MA 571 Qualifying Exam. August 2014. Professor McClure.

Each problem is worth 14 points and you get two points for free.

Please be careful that your handwriting is clear and easy to read.

Unless otherwise stated, you may use anything in Munkres's book—but be careful to make it clear what fact you are using.

When you use a set theoretic fact that isn't stated in Munkres and isn't obvious, be careful to give a clear explanation.

1. Let X be a topological space, let A be a subset of X, and let U be an open subset of X. **Prove** that $U \cap \bar{A} \subset \overline{U \cap A}$.

2. Let X be the following subspace of $\mathbb{R}^2$:

$$((0,1] \times [0,1]) \cup ([2,3) \times [0,1]).$$

Let $\sim$ be the equivalence relation on X with $(1,t) \sim (2,t)$ (that is $(s,t) \sim (s',t') \iff (s,t) = (s',t')$ or $t = t'$ and $\{s, s'\} = \{1, 2\}$); you do *not* have to prove that this is an equivalence relation). **Prove** that $X/\sim$ is homeomorphic to $(0,2) \times [0,1]$. (Hint: construct maps in both directions).

3. **Prove** that there is an equivalence relation $\sim$ on the interval $[0,1]$ such that $[0,1]/\sim$ is homeomorphic to $[0,1] \times [0,1]$. As part of your proof **explain** how you are using one or more properties of the quotient topology.

4. Let D be the closed unit disk in $\mathbb{R}^2$, that is, the set

$$\{(x,y) \,|\, x^2 + y^2 \leq 1\}.$$

Let E be the open unit disk

$$\{(x,y) \,|\, x^2 + y^2 < 1\}.$$

Let X be the one-point compactification of E, and let $f : D \to X$ be the map defined by

$$f(x,y) = \begin{cases} (x,y) & \text{if } x^2 + y^2 < 1 \\ \infty & \text{if } x^2 + y^2 = 1. \end{cases}$$

Prove that f is continuous.

5. Let X and Y be homotopy-equivalent topological spaces. Suppose that X is path-connected. **Prove** that Y is path-connected.

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6. Let a and b denote the points $(-1, 0)$ and $(1, 0)$ in $\mathbb{R}^2$. Let x_0 denote the origin $(0, 0)$.

   Use the Seifert-van Kampen theorem to calculate $\pi_1(\mathbb{R}^2 - \{a, b\}, x_0)$. You may not use any other method.

   You should state where you are using deformation retractions, but you don't have to give formulas for the retractions or the homotopies.

7. Let $p : E \to B$ be a covering map with B locally connected, and let $x \in B$. **Prove** that x has a neighborhood W with the following property: for every connected component C of $p^{-1}(W)$, the map $p : C \to W$ is a homeomorphism.