MA 571 Qualifying Exam. August 2010. Professor R. Kaufmann

Instructions

There are 8 problems, two of them are on the back. Each problem is worth 12 points and you get four points for free.

Unless otherwise stated, you may use anything in Munkres's book—but be careful to make it clear what fact you are using.

When you use a set theoretic fact that isn't obvious, be careful to give a clear explanation.

Problems

1. Let A, B be subsets of a space X. Denote by $Int(A)$ the interior of A, etc.. **Prove or disprove:**

   (a) $Int(A \cap B) = Int(A) \cap Int(B)$.

   (b) $Int(A \cup B) = Int(A) \cup Int(B)$.

2. Let X be the two-point set $\{0, 1\}$ with the discrete topology. Let Y be a countable product of copies of X; thus an element of Y is a sequence of 0's and 1's.

   For each $n \geq 1$, let $y_n \in Y$ be the element $(1, 1, \ldots, 1, 0, 0, \ldots)$, with n 1's at the beginning and all other entries 0. Let $y \in Y$ be the element with all 1's. **Prove** that the set $\{y_n\}_{n \geq 1} \cup \{y\}$ is closed. Give a clear explanation. Do not use a metric.

3. **Prove or disprove** the following:

   (a) If X is path-connected, and $f : X \to Y$ is continuous, then $f(X)$ is path-connected.

   (b) If X is locally path connected, and $A \subset X$ then A is locally path connected.

4. Let $X = [0, 1]/(\frac{1}{4}, \frac{3}{4})$ be the quotient space of the unit interval where the open interval is identified to a single point. Show that

   (a) X is connected.

   (b) X is compact.

   (c) X is not Hausdorff.

5. Let X be a compact space, and suppose there is a finite family of continuous functions $f_i : X \to \mathbb{R}$, $i = 1, \ldots, n$, with the following property: given $x \neq y$ in X there is an i such that $f_i(x) \neq f_i(y)$. **Prove** that X is homeomorphic to a subspace of $\mathbb{R}^n$.

6. **Prove** that $\mathbb{R}^2$ and $\mathbb{R}^2 \setminus \{0, 0\}$ are not homeomorphic. (Be careful to justify each step).

7. Let S^2 be the 2-sphere, that is, the following subspace of $\mathbb{R}^3$:

$$\{\, (x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1 \,\}.$$

Let x_0 be the point $(0, 0, 1)$ of S^2.

Use the Seifert-van Kampen theorem to **prove** that $\pi_1(S^2, x_0)$ is the trivial group. You may use either of the two versions of the Seifert-van Kampen theorem given in Munkres's book. You will **not** get credit for any other method.

8. Let M be a compact, connected, orientable surface of genus 2 with 2 boundary circles (see Figure 1).

Compute $\pi_1(M)$ **and** $H_1(M)$.

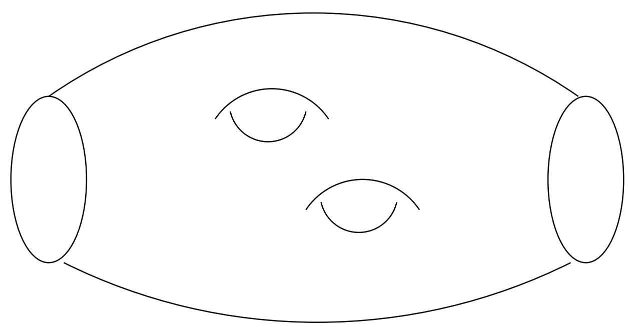

Figure 1: