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PURDUE UNIVERSITY
Study Guide for the Credit Exam in (MA 262)
Linear Algebra and Differential Equations

The topics covered in this exam can be found in "An introduction to differential equations and linear algebra" by Stephen Goode, Chapters 1–11. See also http://www.math.purdue.edu/academic/courses for recent course information and materials.

The exam consists of 25 multiple choice questions with two hour time limit. No books, notes, calculators or any other electronic devices are allowed.

IMPORTANT

1. Read this material thoroughly if you contemplate trying for advanced placement (and extra credit which counts toward graduation).
2. Study all the material listed in the outline.
3. Work the practice problems below. The correct answers are at the end.

Topics covered in the exam

Linear Algebra:

1. Complex numbers, polar representation, roots of complex numbers.
2. Vector spaces: subspace, basis, spanning set, linear combination, linear independence.
3. Matrix operations: addition, multiplication, inverse, determinants.
4. Row reduction of matrices: row-echelon normal form, rank.
5. Systems of linear equations: solve using matrix methods, augmented matrices, Cramer's rule, solution space.
6. Eigenvalues and eigenvectors.

Differential Equations:

1. First-order differential equations with and without initial conditions. Types include - separable variables, exact, linear.
2. Applications of first-order D.E.'s to mixture problems, growth and decay problems, falling bodies, electrical circuits, orthogonal trajectories.
3. Second and higher-order linear D.E.'s with constant coefficients: Undetermined coefficients, initial value problems. Variation of Parameters.
4. Applications of second-order D.E.'s to Newton's 2nd law, spring mass systems, electrical circuits, etc.

Systems of Differential Equations:

1. Matrix Formulation.
2. Solutions to linear systems with constant coefficient using eigenvalues and eigenvectors.
3. Variation of parameters for systems.

Circle the letter corresponding to your choice of correct answer.

1. Find the determinant of A if

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 3 & 4 & 1 \end{bmatrix}$$

A. -1

B. 0

C. 3

D. 7

E. 11

2. If A is a 3×4 matrix and $P = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$ and $Q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$ are solutions of $A \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}$ which of the following is also a solution?

A. $P + Q$

B. $P - Q$

C. $2P - Q$

D. $Q - 2P$

E. all of the above

3. Which of the following complex numbers z satisfies $z^2 = ri$ where r is some real number?

A. i^3

B. $1 + i\sqrt{3}$

C. $\sqrt{3} + i\sqrt{3}$

D. $\sqrt{3} + i$

E. none of these

4. Which of the following are real vector spaces (in each case multiplication by a real number and addition have the usual meanings)?
(i) The set of all real valued functions f such that $f(x+1) = 2f(x)$ for all x.
(ii) The set of all real polynomials p such that $p(1) = p(0) + 1$.
(iii) The set of all 4×4 matrices whose first and last rows are equal.

A. (i) and (iii)

B. none

C. (i) and (ii)

D. only (iii)

E. (ii) and (iii)

5. The smallest subspace of $\mathbb{R}^3$ containing the vectors $(1,2,1)$ and $(5,3,1)$ is the set of all (x,y,z) satisfying which of the following?

A. $x, y, z \geq 0$

B. $x^2 + y^2 + z^2 = 0$

C. $3x + y - z = 0$

D. $x + 5y + 7z = 0$

E. $x - 4y + 7z = 0$

6. Which of the following sets of vectors is linearly dependent?
(i) $(1,0,1)$, $(1,1,0)$, $(0,1,-1)$
(ii) $(1,0,1)$, $(1,1,0)$, $(0,1,1)$
(iii) $(1,0,1)$, $(0,1,0)$, $(0,0,0)$

A. none

B. (ii) only

C. (i) and (ii)

D. (i) and (iii)

E. (ii) and (iii)

7. The characteristic values of $A = \begin{bmatrix} 1 & 1-k \\ k+1 & 1 \end{bmatrix}$ are real if

A. $k^2 > 1$

B. $k^2 \geq 1$

C. $k^2 = 1$

D. $k^2 \leq 1$

E. $k^2 < 1$

8. The system of equations $\begin{bmatrix} 1 & 2 & 6 \\ 3 & 4 & 0 \\ 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has a nonzero solution

A. for all nonzero a, b, c

B. if $a = b + c$

C. if $2c = a + b$

D. if $b \neq a - c$

E. if $c \neq 2a - b$

9. If the 2×2 matrices $B = \begin{bmatrix} 3/2 & 1/2 \\ x & y \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ satisfy $B^2 = A$, then the bottom row of B is equal to

A. $(-1, 0)$

B. $\left(-\frac{1}{2}, \frac{1}{2}\right)$

C. $\left(-\frac{1}{2}, -\frac{1}{2}\right)$

D. $\left(\frac{1}{2}, -\frac{1}{2}\right)$

E. $\left(-\frac{1}{2}, 1\right)$

10. If $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$ then which of the following is not an entry of A^{-1}?

A. $\frac{2}{3}$

B. -2

C. $-\frac{1}{2}$

D. all are entries

E. none are entries

11. The matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{bmatrix}$$

has 1 as a characteristic value. What is the dimension of the vector space spanned by the characteristic vectors corresponding to 1?

A. 2

B. 0

C. 1

D. 3

E. cannot be determined

12. If A and B are 4×4 matrices and $\det A = 4$, $\det B = 6$, then $\det(A^{-1}B^T) =$

A. $\dfrac{1}{24}$

B. $\dfrac{2}{3}$

C. $\dfrac{3}{2}$

D. 24

E. Undefined

13. If $y' + \dfrac{1}{x}y = \dfrac{e^x}{ex}$ and $y(1) = 0$, then $y =$

A. $xe^{x-1} - 1$

B. $(x-1)e^x$

C. $\dfrac{1}{x^2}(e^{x-1} - 1)$

D. $\dfrac{1}{x}(e^{x-1} - 1)$

E. $(2x-2)e^x$

14. The solution of $(\cos^2 x)y' = y^2$ satisfying $y(\pi/4) = 1$ is

A. $y = 2 - \tan x$

B. $y = 2\tan x - 1$

C. $y = \dfrac{1}{2 - \cot x}$

D. $y = \dfrac{1 + \sin x}{1 + \cos x}$

E. $y = \dfrac{\cos x}{2\cos x - \sin x}$

15. The general solution of

$$\frac{dy}{dx} = \frac{2xy + 3x^2}{y^2 - x^2}$$

is

A. $\frac{y^3}{3} + x^2y + x^3 = c$

B. $\frac{y^3}{3} - x^2y - x^3 = c$

C. $2xy + y^2 + 3x^2 = c$

D. $\log(x^2 + y^2) = c$

E. none of the above

16. A tank initially contains 50 gal of water. Alcohol enters at the rate of 2 gal/min and the mixture leaves the tank at the same rate. The time t_0 when there is 25 gal of alcohol in the tank satisfies

A. $25 = e^{t_0/2}$

B. $1 = t_0/50$

C. $t_0 = 25 \ln 2$

D. $\frac{1}{25} = e^{t_0/25}$

E. $t_0 = 25$

17. The general solution of $y^{(4)} + 4y = 0$ is

A. $e^x(c_1 + c_2x) + e^{-x}(c_3 \cos x + c_4 \sin x)$

B. $e^x(c_1 \cos x + c_2 \sin x) + e^{-x}(c_3 \cos x + c_4 \sin x)$

C. $(c_1 + c_2x)e^{-x} + (c_3 + c_4x)e^x$

D. $e^{-x}(c_1 + c_2x + c_3x^2 + c_4x^3)$

E. none of the above

18. For a particular solution of $y'' + 3y' + 2y = xe^{-x}$ one should try a solution of the form

A. $Ae^{-2x} + Be^{-x}$

B. $Ae^{-x} + Bxe^x$

C. $(A + Bx)e^{-x}$

D. $(Ax + Bx^2)e^{-x}$

E. Ax^2e^{-x}

19. A body attached to the lower end of a vertical spring has acceleration $\dfrac{d^2x}{dt^2} = -16x$ ft/sec^2, where $x = x(t)$ is the distance of the body from the equlibrium position at time t. If the body passes through the equilibrium position at $t = 0$ with $v = 12$ ft/sec, then $x =$

A. $3\cos 4t$

B. $\cos 4t + 3\sin 4t$

C. $3\sin 4t$

D. $3e^{4t}$

E. $3\cos 4t - \sin 4t$

20. The substitution $t = e^x$ transforms $y'' + (e^x - 1)y' + e^{2x}y = e^{3x}$ into

A. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = t$

B. $t^2\dfrac{d^2y}{dt^2} + (t-1)\dfrac{dy}{dt} + t^2y = t^3$

C. $t^2\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + t^2y = t^3$

D. $\dfrac{d^2y}{dt^2} + (t-1)\dfrac{dy}{dt} + t^2y = t^3$

E. none of the above

21. The smallest order of a linear homogeneous differential equation with constant coefficients which has $y = xe^{2x}\cos x$ as a solution is

A. 3

B. 4

C. 5

D. 6

E. more than 6

22. A 2×2 matrix A has 3 as a characteristic value with multiplicity 2. Then two linearly independent solutions of $\mathbf{x}' = A\mathbf{x}$ are of the form

A. $\mathbf{k}_1e^{3x}, (\mathbf{k}_2 + \mathbf{k}_1)e^{3x}$

B. $\mathbf{k}_1e^{3x}, (\mathbf{k}_2 + t\mathbf{k}_3)e^{3x}$

C. $\mathbf{k}_1e^{3x}, \mathbf{k}_2te^{2x}$

D. $\mathbf{k}_1e^{3x}, \mathbf{k}_2e^{-x}$

23. If a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$ is $U(t) = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^t \end{bmatrix}$, then a particular solution $\mathbf{u}_p(t)$ of $\mathbf{x}' = A\mathbf{x} + \begin{bmatrix} e^{2t} \\ e^t \end{bmatrix}$, such that $\mathbf{u}_p(0) = \mathbf{0}$ is

A. $\begin{bmatrix} te^t \\ te^{2t} \end{bmatrix}$

B. $\begin{bmatrix} te^{2t} \\ t \end{bmatrix}$

C. $\begin{bmatrix} te^{2t} \\ te^t \end{bmatrix}$

D. $\begin{bmatrix} t^2e^{2t} \\ te^t \end{bmatrix}$

E. $\begin{bmatrix} e^{2t} - 1 \\ e^t - 1 \end{bmatrix}$

24. A 2×2 matrix A has -1 as a characteristic value with corresponding characteristic vector $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$]; and a characteristic value 2 with corresponding characteristic vector $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Then a solution to $\mathbf{x}' = A\mathbf{x}$ is

A. $\begin{bmatrix} 3e^{2t} \\ -e^{-t} \end{bmatrix}$

B. $\begin{bmatrix} 3e^{-t} \\ -e^{2t} \end{bmatrix}$

C. $\begin{bmatrix} e^{2t} \\ -3e^{2t} \end{bmatrix}$

D. $\begin{bmatrix} 3e^{-t} \\ -e^{-t} \end{bmatrix}$

E. none of these

25. The real 2×2 matrix A has characteristic values $1 + i$ and $1 - i$ with corresponding characteristic vectors $\begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -i \\ 1 \end{bmatrix}$. If $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ satisfies $\mathbf{x}' = A\mathbf{x}$ and $x_2(t) = e^t(\cos t + \sin t)$ then $x_1(t) =$

A. $e^t(\sin t - cost)$

B. $e^t(\sin t - 2\cos t)$

C. $e^{-t}(\cos t - \sin t)$

D. $e^t(\cos t - \sin t)$

E. none of these

ANSWER KEY

1.	D	14.	E
2.	C	15.	B
3.	C	16.	C
4.	A	17.	B
5.	E	18.	D
6.	D	19.	C
7.	B	20.	A
8.	C	21.	B
9.	B	22.	B
10.	A	23.	C
11.	A	24.	A
12.	C	25.	D
13.	D.		