June 2011

PURDUE UNIVERSITY
Study Guide for the Credit Exam in (MA 266)
Ordinary Differential Equations

The topics covered in this exam can be found in "Elementary Differential Equations and Boundary Value Problems," by Boyce & DiPrima, Wiley. See also http://www.math.purdue.edu/academic/courses for recent course information and materials.

The exam consists of mostly multiple choice problems with some additional multi–step worked out problems. The time limit is two hours. No books, notes, calculators or any other electronic devices are allowed. A Laplace transform table will be provided in the exam.

You should have both a qualitative and algorithmic understanding of the following topics:

Outline of Topics

First Order Differential Equations

1. Existence and uniqueness of solutions.
2. Solution of first order linear equations.
3. Special techniques - separable, homogeneous, and exact differential equations.
4. Use of direction fields - equilibrium solutions, asymptotic behavior of solutions, interval of definition of solutions.
5. Numerical approximation of solutions to first order equations - the Euler Tangent Line Method.
6. Applications - exponential growth and decay, mixing problems, Newton's Law of Cooling, rising and falling bodies.

Second Order Differential Equations

1. Cases that reduce to first order equations, reduction of order.
2. Existence and uniqueness of solutions for second order linear equations.
3. Linear equations with constant coefficients, characteristic equations
   (a) real distinct roots
   (b) complex roots
   (c) repeated roots
4. Nonhomogeneous Equations
   (a) method of undetermined coefficients
   (b) variation of parameters
5. Applications- harmonic motion, damped and undamped spring-mass systems, Newton's Law of Motion.

Higher Order Differential Equations

1. Linear equations with constant coefficients, characteristic equations
   (a) real distinct roots
   (b) complex roots
   (c) repeated roots
2. Nonhomogeneous Equations - method of undetermined coefficients.

Laplace Transform Methods

1. Definition and theory.
2. Computing Laplace transforms of common functions.
3. Computing inverse Laplace transforms.
4. Using tables of Laplace transforms.
5. Solving initial value probems using Laplace transforms.
6. Laplace transforms of discontinuous functions (step functions, delta functions).
7. Convolutions of functions and their Laplace transforms.

Systems of Differential Equations

1. Matrix formulation; converting higher order differential equations into a first order system.
2. Eigenvalues and eigenvectors for 2×2 systems.
3. Solutions to $\frac{d\vec{x}}{dt} = A\vec{x}$ when A is a 2×2 matrix and
   (a) there are enough eigenvectors
   (b) there are not enough eigenvectors (the matrix A is defective)
4. Using the method of undetermined coefficients for systems to solve nonhomogeneous systems of differential equations $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{g}$.

1. If $y(t) = \sin 2t$ is a solution of $y'' + 9y = f(t)$, then $f(t) =$

A. $\sin 2t$

B. $A \cos 3t$

C. 0

D. $\boxed{5 \sin 2t}$

E. $13 \sin 2t$

2. If $y = y(x)$ is the solution to

$$\frac{dy}{dx} = \frac{4xy}{2+x^2}, \qquad y(0) = 4,$$

then $y(\sqrt{2})=$

A. 4

B. $\boxed{16}$

C. 1

D. 2

E. $2\sqrt{2}$

3. The general solution to $x^2y' + 2xy = e^{5x}$ is

A. $y = \frac{1}{5}e^{5x} + c$

B. $\boxed{y = \frac{1}{5x^2}e^{5x} + cx^{-2}}$

C. $y = \frac{1}{5x^2}e^{5x}$

D. $y = ce^{5x}$

E. $y = \frac{5}{x^2}e^{5x} + c$

4. The solution to the problem

$$(2xy + x^3)dx + (x^2 + y^4)dy = 0, \quad y(0) = 1$$

is

A. $x + \frac{1}{5}y^5 = \frac{1}{5}$

B. $x^2y + 5y^5 = 5$

C. $\boxed{x^2y + \frac{x^4}{4} + \frac{y^5}{5} = \frac{1}{5}}$

D. $x^2y + 4x^4 + 5y^5 = 5$

E. $x^2y + x^4 + \frac{y^5}{5} = \frac{1}{5}$

5. The solution in implicit form of

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$$

is:

A. $x^2 + y^2 = x^3 + C$

B. $\boxed{x^2 + y^2 = Cx^3}$

C. $x^2 + x^3 = y^2 + C$

D. $Cx^2 = x^3 + y^2$

E. $x^2 + y^3 + xy^2 = C$

6. Which of the following best describes the stability of equilibrium solutions for the autonomous differential equation $y' = y(4 - y^2)$?

A. $\boxed{y = 0 \text{ unstable; } y = 2 \text{ and } y = -2 \text{ both stable}}$

B. $y = 0$ unstable; $y = 2$ stable

C. $y = 0$ and $y = 2$ both stable

D. $y = 0$ stable; $y = 2$ unstable; $y = -2$ stable

E. $y = 0$ stable; $y = -2$ and $y = 2$ both unstable

7. Solve the initial value problem $y' - y = e^{-t}$ with $y(0) = a$. For what value(s) of a is the solution bounded (i.e., not tending to infinity as $t \to +\infty$) on the interval $t > 0$?

A. 2

B. $\boxed{-\frac{1}{2}}$

C. $-\frac{1}{3}$

D. all values

E. no values

8. Initially a tank holds 50 gallons of pure water. A salt solution containing $\frac{1}{3}$ lb of salt per gallon runs into the tank at the rate of 5 gallons per minute. The well mixed solution runs out of the tank at a rate of 2 gallons per minute. Let $x(t)$ be the amount of salt in the tank at time t. Find a differential equation satisfied by $x(t)$. (DO NOT SOLVE THE EQUATION)

A. $\boxed{\frac{dx}{dt} = \frac{5}{3} - \frac{2x}{50+3t}}$

B. $\frac{dx}{dt} = \frac{5}{2} - \frac{3x}{50+3t}$

C. $\frac{dx}{dt} = \frac{5}{3} - \frac{3x}{50+2t}$

D. $\frac{dx}{dt} = \frac{5}{2} - \frac{2x}{50+3t}$

E. $\frac{dx}{dt} = \frac{5}{3} - \frac{2x}{50+2t}$

9. The function $y_1 = t^2$ is a solution of the differential equation

$$t^2 \frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2y = 0.$$

Choose a function y_2 from the list below so that the pair y_1, y_2 form a fundamental set of solutions to the differential equation.

A. $y_2 = t^2 \sin t$

B. $y_2 = t^2 e^t$

C. $y_2 = t \sin t$

D. $\boxed{y_2 = t}$

E. $y_2 = t^2$

10. The largest open interval on which the solution to the initial value problem

$$\begin{cases} (\cos t)\, y' + \dfrac{t}{t-3}\, y = \ln(4-t) \\ \\ y(2) = 0 \end{cases}$$

is guaranteed by the Existence and Uniqueness Theorem to exist is

A. $-\frac{\pi}{2} < t < \frac{\pi}{2}$

B. $0 < t < \pi$

C. $\boxed{\frac{\pi}{2} < t < 3}$

D. $2 < t < 4$

E. $4 < t < \infty$

11. If $y(x)$ is the solution of $y'' - y' - 2y = 0$ satisfying $y(0) = 1$ and $y'(0) = -1$, then $y(1) =$

A. $\boxed{e^{-1}}$

B. e^2

C. $e^2 - e^{-1}$

D. $2e^{-1}$

E. $e^{-1} + 2e^2$

12. The general solution $y(t)$ of the differential equation

$$y'' - 3y' + 2y = 2e^t$$

is

A. $y(t) = c_1 e^t + c_2 e^{2t} + 2e^t$

B. $y(t) = c_1 e^{-t} + c_2 e^{2t} + e^t$

C. $y(t) = e^t + e^{2t} + cte^t$

D. $\boxed{y(t) = c_1 e^t + c_2 e^{2t} - 2te^t}$

E. $y(t) = c_1 e^{-t} + c_2 e^{-2t} + 2te^t$

13. The values of the constant r such that $y = x^r$ solves $x^2y'' + xy' - 2y = 0$ for $x > 0$ are

A. $1 \pm \sqrt{2}$

B. $\pm i\sqrt{2}$

C. $\boxed{\pm\sqrt{2}}$

D. $-1, -2$

E. $1, -2$

14. The proper form of the particular solution of the differential equation

$$y''' + 3y'' + 3y' + y = e^{-t}$$

used in the Method of Undetermined Coefficients is

A. Ae^{-t}

B. $A\cos t + B\sin t$

C. $At\cos t + Bt\sin t$

D. At^2e^{-t}

E. $\boxed{At^3e^{-t}}$

15. The inverse Laplace transform of $\frac{s^2+9s+2}{(s+3)(s-1)^2}$ is

A. $4e^t + 2te^t + e^{-3t}$

B. $2e^t + 3te^t - e^{3t}$

C. $2e^{-t} + 3te^{-t} + e^{3t}$

D. $2e^{-t} + 3te^{-t} + te^{-3t}$

E. $\boxed{2e^t + 3te^t - e^{-3t}}$

16. The Laplace transform of

$$f(t) = \int_0^t (t-\tau)e^{t-\tau} \cos\ 2\tau\ d\tau$$

is

A. $F(s) = \dfrac{1}{(s-1)(s^2+4)}$

B. $F(s) = \dfrac{s}{(s^2+1)(s^2+4)}$

C. $\boxed{F(s) = \dfrac{s}{(s-1)^2(s^2+4)}}$

D. $F(s) = \dfrac{2}{(s^2+1)(s+4)^2}$

E. $F(s) = \dfrac{s}{(s-4)^2(s-1)}$

17. Find the Laplace transform of

$$f(t) = \begin{cases} 2, & 0 \le t < 2, \\ (t-2)^2, & t \ge 2. \end{cases}$$

A. $\frac{2}{s} + e^{-2s}(\frac{2}{s^3} - \frac{4}{s^2})$

B. $\frac{2}{s} + e^{-2s}(\frac{2}{s^3} - \frac{4}{s^2} + \frac{2}{s})$

C. $\frac{2}{s} + e^{-2s}(\frac{2}{s^3} + \frac{4}{s^2} + \frac{2}{s})$

D. $\frac{2}{s} + e^{-2s}\frac{2}{s^3}$

E. $\boxed{\frac{2}{s} + e^{-2s}(\frac{2}{s^3} - \frac{2}{s})}$

18. A mass weighing 16 lb stretches a spring $\frac{1}{2}$ ft. The mass is pulled down 1 ft from the equilibrium position, and then set in motion with a downward velocity of 8 ft/sec. Assuming that there is no air resistance and that the downward direction is the position direction. (The gravity constant $g = 32$ ft/sec^2.) Then, the amplitude of the oscillation is:

A. 1

B. $\boxed{\sqrt{2}}$

C. $-\sqrt{2}$

D. 2

E. $\frac{\pi}{4}$

19. The solution of the differential equation

$$y'' - 2y' + 2y = \delta(t-1), \quad y(0) = 0,\ y'(0) = 0$$

is

A. $\boxed{u_1(t)e^{t-1}\sin(t-1)}$

B. $\frac{e^{-s}}{s^2-2s+2}$

C. $\frac{u_1(t)}{t^2-2t+2}$

D. $u_1(t)e^t\sin t$

E. $u_1(t)e^{t+1}\sin(t+1)$

20. The general solution to

$$y^{(4)} + 2y'' + y = 0$$

is

A. $c_1\cos t + c_2\sin t$

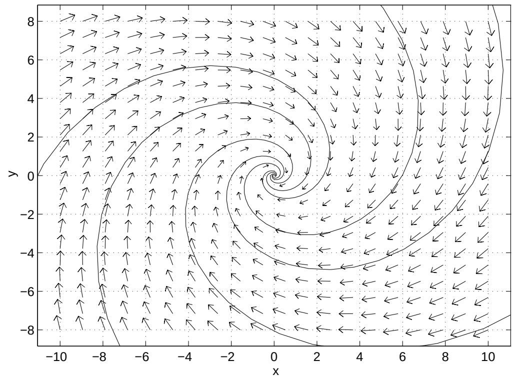
B. $c_1e^t + c_2e^{-t}$

C. $\boxed{c_1\cos t + c_2\sin t + c_3t\cos t + c_4t\sin t}$

D. $c_1e^t\sin t + c_2e^t\cos t$

E. $c_1e^t + c_2e^{-t} + c_3\sin t + c_4\cos t$

21. The phase portrait for a linear system of the form $\vec{x}\,' = \mathbf{A}\vec{x}$, where $\mathbf{A}$ is a 2×2 matrix is as follows.



If r_1 and r_2 denote the eigenvalues of $\mathbf{A}$, then what can you conclude about r_1 and r_2 by examining the phase portrait?

A. r_1 and r_2 are distinct and positive

B. r_1 and r_2 are distinct and negative

C. r_1 and r_2 have opposite signs

D. r_1 and r_2 are complex and have positive real part

E. $\boxed{r_1 \text{ and } r_2 \text{ are complex and have negative real part}}$

22. The function $x_2(t)$ determined by the initial value problem

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 \end{aligned}$$

with initial conditions $x_1(0) = 1$ and $x_2(0) = 1$ is given by

A. $\boxed{x_2(t) = -\sin t + \cos t}$

B. $x_2(t) = \sin t + \cos t$

C. $x_2(t) = \frac{1}{2}(e^t + e^{-t})$

D. $x_2(t) = \cos t$

E. $x_2(t) = ie^{it} - ie^{-it}$

23. Find the general solution of the first order system

$$\vec{x}\,' = \mathbf{A}\vec{x} \text{ where } \mathbf{A} = \begin{pmatrix} 6 & -4 \\ 1 & 2 \end{pmatrix}$$

given that $\vec{\xi} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector associated to the repeated eigenvalue $r = 4$ for the matrix $\mathbf{A}$, and that $\vec{\eta} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ satisfies $(\mathbf{A} - 4\mathbf{I})\vec{\eta} = \vec{\xi}$.

A. $c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + c_2 t \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t}$

B. $\boxed{c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 2t+1 \\ t \end{pmatrix} e^{4t}}$

C. $c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{4t}$

D. $c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} t+2 \\ 1 \end{pmatrix} e^{4t}$

E. $c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + c_2 t \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{4t}$

24. Find the solution of the initial value problem

$$\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x} \text{ with } \vec{x}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

A. $\boxed{\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}}$

B. $\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}$

C. $-\frac{1}{2} \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{3t} + \frac{1}{2} \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}$

D. $\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{3t}$

E. $\frac{1}{2} \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t} - \frac{1}{2} \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t}$

25. Consider the system

$$\vec{x}' = \begin{pmatrix} \alpha & 1 \\ 1 & \alpha \end{pmatrix} \vec{x}$$

For what values of α is the equilibrium solution $\vec{x} = 0$ an asymptotically stable node?

A. no value of α

B. $\boxed{\alpha < -1}$

C. $\alpha > 1$

D. $-1 < \alpha < 1$

E. all real α

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$
2.	e^{at}	$\dfrac{1}{s-a}$
3.	t^n	$\dfrac{n!}{s^{n+1}}$
4.	$t^p \ (p > -1)$	$\dfrac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sin at$	$\dfrac{a}{s^2+a^2}$
6.	$\cos at$	$\dfrac{s}{s^2+a^2}$
7.	$\sinh at$	$\dfrac{a}{s^2-a^2}$
8.	$\cosh at$	$\dfrac{s}{s^2-a^2}$
9.	$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}$
10.	$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}$
11.	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
12.	$u_c(t)$	$\dfrac{e^{-cs}}{s}$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$
16.	$\int_0^t f(t-\tau)\,g(\tau)\,d\tau$	$F(s)\,G(s)$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$
19.	$(-t)^n f(t)$	$F^{(n)}(s)$