MA 571 Qualifying Exam. January 2008. Professor McClure.

Each problem is worth 14 points and you get two points for free.

Unless otherwise stated, you may use anything in Munkres's book—but be careful to make it clear what fact you are using.

When you use a set theoretic fact that isn't obvious, be careful to give a clear explanation.

1. Let X be a topological space with a countable basis. Prove that every open cover of X has a countable subcover.

2. Let X be a compact space, and suppose there is a finite family of continuous functions $f_i : X \to \mathbb{R}$, $i = 1, \ldots, n$, with the following property: given $x \neq y$ in X there is an i such that $f_i(x) \neq f_i(y)$. **Prove** that X is homeomorphic to a subspace of $\mathbb{R}^n$.

3. Let X be any topological space and let Y be a Hausdorff space. Let $f, g : X \to Y$ be continuous functions.

   Prove that the set $\{x \in X \mid f(x) = g(x)\}$ is closed.

4. Let X be the two-point set $\{0, 1\}$ with the discrete topology. Let Y be a countable product of copies of X; thus an element of Y is a sequence of 0's and 1's.

   For each $n \geq 1$, let $y_n \in Y$ be the element $(1, 1, \ldots, 1, 0, 0, \ldots)$, with n 1's at the beginning and all other entries 0. Let $y \in Y$ be the element with all 1's. **Prove** that the set $\{y_n\}_{n \geq 1} \cup \{y\}$ is closed. Give a clear explanation. Do not use a metric.

5. Let X be a connected space. Let $\mathfrak{U}$ be an open covering of X and let U be a nonempty set in $\mathfrak{U}$. Say that a set V in $\mathfrak{U}$ is *reachable from* U if there is a sequence

   $$U = U_1, U_2, \ldots, U_n = V$$

   of sets in $\mathfrak{U}$ such that $U_i \cap U_{i+1} \neq \emptyset$ for each i from 1 to $n - 1$.

   Prove that every nonempty V in $\mathfrak{U}$ is reachable from U.

6. Let $p : E \to B$ be a covering map. Suppose that points are closed in B. Let $A \subset E$ be compact. **Prove** that for every $b \in B$ the set $A \cap p^{-1}(b)$ is finite.

7. Let $p : E \to B$ be a covering map.

   Let Y be a path-connected space and let y_0 be a point of Y.

   Let $h, k : Y \to E$ be continuous functions with $h(y_0) = k(y_0)$ and $p \circ h = p \circ k$.

   Prove that h and k are the same function.