MA 571- Qualifying Exam- January 2022

Every problem is worth 14 points. Please be careful that your handwriting is clear and easy to read.

Unless otherwise stated, you may use any theorem from Munkres' book but be careful to make it clear what fact you are using.

Problem 1: Recall a topological space X is *locally compact* if for any point $p \in X$ there is an open pset U and a compact set K such that $p \in U \subseteq K$. Let $\{X_n\}_{n=1}^{\infty}$ be non-empty topological spaces. Prove that if the product $\prod_{n=1}^{\infty} X_n$ is locally compact, then each X_n is locally compact and X_n is compact for all but finitely many n.

Problem 2: Prove that a space X is Hausdorff if and only if the diagonal $\Delta_X = \{x \times x \in X \times X | x \in X\}$ is closed in $X \times X$.

Problem 3: Let (M, d) be a metric space with metric $d : M \times M \to \mathbb{R}$. A map $f : M \to M$ is called a *contraction* if there is a real number $\lambda < 1$ such that

$$d(f(x), f(y)) = \lambda d(x, y)$$

for all $x, y \in M$. If f is a contraction and M is compact, prove f has unique fixed point, i.e. a point $p \in M$ such that $f(p) = p$. [Hint: Define $f^1 = f$ and $f^{n+1} = f \circ f^n$ and consider the intersection of the sets $A_n = f^n(X)$].

Problem 4: Let X be a locally compact Hausdorff space and Y any topological space. Consider the space $\mathcal{C}(X, Y)$ of continuous functions from X to Y equipped with the compact-open topology. Prove the map

$$e : X \times \mathcal{C}(X, Y) \to Y \to Y$$

defined by

$$e(x, f) = f(x)$$

is continuous.

Problem 5: Let S^2 be the standard 2-sphere in $\mathbb{R}^3$. Let $p_1, ..., p_n \in S^2$ be any n distinct points. Consider the space X obtained by identifying all $p_1, ..., p_n$ into a single point. More precisely, X is the quotient space $S^2/\sim$ where $\sim$ is the equivalence relation generated by declaring $x \sim y$ for $x \neq y$ if and only if $x = p_i$ and $y = p_j$ for two distinct $i, j \in \{1, ..., n\}$. Compute the fundamental group of X.

Problem 6: Let $p : E \to B$ be a covering map, A a connected space and $a \in A$. Prove that if two continuous functions $f, g : A \to E$ have the property that $f(a) = g(a)$ and $p \circ f = p \circ g$ then $f = g$.

Problem 7: Let X be the quotient space obtained from an octagon P by pasting its edges together according to the labelling scheme $aabbcdc^{-1}d^{-1}$.
a) Calculate the first homology group $H_1(X)$.
b) Assuming X is homeomorphic to one of the standard surfaces in the classification theorem, which surface is it?