MA 571 Qualifying Exam. August 2013 Professor R. Kaufmann

Instructions

There are 7 problems, two of them are on the back. Each problem is worth 10 points.

Unless otherwise stated, you may use anything in Munkres's book—but be careful to make it clear what fact you are using.

When you use a set theoretic fact that isn't obvious, be careful to give a clear explanation. To disprove a statement, give a counterexample and prove that it has the necessary properties.

Problems

1. (a) Give the definition of a sequence of points x_n converging to a point x in a topological space X.

   (b) **Prove or disprove!** A function $f : X \to Y$ is continuous if and only if for all sequences x_n converging to a point x in X the points $f(x_n)$ converge to to $f(x)$.

2. For an uncountable product $\mathbb{R}^J$ in the product topology let A be the subspace consisting of all points (x_α), for which $x_\alpha = 1$ for all but finitely many $\alpha \in J$. Show that there is a point in the closure of A which is not the limit of a sequence in A. Conclude using the Sequence Lemma that an uncountable product $\mathbb{R}^J$ in the product topology is not metrizable.

3. If f and g are continuous functions on a topological space X with values in a Hausdorff space Y and f and g agree on a dense subset of X, then $f = g$.

4. Let X be a locally compact Hausdorff space, let Y be any space, and let the function space $\mathcal{C}(X, Y)$ have the compact-open topology.

   Prove that the map
   $$e : X \times \mathcal{C}(X, Y) \to Y$$
   defined by the equation
   $$e(x, f) = f(x)$$
   is continuous.

5. Let X be the subspace $\{(x, y) \in \mathbb{R}^2 : xy = 0\}$, and let $f : X \to \mathbb{R}$ be the function defined by $f(x, y) = x$. **Prove or disprove!**

   (a) f is a continuous map.

   (b) f is an open map.

   (c) f is a closed map.

   (d) f is a quotient map.

6. Let X be the space obtained by attaching two discs to S^1, where the first disc D_1 is attached via the map $\partial D_1 = S^1 \to S^1, z \to z^3$ and the second disc is attached by $g : \partial D_2 = S^1 \to S^1 : z \to z^5$. Compute $\pi_1(X)$ and $H_1(X)$. (Hint use the special case of Seifert–van Kampen for adjoining 2–cells).

7. The Klein bottle K is the space obtained from a square by the labelling scheme $aba^{-1}b$.

   (a) Give a presentation of the fundamental group of $S = K\#K$, that is the connected sum of two copies of K.

   (b) Give the standard surface in the classification of surfaces that S is homeomorphic to and prove your answer.