Math 571 Qualifying Exam **January 2016**

Each problem is worth ten points, for a total of sixty possible points.

1. Let X be a set and let $\mathcal{P}(X)$ denote the power set of X (the set of all subsets of X). Suppose given a function $f : \mathcal{P}(X) \to \mathcal{P}(X)$ such that

   (a) $f(\emptyset) = \emptyset$,

   (b) for any Y in $\mathcal{P}(X)$, $Y \subseteq f(Y)$,

   (c) for any Y and Z in $\mathcal{P}(X)$, $f(Y \cup Z) = f(Y) \cup f(Z)$, and

   (d) $f \circ f = f$.

   Show that f determines a topology on X in which the closed subsets are precisely those subsets $Z \subseteq X$ such that $f(Z) = Z$, and conversely that if X is a topological space then the assignment $f(Y) = \overline{Y}$ determines a function $f : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying these properties.

2. Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces X_α, let $X = \prod_{\alpha \in A} X_\alpha$ be the cartesian product of the X_α, equipped with the product topology, and let $p_\alpha : X \to X_\alpha$ denote the projection map. Suppose given points $x_\alpha \in X_\alpha$ for all $\alpha \in A$, and let $Y \subseteq X$ denote the subspace of X consisting of those points $x \in X$ such that $p_\alpha(x) = x_\alpha$ for all but finitely many $\alpha \in A$. Show that Y is a dense subset of X (i.e. the closure of Y is equal to X).

3. An *embedding* is a continuous injection $f : X \to Y$ which is a homeomorphism onto its image; in other words, writing $f(X) \subseteq Y$ for the subspace of Y given by the image of f, f is an embedding if $X \to f(X)$ is a homeomorphism. Show that, if $f : X \to Y$ is a continuous injection of topological spaces such that X is compact and Y is Hausdorff, then f is an embedding.

4. Let $q : X \to Y$ be a continuous surjection. Show that q is a quotient map if, for any topological space Z and any function $f : Y \to Z$, f is continuous if and only if $f \circ q : X \to Z$ is continuous. Show additionally that if q is either open or closed then q is a quotient map.

5. Let $p : Y \to X$ be a covering space such that Y is simply connected and let $x_0 \in X$. Show that there exists a bijection of sets $\pi_1(X, x_0) \cong p^{-1}(x_0)$.

6. Let n be a positive integer and let $s_1, \ldots, s_n \in S^2$ be a sequence of n distinct points on the 2-sphere S^2. Let $X = S^2 - \{s_1, \ldots, s_n\}$ be the subspace of S^2 obtained as the complement of $\{s_1, \ldots, s_n\} \subset S^2$. Calculate $\pi_1(X, x)$ for a choice of basepoint $x \in X$.