilde{f}(0) = 0$.

(a) (14 points) **Prove** that ϕ is 1-1.

(b) (14 points) **Prove** that ϕ is a group homomorphism.

6. Let X be a topological space and let $x_0 \in X$.

Let U and V be open sets containing x_0, and suppose that the hypotheses of the Seifert-van Kampen theorem are satisfied (that is,

$$U \cup V = X,$$

and U, V, $U \cap V$ are path-connected).

Let $i_1 : U \cap V \to U$, $i_2 : U \cap V \to V$, $j_1 : U \to X$ and $j_2 : V \to X$ be the inclusion maps.

Suppose that $(i_1)_* : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ is onto.

Prove, using the Seifert-van Kampen theorem, that $(j_2)_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$ is onto.