MA 571 Qualifying Exam. January 2016. Professor McClure.

Each problem is worth 14 points and you get two points for free.

Please be careful that your handwriting is clear and easy to read.

Unless otherwise stated, you may use anything in Munkres's book—but be careful to make it clear what fact you are using.

When you use a set theoretic fact that isn't stated in Munkres and isn't obvious, be careful to give a clear explanation.

1. Let $X = A \cup B$ and let C be a subset of $A \cap B$ which is closed in the subspace topology of A and also in the subspace topology of B. **Prove** that C is closed in the topology of X.

2. Let $\sim$ be the equivalence relation on $[0,1] \times [0,1]$ with $(s,0) \sim (s,1)$ (that is, two points (s,t) and (s',t') are equivalent if they are equal or if $s = s'$ and $\{t,t'\} = \{0,1\}$). Let X be the quotient space $([0,1] \times [0,1])/\sim$ and define $f : X \to X$ by

$$f([s,t]) = \begin{cases} [s, t + \frac{1}{2}] \text{ if } t \le \frac{1}{2} \\ [s, t - \frac{1}{2}] \text{ if } t \ge \frac{1}{2} \end{cases}$$

   Prove that f is well-defined and continuous.

3. Let X be a compact space and let $\{C_\alpha\}_{\alpha \in A}$ be a collection of closed sets in X. Let $C = \bigcap_{\alpha \in A} C_\alpha$ and let U be an open set containing C. **Prove** that there is a finite set $\alpha_1, \cdots, \alpha_n$ in A with $C_{\alpha_1} \cap \cdots \cap C_{\alpha_n} \subset U$.

4. Recall that a space is locally compact if every point has a neighborhood which is contained in a compact set. Suppose that X is locally compact, and let $f : X \to Y$ be a closed continuous surjective map such that $f^{-1}(y)$ is compact for every $y \in Y$. **Prove** that Y is locally compact. (Hint: first prove that if $U \subset X$ is open then $Y - f(X - U)$ is open.)

5. Let X and Y be topological spaces and let $x_0 \in X$, $y_0 \in Y$. **Prove** that there is a function from $\pi_1(X \times Y, (x_0, y_0))$ to $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ which is 1–1 and onto (you do **NOT** have to show that it is a homomorphism).

6. Let a and b denote the points $(-1,0)$ and $(1,0)$ in $\mathbb{R}^2$. Let x_0 denote the origin $(0,0)$.

   Use the Seifert-van Kampen theorem to calculate $\pi_1(\mathbb{R}^2 - \{a,b\}, x_0)$. You may not use any other method.

   You should state where you are using deformation retractions, but you don't have to give formulas for the retractions or the homotopies.

7. Let $p : E \to B$ be a covering map with B locally connected. Let D be a component of E. **Prove** that $p(D)$ is open **and** that $p : D \to p(D)$ is a covering map.