Degenerate nonlinear parabolic equations with discontinuous diffusion coefficients

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(based on joint works with Dohyun Kwon, UCLA)

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Mathematical motivation

→ Study the well-posedness and structure of solutions to diffusion equations with discontinuous nonlinearlies.

→ Model problem:

$$\begin{cases} \partial_t \rho - \Delta\varphi(\rho) - \nabla\cdot(\nabla\Phi\rho) = 0, & \text{in } (0,T)\times\Omega, \\ (\nabla\varphi(\rho) + \nabla\Phi\rho)\cdot\mathbf{n} = 0, & \text{on } (0,T)\times\partial\Omega \\ \rho(0,\cdot) = \rho_0, & \text{in } \Omega, \end{cases} \tag{NDE}$$

where $T > 0$, $\Omega \subset \mathbb{R}^d$ smooth, bounded convex domain, $\rho_0 \in \mathscr{P}^{\mathrm{ac}}(\Omega)$ and $\Phi : \Omega \to \mathbb{R}$ is a given Lipschitz continuous potential.

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→ Example of a nonlinearity

$$\varphi : [0,+\infty) \to \mathbb{R},\ \varphi(s) = \begin{cases} \rho, & \rho \in [0,1), \\ [\rho, 2\rho], & \rho = 1, \\ 2\rho, & \rho > 1, \end{cases}$$

Motivation: Starvation driven diffusion in mathematical biology

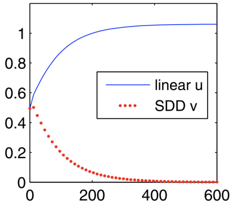
→ A competition between a linear diffusion and a starvation driven diffusion:

$$\partial_t u = d\Delta u + u(m-u-v), \quad \partial_t v = \Delta\varphi(v;m) + v(m-u-v).$$

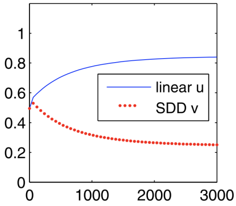
where u, v represent two population densities and m stands for the resource density.

→ For $0 < l < h$, $\varphi(v;m) := \begin{cases} lv, & \text{if } v < m, \\ hv, & \text{if } v > m. \end{cases}$

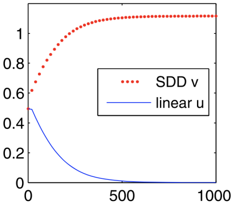
→ Cho-Kim [2013, Bull. Math. Biol.] ("Starvation driven diffusion as a survival strategy of biological organisms") (Ex: $\Omega = (0,1)$, m discontinuous with two constant values and $u(0,\cdot) = v(0,\cdot) = m/2$; $l = 0.002$, $h = 0.004$)



(a) $d = 0.0005$



(b) $d = 0.001$



(c) $d = 0.0015$

Motivation: self-organized criticality in physics

- → Bántay-Jánosi [1992, Phys. Rew. Let.] ("Avalanche dynamics from anomalous diffusion" - self organized criticality in sandpile models).
- → Same problem as (NDE), with $\Phi = 0$, $\varphi(\rho) = f(\rho)H(\rho - \rho_c)$, where f is some given function (either identity, or a constant), H is the Heaviside function and ρ_c stands for the critical density value.
- →

Figure: Avalanches in the Himalayas

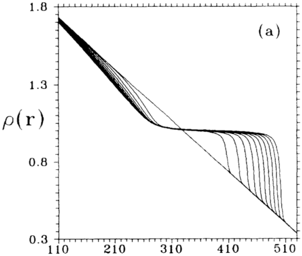


Figure: Time evolution of ρ, $\rho_c = 1$ [Bántay-Jánosi, 1992]

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→ Blanchard-Röckner-Russo [2010, Ann. Probab.] →$|\varphi(\rho)| \leq C\rho$; probabilistic approach in 1D; non-degenerate case.

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→ Barbu-Röckner [2018, SIMA] → higher dimensions; probabilistic approach; nonlinear semigroup theory → maximal monotone operators, parabolic approximation, i.e. $\varphi_\varepsilon \to \varphi$.

→ Notion of solution: generalized entropic solutions à la Kruzkov.

→ This heuristically can be written as pairs (ρ, η_ρ) belonging to well-chosen function spaces, such that

$$\partial_t \rho - \Delta(\eta_\rho) - \nabla \cdot (\nabla \Phi \rho) = 0$$

is fulfilled and $\rho(t,x) \in \eta_\rho(t,x)$ a.e.

Our main objectives

(1) Find a unified way to treat general discontinuous nonlinearities.

(2) Give a fine characterization of the emerging critical regions $\{\rho = 1\}$ observed in numerical experiments.

Our approach: optimal transport and gradient flows

OT toolbox

→ for $\mu, \nu \in \mathscr{P}(\Omega)$ we define the 2-Wasserstein distance W_2 as

$$W_2^2(\mu,\nu) := \inf\left\{ \int_{\Omega\times\Omega} |x-y|^2 \, \mathrm{d}\gamma : \ \gamma \in \mathscr{P}(\Omega\times\Omega), \ (\pi^x)_{\#}\gamma = \mu, \ (\pi^y)_{\#}\gamma = \nu \right\}$$

where for $T : X \to Y$ Borel function $T_{\#}\mu = \nu$ means that $\nu(A) = \mu(T^{-1}(A))$ for any $A \subseteq Y$ Borel set.

→ we have the dual formulation

$$W_2^2(\mu,\nu) := \sup\left\{ \int_{\Omega} \phi \, \mathrm{d}\mu + \int_{\Omega} \psi \, \mathrm{d}\nu : \ \phi, \psi \in C_b(\Omega), \phi(x) + \psi(y) \le |x-y|^2 \right\}.$$

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→ for any finite measure χ s.t. $\chi(\Omega) = 0$ we have the first variation formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{1}{2} W_2^2(\mu + t\chi, \nu) = \int_\Omega \phi \,\mathrm{d}\chi.$$

→ Brenier [1991, CPAM]: if $\mu \in \mathscr{P}^{\mathrm{ac}}(\Omega)$, then $\gamma_{\mathrm{opt}} = (\mathrm{id}, T)_\#\mu$, with $T = \mathrm{id} - \nabla\phi_{\mathrm{opt}}$.

Gradient flows in $(\mathscr{P}(\Omega), W_2)$

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- $\rightarrow$ for example $\partial_t\rho - \Delta\rho = 0$ can be seen as the GF of the Boltzmann entropy $\mathcal{J}(\rho) = \int_\Omega \rho\log(\rho)\,\mathrm{d}x$.

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De Giorgi's minimizing movements scheme (cf. Jordan-Kinderlehrer-Otto [1998, SIMA])

- $\rightarrow$ let ρ_0 be given and $N \in \mathbb{N}$ and $\tau > 0$ be such that $T = N\tau$. Construct the recursive sequence for all $k \in \{1, \ldots, N\}$

$$\rho_k \in \mathrm{argmin}_{\rho \in \mathscr{P}(\Omega)} \left\{ \mathcal{J}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \rho_{k-1}) \right\} \qquad \text{(MM)}$$

- $\rightarrow$ optimality condition $\log(\rho_k) + 1 + \frac{\phi_k}{\tau} = \text{const on } \mathrm{spt}(\rho_k)$.
- $\rightarrow$ approximate velocity $\mathbf{v}_k^\tau := \frac{x - T_k(x)}{\tau} = \frac{\nabla \phi_k}{\tau} = -\frac{\nabla \rho_k}{\rho_k}$.
- $\rightarrow$ after interpolations, the limit curve, as $\tau \downarrow 0$ solves $\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$.

Back to our problems

→ We define the energy associated to our models as

$$\mathcal{J}(\rho) := \mathcal{S}(\rho) + \mathcal{F}(\rho) := \int_\Omega S(\rho(x))\,\mathrm{d}x + \int_\Omega \Phi\,\mathrm{d}\rho(x).$$

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→ We consider the GF of the functional $\mathcal{J}$ in $(\mathscr{P}(\Omega), W_2)$.

→ $\mathcal{J}$ fails to be differentiable. Therefore the classical theory does not imply directly; one needs to work with subdifferential calculus.

→ We need to rely on the scheme (MM). To write optimality conditions, we characterize the Wasserstein subdifferential of $\mathcal{J}$.

Estimates

→ We need to choose carefully the function spaces: we work in $L^p(\Omega)$, $1 < p \leq +\infty$.

Lemma (L^∞ estimates)

Let $\rho_0 \in L^\infty(\Omega)$. *Let* $(\rho_k)_{k=1}^N$ *be constructed via the scheme* (MM). *Then we have*

$$\|\rho_k\|_{L^\infty} \leq C(T, \Phi)\|\rho_0\|_{L^\infty}, \ \forall k \in \{1, \dots, N\}.$$

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Lemma (L^β estimates)

Let $\rho_0 \in \mathscr{P}(\Omega)$ *such that* $\mathcal{J}(\rho_0) < +\infty$. *Let* $S''(\rho) \geq C\rho^{r-2}$, *if* $\rho \in (1,+\infty)$ *for some* $r \geq 1$. *Let* $(\rho_k)_{k=1}^N$ *be constructed via the scheme* (MM). *Then we have*

$$\|\rho_k\|_{L^\beta} \leq C(T,\Phi,1/\tau), \ \forall k \in \{1,\dots,N\},$$

where $\beta := \begin{cases} (2r-1)\frac{d}{d-2}, & d \geq 3, \\ < +\infty, & d = 2, \\ +\infty, & d = 1. \end{cases}$

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Theorem

For all $k\in\{1,\dots,N\}$, there exists $C=C(k)\in\mathbb{R}$ and ϕ_k such that

$$\begin{cases} C-\frac{\phi_k}{\tau}-\Phi\le S'(0+) & \text{in } \{\rho_k=0\},\\ C-\frac{\phi_k}{\tau}-\Phi\in[S'(1-),S'(1+)], & \text{in } \{\rho_k=1\},\\ C-\frac{\phi_k}{\tau}-\Phi=S'\circ\rho_k, & \text{otherwise.}\end{cases}$$

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Theorem

For ρ_k is given in (MM), *if $\xi\in\partial\mathcal{S}(\rho_k)\cap L^1(\Omega)$, then it holds that*

$$\xi\in\begin{cases}[-\infty,S'(0+)] & \text{in } \{\rho_k=0\},\\ [S'(1-),S'(1+)] & \text{in } \{\rho_k=1\},\\ S'\circ\rho_k & \text{in } \{\rho_k\neq 1\},\end{cases} \tag{1}$$

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→ Question: how do we identify the approximate velocity, i.e. $\frac{\nabla \phi_k}{\tau}$?

→ Answer: inspired by the analysis of Maury-Roudneff-Chupin-Santambrogio [2010, M3AM] (also [M.-Santambrogio, 2016, APDE]), we introduce a new variable:

→ For $k \in \{1, \ldots, N\}$, we define $p_k : \Omega \to \mathbb{R}$ as

$$p_k := \begin{cases} \max\left\{C - \frac{\phi_k}{\tau} - \Phi, S'(1-)\right\} & \text{in } \{\rho_k < 1\}, \\ C - \frac{\phi_k}{\tau} - \Phi & \text{in } \{\rho_k = 1\}, \\ \min\left\{C - \frac{\phi_k}{\tau} - \Phi, S'(1+)\right\} & \text{in } \{\rho_k > 1\}. \end{cases}$$

→ Or, equivalently

$$p_k = \min\left\{\max\left\{C - \frac{\phi_k}{\tau} - \Phi, S'(1-)\right\}, S'(1+)\right\}.$$

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Lemma

For all $k \in \{1, \ldots, N\}$, there exists $C \in \mathbb{R}$ such that

$$p_k(1 + \log \rho_k) + \frac{\phi_k}{\tau} + \Phi = C \text{ a.e.}$$

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In particular, both p_k and ρ_k are Lipschitz continuous and $\rho_k > 0$ a.e.

- → As a consequence, $\frac{\nabla \phi_k}{\tau} = -\nabla \Phi - \nabla p_k - p_k \frac{\nabla \rho_k}{\rho_k}$ (since $\nabla p_k \log(\rho_k) = 0$).

Uniform estimates and passing to the limit at $\tau \downarrow 0$

→ Let us notice that

$$\frac{1}{\tau}\sum_{k=1}^{N} W_2^2(\rho_k, \rho_{k-1}) = \frac{1}{\tau}\sum_{k=1}^{N}\int_\Omega |\nabla \phi_k|^2 \le \mathcal{J}(\rho_0) - \inf \mathcal{J}.$$

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Theorem

Suppose that $\rho_0 \in L^\infty(\Omega)$ and $\nabla\Phi \cdot \mathbf{n} > 0$ on $\partial\Omega$. Then, there exists $\rho, p \in L^\infty([0,T]\times\Omega) \cap L^2([0,T];H^1(\Omega))$ such that (ρ, p) is a unique solution to

$$\begin{cases} \partial_t \rho - \Delta(p\rho) - \nabla\cdot(\nabla\Phi\rho) = 0, & \text{in } (0,T)\times\Omega, \\ (\nabla(p\rho) + \nabla\Phi\rho)\cdot\mathbf{n} = 0, & \text{on } (0,T)\times\partial\Omega, \\ \rho(\cdot,0) = \rho_0, & \text{in } \Omega, \end{cases} \tag{2}$$

in the sense of distribution.

Some remarks

Remark

(ρ, p) *satisfies*

$$\begin{cases} p = 1 & \text{a.e. in } \{0 < \rho < 1\}, \\ p \in [1,2] & \text{a.e. in } \{\rho = 1\}, \\ p = 2 & \text{a.e. in } \{\rho > 1\}. \end{cases}$$

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$$\begin{cases} p = 1 & \text{a.e. in } \{0 < \rho < 1\}, \\ p \in [1,2] & \text{a.e. in } \{\rho = 1\}, \\ p = 2 & \text{a.e. in } \{\rho > 1\}. \end{cases}$$

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If we consider more general initial, i.e. $\rho_0 \in \mathscr{P}(\Omega)$ such that $\mathcal{E}(\rho_0) < +\infty$, we find a solution

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Some remarks

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$\rightarrow$ In the proof, to gain compactness we use an Aubin-Lions type argument for ρ^τ.

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where $m > 1$.

→ Our main theorem for the associated entropy can be formulated as follows.

Main theorem for the PM type model problem

Theorem (Kwon-M., 2021)

For $\rho_0 \in \mathscr{P}(\Omega)$ such that $\mathcal{J}(\rho_0) < +\infty$, there exists $\rho \in L^\beta([0,T]\times\Omega)$ and $p \in L^2([0,T];H^1(\Omega)) \cap L^\infty([0,T]\times\Omega)$ with $\rho^{m-\frac{1}{2}} \in L^2([0,T];H^1(\Omega))$ such that (ρ,p) is a weak solution of

$$\begin{cases} \partial_t\rho - \Delta([(m-1)\rho^m + 1]\frac{p}{m}) - \nabla\cdot(\nabla\Phi\rho) = 0, & \text{in } (0,T)\times\Omega, \\ \rho(0,\cdot) = \rho_0, & \text{in } \Omega, \\ (\nabla([(m-1)\rho^m + 1]\frac{p}{m}) + \nabla\Phi\rho)\cdot\mathbf{n} = 0, & \text{in } [0,T]\times\partial\Omega, \end{cases} \tag{3}$$

in the sense of distribution. Furthermore, (ρ,p) satisfies

$$\begin{cases} p(t,x) = \frac{m}{m-1} & \text{a.e. in } \{0<\rho<1\}, \\ p(t,x) \in \left[\frac{m}{m-1}, \frac{2m}{m-1}\right] & \text{a.e. in } \{\rho = 1\}, \\ p(t,x) = \frac{2m}{m-1} & \text{a.e. in } \{\rho > 1\}. \end{cases}$$

In addition, if $\rho_0 \in L^\infty(\Omega)$ and $\nabla\Phi\cdot\mathbf{n} > 0$ on $\partial\Omega$, then $\rho \in L^\infty([0,T]\times\Omega)$ and $\rho^m \in L^2([0,T];H^1(\Omega))$.

The ‘fully’ general problem

→ Recall that if S is differentiable, then we have

$$\varphi(\rho) = \rho S'(\rho) - S(\rho) + S(1)$$

→ Based on the observation and the derivation of p, we define the operator L_S pointwisely for functions $(\rho, p) : [0, T] \times \Omega \to \mathbb{R}$ by

$$\begin{aligned} L_S(\rho, p)(t, x) := {} & [\rho(t,x) S'(\rho(t,x)) - S(\rho(t,x)) + S(1)] \, \mathbb{1}_{\{\rho \neq 1\}}(t,x) \\ & + p(t,x) \mathbb{1}_{\{\rho = 1\}}(t,x) \end{aligned}$$

→ Recall that for a.e. $(t,x) \in [0,T] \times \Omega$ the pressure variable $p : [0,T] \times \Omega \to \mathbb{R}$ satisfies a.e.

$$\begin{cases} p(t,x) = S'(1-) & \text{if } 0 \leq \rho(t,x) < 1, \\ p(t,x) \in [S'(1-), S'(1+)] & \text{if } \rho(t,x) = 1, \\ p(t,x) = S'(1+) & \text{if } \rho(t,x) > 1. \end{cases} \tag{P}$$

→ We aim to find a solution to the PDE

$$\begin{cases} \partial_t \rho - \Delta(L_S(\rho, p)) - \nabla \cdot (\nabla \Phi \rho) = 0, & \text{in } (0,T) \times \Omega, \\ \rho(0, \cdot) = \rho_0, & \text{in } \Omega, \\ (\nabla(L_S(\rho, p)) + \nabla \Phi \rho) \cdot \mathbf{n} = 0, & \text{in } [0,T] \times \partial\Omega. \end{cases} \tag{G}$$

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→ Assume that $S \sim \rho^m$ in $(0,1)$ and $S \sim \rho^r$ in $(1,+\infty)$, for some $m \geq 1, r \geq 1$. Set $\beta \geq 1$ as before, i.e.

$$\beta := \begin{cases} (2r-1)\frac{d}{d-2} & \text{if } d \geq 3, \\ [1,\infty) & \text{if } d = 2, \\ +\infty & \text{if } d = 1. \end{cases}$$

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→ Our main theorem reads as

Theorem (Kwon-M., 2021)

Suppose that the above growth conditions are fulfilled and

$$m < r + \frac{\beta}{2}$$

holds true. For $\rho_0 \in \mathscr{P}(\Omega)$ such that $\mathcal{E}(\rho_0) < +\infty$, there exists $\rho \in L^\beta([0, T] \times \Omega)$, $\rho^{m-\frac{1}{2}} \in L^2([0, T]; H^1(\Omega))$ and $p \in L^2([0, T]; H^1(\Omega)) \cap L^\infty([0, T] \times \Omega)$ such that (ρ, p) is a solution of (G)-(P) in the sense of distributions.

The main idea of the proof ($m = 1$, the less involved case)

→ The hypothesis $m < r + \frac{\beta}{2}$ is always true.

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→ We define the auxiliary functions S_a and $S_b : [0, +\infty) \to \mathbb{R}$ by

$$S_a(\rho) := \begin{cases} S'(1-)\rho \log \rho, & \text{for } \rho \in [0,1], \\ S'(1+)\rho \log \rho, & \text{for } \rho \in (1, +\infty), \end{cases}$$

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→ We obtain that $\rho_k > 0$ a.e. and the optimality condition,

$$p_k(1 + \log \rho_k) + S_b'(\rho_k) + \frac{\phi_k}{\tau} + \Phi = C \text{ a.e.}$$

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Lemma

For all $k \in \{1,\dots,N\}$, there exists $C \in \mathbb{R}$ such that

$$\rho_k^{m-1} p_k = \left(C - \frac{\phi_k}{\tau} - \Phi\right)_+ \quad \text{a.e.}$$

In particular, p_k and ρ_k^{m-1} are Lipschitz continuous.

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Lemma

(1) *If* $r \geq m$, *then* $((\rho^\tau)^{m-\frac{1}{2}})_{\tau>0}$ *is uniformly bounded in* $L^2([0,T];H^1(\Omega))$.

(2) *If* $r < m < r + \frac{\beta}{2}$, *then* $((\rho^\tau)^{m-\frac{1}{2}})_{\tau>0}$ *is uniformly bounded in* $L^q([0,T];W^{1,q}(\Omega))$ *for some* $q \in (1,2)$.

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→ Together with the previous estimates, these are enough to pass to the limit, using again a refined version of the Aubin-Lions lemma.

Representation as continuity equations

Under suitable additional assumptions, our main equation (G) also reads as

$$\begin{cases} \partial_t \rho - \nabla \cdot \left(\rho \nabla \left(S'(\rho)\mathbb{1}_{\{\rho \neq 1\}} + p\mathbb{1}_{\{\rho=1\}}\right)\right) - \nabla \cdot (\rho \nabla \Phi) = 0, & \text{in } (0,T) \times \Omega, \\ \rho(0,\cdot) = \rho_0, & \text{in } \Omega, \\ \rho\left[\nabla\left(S'(\rho)\mathbb{1}_{\{\rho\neq 1\}} + p\mathbb{1}_{\{\rho=1\}}\right) + \nabla\Phi\right] \cdot \mathbf{n} = 0, & \text{in } [0,T] \times \partial\Omega. \end{cases} \tag{4}$$

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Corollary

If

$$m < r + \frac{1}{2},\ \beta > 2 \text{ and } m < \frac{\beta}{2} + \frac{1}{2},$$

then (ρ, p) *is a weak solution of* (4) *in the sense of distribution.*

We underline that additional assumptions are needed to guarantee Sobolev estimates on $S'(\rho)$.

The emergence of the region $\{\rho = 1\}$

The phenomenon observed in [Bántay-Jánosi, 1992] (they use Dirichlet boundary conditions):

$\rightarrow$

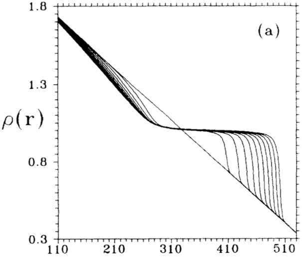

Figure: Time evolution of ρ

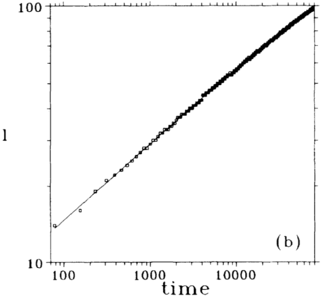

Figure: The growth of the critical region on a log-log scale

Confirming such a phenomenon

Our results support such phenomena by the simple reasoning below.

Lemma

If $t \in (0,T)$ is a Lebesgue point both for $t \mapsto \rho_t$ and $t \mapsto p_t$ with $\mathscr{L}^1(\{\rho_t < 1\}) > 0$ and $\mathscr{L}^1(\{\rho_t > 1\}) > 0$ then $\mathscr{L}^1(\{\rho_t = 1\}) > 0$.

$\rightarrow$ The proof is based on $p(t,\cdot) \in C^{0,\frac{1}{2}}(\Omega)$ (coming from the H^1 spacial regularity in 1D) for all Lebesgue point t for $t \mapsto \rho_t$ and $t \mapsto p_t$.

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- $\rightarrow$ $p = S'(1-)$ a.e. on $\{\rho < 1\}$, $p = S'(1+)$ a.e. on $\{\rho > 1\}$ and $S'(1-) < S'(1+)$.

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→ We computed one minimizing movement step in 1D, for $\Phi(x) = 2x$, $\Omega = [0, 1]$ and S in the logarithmic entropy.

$$\rho_k := \operatorname{argmin}_{\rho \in \mathscr{P}(\Omega)} \left\{ \int_\Omega S(\rho(x))\,\mathrm{d}x + \int_\Omega 2x\,\mathrm{d}\rho(x) + \frac{1}{2\tau} W_2^2(\rho, \rho_{k-1}) \right\},$$

$$\begin{cases} p_k(x) = 1 & \text{a.e. in } \{0 < \rho_k(x) < 1\}, \\ p_k(x) \in [1, 2] & \text{a.e. in } \{\rho_k(x) = 1\}, \\ p_k(x) = 2 & \text{a.e. in } \{\rho_k(x) > 1\}. \end{cases}$$

→

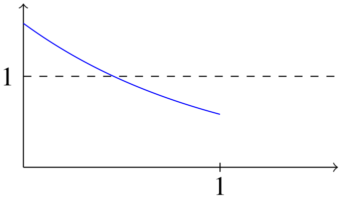

Figure: ρ_0

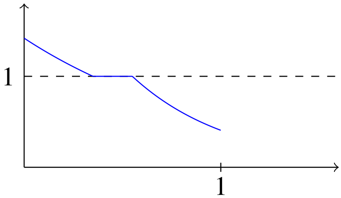

Figure: ρ_1

Uniqueness of solutions

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Theorem

Let $(\rho^1, p^1), (\rho^2, p^2)$ be solutions to (G)-(P) with initial conditions $\rho_0^1, \rho_0^2 \in \mathscr{P}(\Omega)$ such that $\mathcal{J}(\rho_0^i) < +\infty$, $i = 1, 2$. Suppose that $L_S(\rho^i, p^i) \in L^2([0,T] \times \Omega)$, $i = 1, 2$. Then we have

$$\|\rho_t^1 - \rho_t^2\|_{L^1(\Omega)} \leq \|\rho_0^1 - \rho_0^2\|_{L^1(\Omega)}, \ \mathscr{L}^1 - \text{a.e. } t \in [0,T].$$

→ The assumption $L_S(\rho, p) \in L^2([0,T] \times \Omega)$ seems natural in the context of the PME equation.

→ This is not needed if $\rho_0^i \in L^\infty(\Omega)$.

→ Because of the $L^\beta([0,T] \times \Omega)$ estimates on ρ^i, this assumption is fulfilled already if $\beta \geq 2r$.

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Open question: can one obtain $W_2(\rho_t^1, \rho_t^2) \leq C(t) W_2(\rho_0^1, \rho_0^2)$? (cf. [Bolley-Carrillo, CPDE, 2014]).

Singular limits

→ For $\varepsilon_1, \varepsilon_2 > 0$, consider $\mathcal{E}_{\varepsilon_1,\varepsilon_2} : \mathscr{P}(\Omega) \to \mathbb{R} \cup \{+\infty\}$, defined as

$$\mathcal{E}_{\varepsilon_1,\varepsilon_2}(\rho) := \begin{cases} \int_\Omega S_{\varepsilon_1,\varepsilon_2}(\rho(x))\,\mathrm{d}x, & \text{if } S_{\varepsilon_1,\varepsilon_2}(\rho) \in L^1(\Omega), \\ +\infty, & \text{otherwise,} \end{cases}$$

where $S_{\varepsilon_1,\varepsilon_2} : \mathbb{R} \to \mathbb{R}$ is convex and has the form

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→ It turns out that we have uniform estimates w.r.t $\varepsilon_1, \varepsilon_2 > 0$.

→ One can take $\varepsilon_1 \downarrow 0$ (and ε_2 fixed) to obtain the well-posedness of the original sandpile model.

→ One can take $\varepsilon_2 \to +\infty$ (and ε_1 fixed) to obtain well-posedness results for (parabolic) problems under density constraints $\rho \leq 1$.

Open question #1

$\rightarrow$ Can we obtain the higher regularity of ρ and p?

$\rightarrow$ More properties of the critical region $\{\rho = 1\}$?

$\rightarrow$ Can we obtain the regularity of the interface $\partial\{\rho = 1\}$?

Free boundary approach

$\rightarrow$

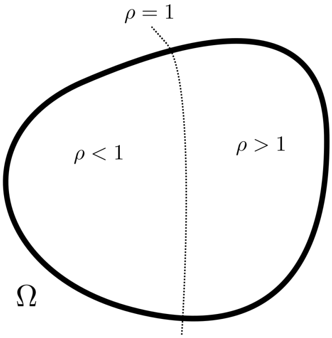

Figure: Two phases

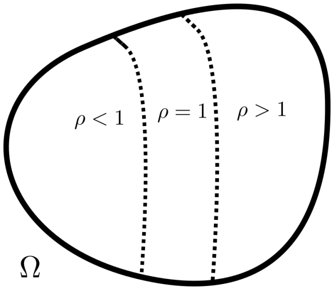

Figure: Three phases

Free boundary approach

→ Formally, we can write the *three phase free boundary problem*

$$\Delta p = -\Delta\Phi,\ \text{in}\ \{\rho = 1\},\quad p = \mathcal{S}'(1-)\ \text{in}\ \{\rho < 1\}\ \text{and}\ p = S'(1+)\ \text{in}\ \{\rho > 1\},$$

Free boundary approach

→ Formally, we can write the *three phase free boundary problem*

$$\Delta p = -\Delta\Phi, \text{ in } \{\rho = 1\}, \quad p = \mathcal{S}'(1-) \text{ in } \{\rho < 1\} \text{ and } p = S'(1+) \text{ in } \{\rho > 1\},$$

or more in details for our first example as

$$\begin{cases} \partial_t\rho = \Delta\rho + \nabla\cdot(\nabla\Phi\rho) & \text{in } \{p\rho < 1\}, \\ -\Delta p = \Delta\Phi, & \text{in } \{1 < p\rho < 2\}, \\ \partial_t\rho = 2\Delta\rho + \nabla\cdot(\nabla\Phi\rho) & \text{in } \{p\rho > 2\}, \end{cases}$$

with boundary conditions

$$\begin{cases} |D(p\rho)^{1+}| - |D(p\rho)^{1-}| = 0 \text{ on } \{p\rho = 1\}. \\ |D(p\rho)^{2+}| - |D(p\rho)^{2-}| = 0 \text{ on } \{p\rho = 2\} \end{cases}$$

and

$$\begin{cases} p = 1 & \text{in } \{p\rho < 1\}, \\ \rho = 1, & \text{in } \{1 < p\rho < 2\}, \\ p = 2 & \text{in } \{p\rho > 2\}, \end{cases}$$

Open questions #2

Recall the growth of S: $S \sim \rho^m$ in $(0, 1)$ and $S \sim \rho^r$ in $(1, +\infty)$.

- $\to$ What happens if $m >> r$?
- $\to$ Can we obtain Sobolev estimates?
- $\to$ If not, can we observe some singular phenomena as below?
- $\to$

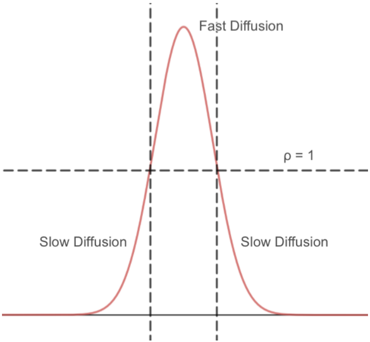

Figure: $t = 0$

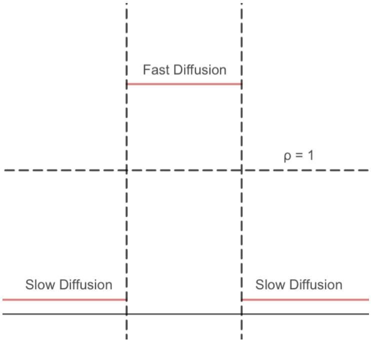

Figure: $t = t^* > 0$

Thank you for your attention!