June 2011

PURDUE UNIVERSITY
Study Guide for the Credit Exam in (MA 265)
Linear Algebra

This study guide describes briefly the course materials to be covered in MA 265. In order to be qualified for the credit, one is expected not only to "know" these materials but also to demonstrate the skills to solve the quantitative and numerical problems.

The current textbook used in MA 265 is: Introduction to Linear Algebra with Applications by Bernard Kolman. See also http://www.math.purdue.edu/academic/courses for recent course information and materials.

The exam consists of short answer questions, multiple choice and several multi–step worked out problems. The time limit is two hours. No books, notes, calculators or any other electronic devices are allowed.

Topics to be covered:

1. Linear Systems of Algebraic Equations
   - Method of elimination

     Elementary Operations
   - Description of the set of solutions
2. Matrices
   - Definition of a matrix
   - Matrix Operations

     Addition

     Scalar multiplication

     Transpose

     Multiplication of matrices
   - Properties of Matrix Operations
   - Inverse Matrix

     How to find the inverse via Gauss–Jordan method

     The formula $A^{-1} = \frac{1}{\det A}\,\mathrm{adj}(A)$.
3. Solutions of Linear Systems in terms of Matrices
   - Augmented Matrix
   - Reduced Row Echelon Form (RREF)
   - Gauss-Jordan reduction

     Description of the set of solutions via RREF
   - Homogeneous System

     Dimension of the set of solutions (the null space)
   - Non-Homogeneous System

     Reduction to the associated homogeneous system via a particular solution
4. Determinants

- Definition of a determinant
  - Permutation
  - The number of inversion
- Properties of determinants
- Cofactor Expansion
  - $A\cdot$ adj$(A) = \det A \cdot I$
  - Cramer's rule

5. Linear (Affine) Geometry of $\mathbb{R}^n$
   - Vectors in $\mathbb{R}^n$
   - Operations among vectors
     - Addition
     - Scalar multiplication
     - Dot Product
       - Orthogonality in terms of dot product
   - Lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$
     - Parametric and symmetric equations of a line
     - Equation for a plane with a given normal vector and a point on it
6. Vector spaces
   - Definition of an abstract vector space
   - Subspaces
   - Linear Independence and Dependence
   - Basis and Dimension
     - Column and row space of a matrix
     - Rank of a matrix
   - Inner Products
   - Orthonormal basis
     - Gram-Schmidt process
     - Orthogonal complements
       - Applications to the method of least squares
7. Linear Transformations
   - Definition of a linear transformation
   - Kernel and range of a linear transformation
   - Matrix representing a linear transformation
   - Coordinates and change of basis
8. Eigenvalues and Eigenvectors
   - Definition of eigenvalues and eigenvectors
     - Characteristic polynomial
   - Diagonalization
     - Diagonalization of a matrix (similar to a diagonal matrix)

Diagonalization of a symmetric matrix

9. Linear Differential Equations
    - Solutions to linear differential equations (with diagonalizable matrices)

      real eigenvalues

      real solutions with complex eigenvalues

        Complex numbers and Euler's formula
    - Solution of the form $X = e^{tA}$ for $\frac{dX}{dt} = AX$

      exponential of a matrix
10. Matlab (computer)
    - Understanding of basic commands
    - Matrix operations in Matlab
    - Elementary Row Operations in Matlab
    - Vectors in Matlab

Note: The use of Matlab is an essential and active part of MA 265, though the real use of computers in the exams is not required or practiced under the current curriculum. One should know the basic commands, such as $rref$ and inv, and be able to interpret the output depending on the situation. The skills to read the necessary information off the output of Matlab will be tested.

Math 265 Linear Algebra **Credit Exam Practice**

Student Name (print):	Student ID:

Do not write below this line.

Please be neat and show all work.
Write each answer in the provided box.
Use the back of the sheets and the last 3 pages for extra scratch space.
Return this entire booklet to your instructor.
No books. No notes. No calculators.

Problem #	Max pts.	Earned points
1	20	
2	8	
3	8	
4	8	
5	8	
6	8	
7	8	
8	8	
9	8	
10	8	
11	8	
Section I	**100**	

13	10	
14	10	
15	10	
Section II	**30**	
16	20	
17	15	
18	15	
19	20	
Section III	**70**	
TOTAL	**200**	

Section I: Short problems

No partial credit on this part, but show all your work anyway. It might help you if you come close to a borderline. Please be neat. Write your answer in the provided box.

1. It is given that $A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$, $\mathbf{rref}(A) = \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find the rank of A.

(b) Find a basis for the null space of A.

(c) Find a basis for the column space of A. We require that you choose the vectors for the basis from the column vectors of A.

(d) Find a basis for the row space of A. We require that you choose the vectors for the basis from the row vectors of A.

2. Determine the value(s) of a so that the following linear system has no solution.

$$\begin{cases} x_1 + 2x_2 + x_3 = a \\ x_1 + x_2 + ax_3 = 1 \\ 3x_1 + 4x_2 + (a^2-2)x_3 = 1. \end{cases}$$

3. Find the standard matrix for the linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}, L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}, L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\-2\end{bmatrix}.$$

4. Determine the value(s) of a so that the line whose parametric equations are given by

$$\begin{cases} x = -3 + t \\ y = 2 - t \\ z = 1 + at \end{cases}$$

is parallel to the plane

$$3x - 5y + z + 3 = 0.$$

5. Find the symmetric equations of the line which is the intersection of the following two planes: $x + y - z = 2, 3x + 4y + z = 5$.

6. Compute the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

7. E is a 3×3 matrix of the form

$$E = \begin{bmatrix} 1 & 8 & 3 \\ x & y & z \\ -3 & 7 & 2 \end{bmatrix}.$$

Given $\det(E) = 5$, compute the determinant of the following matrix

$$F = \begin{bmatrix} x & y & z \\ 1 & 8 & 3 \\ -3+4x & 7+4y & 2+4z \end{bmatrix}$$

8. Find the matrix G such that

$$adj(G) = \begin{bmatrix} 2 & 4 \\ -5 & 7 \end{bmatrix}.$$

9. Find the dimension of the subspace $V = span\{v_1, v_2, v_3, v_4\}$ in $\mathbb{R}^3$ where

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

10. Find the projection $\mathrm{Proj}_W v$ of the vector $v = \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}$ onto the subspace W spanned by

$$\left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

11. We have a subspace W in $\mathbb{R}^4$ spanned by the following three linearly independent vectors

$$\left\{ u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 3 \\ -3 \\ 0 \\ -2 \end{bmatrix} \right\}.$$

Find an orthonormal basis of W.

Section II: Multiple choice problems

For Problems 12 through 15, circle only one (the correct) answer for each part. No partial credit.

12. Let A be a 3×3 matrix with $\det(A) = 0$. Determine if each of the following statements is true or false.

(a) $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. True False

(b) $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$. True False

(c) For every 3×3 matrix B, we have $\det(A + B) = \det(B)$. True False

(d) For every 3×3 matrix B, we have $\det(AB) = 0$. True False

(e) There is a vector $\mathbf{b}$ in $\mathbb{R}^3$ such that $rank([A\ \mathbf{b}]) > rank(A)$. True False

13. For each of the following sets, determine if it is a vector (sub)space:

(a) The set of all vectors (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ with the property $2x_1 - x_2 = 0, 3x_3 - x_4 = 0$;

Yes No

(b) The set of all vectors (x_1, x_2, x_3) in $\mathbb{R}^3$ with the property $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$;

Yes No

(c) The set of all vectors (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ with the property $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$;

Yes No

(d) The set of all vectors of the form $(a + b - 1, 2a + 3c - 1, b - c, a + b + c + 2)$ in $\mathbb{R}^4$ where a, b and c are arbitrary real numbers;

Yes No

(e) The set of all solutions to the linear system of differential equations $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ where $A = \begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix}$;

Yes No

14. For the problems (a), (b) and (c), determine if the given set of vectors is linearly independent or linearly dependent:

(a) $\left\{ \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$; Independent Dependent

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$; Independent Dependent

(c) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix} \right\}$; Independent Dependent

For the problems (d) and (e), determine if the given set of vectors spans $\mathbb{R}^3$:

(d) $\left\{ \begin{bmatrix} \pi \\ 2\pi \\ -1\pi \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$; span not span

(e) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$; span not span

Section III: Multi-Step problems

Show all work (no work - no credit!) and display computing steps. Write clearly.

15. Let

$$A = \begin{bmatrix} -15 & 28 \\ -8 & 15 \end{bmatrix}.$$

(a) Find the eigenvalues and compute an eigenvector for each eigenvalue.

(b) Find an invertible matrix P and a diagonal matrix D such that

$$P^{-1}AP = D.$$

(c) Compute A^{37}.

16. Find the least squares fit line for the points

$$(-2,1),(-1,3),(0,2),(1,3),(2,1).$$

17. Let

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \\ \dfrac{dx_2}{dt} \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

be the linear system of differential equations where

$$A = \begin{bmatrix} 3 & -5 \\ 5 & 3 \end{bmatrix}.$$

(a) Find the eigenvalues and find an eigenvector for each eigenvalue for A.
Note: The eigenvalues are COMPLEX-valued.

(b) Find the general REAL solution to the linear system of differential equations.

18. Let

$$\begin{cases} \dfrac{dx_1}{dt} = 2x_1 + 5x_2 \\ \dfrac{dx_2}{dt} = 3x_1 + x_2 + 3x_3 \\ \dfrac{dx_3}{dt} = -x_1 \end{cases}$$

be a linear system of differential equations.

(a) Find the eigenvalues and find an eigenvector for each eigenvalue for the coefficient matrix of the linear system of differential equations.

(b) Find the general solution to the linear system of differential equations.

(c) Find the solution to the initial value problem

$$x_1(0) = 4, x_2(0) = 16, x_3(0) = 0.$$

1. (a) 3. (b) $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. (c) A_1, A_2, A_4. (d) A^1, A^2, A^3.

2. $a = 3$. **3.** $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -2 \end{bmatrix}$. **4.** $a = -8$. **5.** $\dfrac{x-3}{5} = \dfrac{y+1}{-4} = z$.

6. $\begin{bmatrix} 0 & 1 & 0 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$. **7.** -5. **8.** $\begin{bmatrix} 7 & -4 \\ 5 & 2 \end{bmatrix}$. **9.** 3.

10. $\begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$. **11.** $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$.

12. (a) True. (b) False. (c) False. (d) True. (e) True.

13. (a) Yes. (b) No. (c) No. (d) Yes. (e) Yes.

14. (a) Independent. (b) Dependent. (c) Dependent. (d) not span. (e) span.

15. (a) $\lambda_1 = -1$, $P_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. $\lambda_2 = 1$, $P_2 = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$. (b) $P = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}$, $\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

16. $y = 2$.

17. (a) $\lambda_1 = 3 + 5i$, $P_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$. $\lambda_2 = 3 - 5i$, $P_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

(b) $\mathbf{x}(t) = c_1 e^{3t} \begin{bmatrix} -\sin 5t \\ \cos 5t \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} \cos 5t \\ \sin 5t \end{bmatrix}$.

18. (a) (a) $\lambda_1 = -3$, $P_1 = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$. $\lambda_2 = 1$, $P_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix}$. $\lambda_3 = 5$, $P_3 = \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$.

(b) $\mathbf{x}(t) = c_1 e^{-3t} \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix} + c_3 e^{5t} \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$.

(c) $\mathbf{x}(t) = -2e^{-3t} \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} + e^{t} \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix} + -3e^{5t} \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$.