Degenerate nonlinear parabolic equations with discontinuous diffusion coefficients

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(based on joint works with Dohyun Kwon, UCLA)

Geometric and Functional Inequalities and Recent Topics in Nonlinear PDEs
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Mathematical motivation

→ Study the well-posedness and structure of solutions to diffusion equations with discontinuous nonlinearities.

→ Model problem:

\[
\begin{align*}
\partial_t \rho - \Delta \varphi(\rho) - \nabla \cdot (\nabla \Phi \rho) &= 0, \quad \text{in } (0, T) \times \Omega, \\
(\nabla \varphi(\rho) + \nabla \Phi \rho) \cdot \mathbf{n} &= 0, \quad \text{on } (0, T) \times \partial\Omega \quad \text{(NDE)} \\
\rho(0, \cdot) &= \rho_0, \quad \text{in } \Omega,
\end{align*}
\]

where \( T > 0, \Omega \subset \mathbb{R}^d \) smooth, bounded convex domain, \( \rho_0 \in \mathcal{P}^{\text{ac}}(\Omega) \) and \( \Phi : \Omega \to \mathbb{R} \) is a given Lipschitz continuous potential.
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→ Example of a nonlinearity

\[
\varphi : [0, +\infty) \to \mathbb{R}, \quad \varphi(s) = \begin{cases} 
\rho, & \rho \in [0, 1), \\
[\rho, 2\rho], & \rho = 1, \\
2\rho, & \rho > 1,
\end{cases}
\]
Motivation: Starvation driven diffusion in mathematical biology

→ A competition between a linear diffusion and a starvation driven diffusion:

\[ \partial_t u = d \Delta u + u(m - u - v), \quad \partial_t v = \Delta \varphi(v; m) + v(m - u - v). \]

where \( u, v \) represent two population densities and \( m \) stands for the resource density.

→ For \( 0 < l < h \), \( \varphi(v; m) := \begin{cases} \lv, & \text{if } v < m, \\ hv, & \text{if } v > m. \end{cases} \)

→ Cho-Kim [2013, Bull. Math. Biol.] (“Starvation driven diffusion as a survival strategy of biological organisms”) (Ex: \( \Omega = (0, 1) \), \( m \) discontinuous with two constant values and \( u(0, \cdot) = v(0, \cdot) = m/2; l = 0.002, h = 0.004 \))
Motivation: self-organized criticality in physics


→ Same problem as (NDE), with $\Phi = 0$, $\varphi(\rho) = f(\rho)H(\rho - \rho_c)$, where $f$ is some given function (either identity, or a constant), $H$ is the Heaviside function and $\rho_c$ stands for the critical density value.

Figure: Avalanches in the Himalayas

Figure: Time evolution of $\rho$, $\rho_c = 1$ [Bántay-Jánosi, 1992]
Mathematical literature on the previous models

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Mathematical literature on the previous models

→ Then case when \( \varphi \) is continuous has been studied extensively by many-many authors.

For \( \varphi \) discontinuous:

→ Blanchard-Röckner-Russo [2010, Ann. Probab.] \( \rightarrow |\varphi(\rho)| \leq C\rho \); probabilistic approach in 1D; non-degenerate case.

→ Barbu-Röckner-Russo, [2011, PTRF] \( \rightarrow \) same model, probabilistic approach in 1D; degenerate case.
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→ Barbu-Röckner [2018, SIMA] $\rightarrow$ higher dimensions; probabilistic approach; nonlinear semigroup theory $\rightarrow$ maximal monotone operators, parabolic approximation, i.e. $\varphi_\varepsilon \rightarrow \varphi$.

→ Notion of solution: generalized entropic solutions à la Kruzkov.

→ This heuristically can be written as pairs $(\rho, \eta_\rho)$ belonging to well-chosen function spaces, such that

$$\partial_t \rho - \Delta(\eta_\rho) - \nabla \cdot (\nabla \Phi \rho) = 0$$

is fulfilled and $\rho(t, x) \in \eta_\rho(t, x)$ a.e.
Our main objectives

(1) Find a unified way to treat general discontinuous nonlinearities.

(2) Give a fine characterization of the emerging critical regions \( \{ \rho = 1 \} \) observed in numerical experiments.
Our approach: optimal transport and gradient flows

OT toolbox

→ for $\mu, \nu \in \mathcal{P}(\Omega)$ we define the $2$-Wasserstein distance $W_2$ as

$$W_2^2(\mu, \nu) := \inf \left\{ \int_{\Omega \times \Omega} |x - y|^2 \, d\gamma : \gamma \in \mathcal{P}(\Omega \times \Omega), (\pi^x)^\# \gamma = \mu, (\pi^y)^\# \gamma = \nu \right\}$$

where for $T : X \to Y$ Borel function $T^\# \mu = \nu$ means that $\nu(A) = \mu(T^{-1}(A))$ for any $A \subseteq Y$ Borel set.

→ we have the dual formulation

$$W_2^2(\mu, \nu) := \sup \left\{ \int_{\Omega} \phi \, d\mu + \int_{\Omega} \psi \, d\nu : \phi, \psi \in C_b(\Omega), \phi(x) + \psi(y) \leq |x - y|^2 \right\}.$$
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→ for any finite measure $\chi$ s.t. $\chi(\Omega) = 0$ we have the first variation formula

$$\frac{d}{dt} \bigg|_{t=0} \frac{1}{2} W_2^2(\mu + t\chi, \nu) = \int_{\Omega} \phi \, d\chi.$$  

→ Brenier [1991, CPAM]: if $\mu \in \mathcal{P}^{ac}(\Omega)$, then $\gamma_{opt} = (\text{id}, T)_\# \mu$, with $T = \text{id} - \nabla \phi_{opt}$. 

Gradient flows in \((\mathcal{P}(\Omega), W_2)\)

→ noticed by Otto (see [2001, CPDE]), and Ambrosio-Gigli-Savaré (see [2005, Birkhäuser, Springer]) \((\mathcal{P}(\Omega), W_2)\) has a differential geometric structure

Let \( \rho_0 \) be given and \( N \in \mathbb{N} \) and \( \tau > 0 \) be such that \( T = N \tau \). Construct the recursive sequence for all \( k \in \{1, \ldots, N\} \)

\[
\rho_k \in \text{argmin}_{\rho \in \mathcal{P}(\Omega)} J(\rho) + \frac{1}{2} \tau W_2^2(\rho, \rho_{k-1})
\]

→ optimality condition \( \log(\rho_k) + \phi_k \tau = \text{const on spt}(\rho_k) \).

→ approximate velocity \( v_{\tau k} := x - T_k(x) \tau = r \phi_k \tau = -r \rho_k \rho_k \).

→ after interpolations, the limit curve, as \( \tau \downarrow 0 \) solves

\[
\partial_t \rho + r \cdot (\rho v) = 0
\]

Optimal transport and gradient flows

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→ for example \( \partial_t \rho - \Delta \rho = 0 \) can be seen as the GF of the Boltzmann entropy \( \mathcal{I}(\rho) = \int_\Omega \rho \log(\rho) \, dx \).
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\]

→ optimality condition \(\log(\rho_k) + 1 + \frac{\phi_k}{\tau} = \text{const on spt}(\rho_k)\).

→ approximate velocity \(v_k^T := \frac{x - T_k(x)}{\tau} = \frac{\nabla \phi_k}{\tau} = -\frac{\nabla \rho_k}{\rho_k}\).

→ after interpolations, the limit curve, as \(\tau \downarrow 0\) solves \(\partial_t \rho + \nabla \cdot (\rho v) = 0\).
We define the energy associated to our models as

\[
\mathcal{J}(\rho) := S(\rho) + \mathcal{F}(\rho) := \int_{\Omega} S(\rho(x)) \, dx + \int_{\Omega} \Phi \, d\rho(x).
\]
Back to our problems

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\[ \varphi(\rho) = \rho S'(\rho) - S(\rho) + S(1). \]
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→ We consider the GF of the functional \( J \) in \( (P(\Omega), W_2) \).

→ \( J \) fails to be differentiable. Therefore the classical theory does not imply directly; one needs to work with subdifferential calculus.

→ We need to rely on the scheme (MM). To write optimality conditions, we characterize the Wasserstein subdifferential of \( J \).
Estimates

→ We need to choose carefully the function spaces: we work in $L^p(\Omega)$, $1 < p \leq +\infty$.

Lemma ($L^\infty$ estimates)

Let $\rho_0 \in L^\infty(\Omega)$. Let $(\rho_k)_{k=1}^N$ be constructed via the scheme (MM). Then we have

$$\|\rho_k\|_{L^\infty} \leq C(T, \Phi)\|\rho_0\|_{L^\infty}, \forall k \in \{1, \ldots, N\}.$$
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→ Proof: easy argument combining [Santambrogio, 2015, Springer] and [Carrillo-Santambrogio, QAM, 2018].
The model problem via GF in \((\mathcal{P}(\Omega), W_2)\)

# Estimates

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**Lemma (\(L^\infty\) estimates)**

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**Lemma (\(L^\beta\) estimates)**

Let \(\rho_0 \in \mathcal{P}(\Omega)\) such that \(\mathcal{J}(\rho_0) < +\infty\). Let \(S''(\rho) \geq C\rho^{r-2}\), if \(\rho \in (1, +\infty)\) for some \(r \geq 1\). Let \((\rho_k)_{k=1}^N\) be constructed via the scheme (MM). Then we have

\[
\|\rho_k\|_{L^\beta} \leq C(T, \Phi, 1/\tau), \quad \forall k \in \{1, \ldots, N\},
\]

where \(\beta := \begin{cases} (2r - 1) \frac{d}{d-2}, & d \geq 3, \\ +\infty, & d = 2, \\ +\infty, & d = 1. \end{cases} \)
Estimates and optimality conditions

As a consequence, we have uniform $L^\beta([0, T] \times \Omega)$ estimates on the piecewise constant interpolations $(\rho^\tau)_{\tau > 0}$. 
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→ We compute subdifferentials in $L^p(\Omega)^*$ (including $p = +\infty$). We have

Theorem

For all $k \in \{1, \ldots, N\}$, there exists $C = C(k) \in \mathbb{R}$ and $\phi_k$ such that

\[
\begin{cases}
C - \frac{\phi_k}{\tau} - \Phi \leq S'(0+) & \text{in } \{\rho_k = 0\}, \\
C - \frac{\phi_k}{\tau} - \Phi \in [S'(1-), S'(1+)] & \text{in } \{\rho_k = 1\}, \\
C - \frac{\phi_k}{\tau} - \Phi = S' \circ \rho_k & \text{otherwise}.
\end{cases}
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**Theorem**

*For $\rho_k$ is given in (MM), if $\xi \in \partial S(\rho_k) \cap L^1(\Omega)$, then it holds that*

\[
\xi \in \begin{cases}
[-\infty, S'(0+)] & \text{in } \{\rho_k = 0\}, \\
[S'(1-), S'(1+)] & \text{in } \{\rho_k = 1\}, \\
S' \circ \rho_k & \text{in } \{\rho_k \neq 1\},
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More on optimality conditions and a new variable

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More on optimality conditions and a new variable

→ proof uses a theorem of Rockafellar [1971, PJM]; we can also show that $\xi^s = 0$.
→ **Question:** how do we identify the approximate velocity, i.e. $\frac{\nabla \phi_k}{\tau}$?
→ **Answer:** inspired by the analysis of Maury-Roudneff-Chupin-Santambrogio [2010, M3AM] (also [M.-Santambrogio, 2016, APDE]), we introduce a new variable:

→ For $k \in \{1, \ldots, N\}$, we define $p_k : \Omega \to \mathbb{R}$ as

$$p_k := \begin{cases} \max \left\{ C - \frac{\phi_k}{\tau} - \Phi, S'(1-) \right\} & \text{in } \{\rho_k < 1\}, \\ C - \frac{\phi_k}{\tau} - \Phi & \text{in } \{\rho_k = 1\}, \\ \min \left\{ C - \frac{\phi_k}{\tau} - \Phi, S'(1+) \right\} & \text{in } \{\rho_k > 1\}. \end{cases}$$

→ Or, equivalently

$$p_k = \min \left\{ \max \left\{ C - \frac{\phi_k}{\tau} - \Phi, S'(1-) \right\}, S'(1+) \right\}.$$
A model problem

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→ Let us illustrate this in the example of $S(\rho) := \begin{cases} 
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→ We have

$$p_k := \begin{cases} 
1 & \text{in } \{\rho_k < 1\}, \\
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Lemma

For all $k \in \{1, \ldots, N\}$, there exists $C \in \mathbb{R}$ such that

$$p_k(1 + \log \rho_k) + \frac{\phi_k}{\tau} + \Phi = C \text{ a.e.}$$

In particular, both $p_k$ and $\rho_k$ are Lipschitz continuous and $\rho_k > 0$ a.e.
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Lemma

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p_k(1 + \log \rho_k) + \frac{\phi_k}{\tau} + \Phi = C \text{ a.e.}
\]

*In particular, both $p_k$ and $\rho_k$ are Lipschitz continuous and $\rho_k > 0$ a.e.*

→ As a consequence, $\frac{\nabla \phi_k}{\tau} = -\nabla \Phi - \nabla p_k - p_k \frac{\nabla \rho_k}{\rho_k}$ (since $\nabla p_k \log(\rho_k) = 0$).
Uniform estimates and passing to the limit at $\tau \downarrow 0$

→ Let us notice that

$$\frac{1}{\tau} \sum_{k=1}^{N} W_2^2(\rho_k, \rho_{k-1}) = \frac{1}{\tau} \sum_{k=1}^{N} \int_{\Omega} |\nabla \phi_k|^2 \leq J(\rho_0) - \inf J.$$
Uniform estimates and passing to the limit at $\tau \downarrow 0$

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$$

$\rightarrow$ From here, the piecewise constant interpolations satisfy: $(\sqrt{\rho^\tau})_{\tau > 0}$ and $(p^\tau)_{\tau > 0}$ are uniformly bounded in $L^2([0, T]; H^1(\Omega))$. 
Theorem
Suppose that $\rho_0 \in L_\infty(\Omega)$ and $r \Phi \cdot n > 0$ on $\partial \Omega$. Then, there exists $\rho, p \in L_\infty([0, T] \times \Omega) \cap L_2([0, T]; H_1(\Omega))$ such that $(\rho, p)$ is a unique solution to
\[
\begin{cases}
\partial_t \rho - \Delta (p \rho) - r \cdot (r \Phi \rho) = 0, & \text{in } (0, T) \times \Omega,
\\
(r(p \rho) + r \Phi \rho) \cdot n = 0, & \text{on } (0, T) \times \partial \Omega,
\\
\rho(\cdot, 0) = \rho_0, & \text{in } \Omega,
\end{cases}
\]
in the sense of distribution.

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→ If in addition, $\rho_0 \in L^\infty(\Omega)$, then $(\rho^\tau)_{\tau > 0}$ is uniformly bounded in $L^2([0, T]; H^1(\Omega))$. 
The model problem via GF in $\mathcal{P}(\Omega), W_2$

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Suppose that $\rho_0 \in L^\infty(\Omega)$ and $\nabla \Phi \cdot n > 0$ on $\partial \Omega$. Then, there exists $\rho, p \in L^\infty([0, T] \times \Omega) \cap L^2([0, T]; H^1(\Omega))$ such that $(\rho, p)$ is a unique solution to

$$\begin{cases}
\partial_t \rho - \Delta (p \rho) - \nabla \cdot (\nabla \Phi \rho) = 0, & \text{in } (0, T) \times \Omega, \\
(\nabla (p \rho) + \nabla \Phi \rho) \cdot n = 0, & \text{on } (0, T) \times \partial \Omega, \\
\rho(\cdot, 0) = \rho_0, & \text{in } \Omega,
\end{cases} \quad (2)$$

in the sense of distribution.
Some remarks

Remark

$(\rho, p)$ satisfies

\[
\begin{cases}
  p = 1 & \text{a.e. in } \{0 < \rho < 1\}, \\
  p \in [1, 2] & \text{a.e. in } \{\rho = 1\}, \\
  p = 2 & \text{a.e. in } \{\rho > 1\}.
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Remark

*If we consider more general initial, i.e. $\rho_0 \in \mathcal{P}(\Omega)$ such that $\mathcal{E}(\rho_0) < +\infty$, we find a solution*

\[
\rho \in L^\beta([0, T] \times \Omega) \text{ and } p \in L^2([0, T]; H^1(\Omega)) \cap L^\infty([0, T] \times \Omega)
\]

*with $\sqrt{\rho} \in L^2([0, T]; H^1(\Omega))$.***
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with \(\sqrt{\rho} \in L^2([0, T]; H^1(\Omega))\).

→ In the proof, to gain compactness we use an Aubin-Lions type argument for \(\rho^\tau\).
What about more general problems?

→ The corresponding ‘porous medium example’ follows similar arguments with some additional care, since the sequence $(\rho_k)_k$ in general fails to be fully supported on $\Omega$. 
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→ Let

\[
S(\rho) := \begin{cases} 
\frac{\rho^m}{m-1}, & \text{for } \rho \in [0, 1], \\
\frac{2\rho^m}{m-1} - \frac{1}{m-1}, & \text{for } \rho \in (1, +\infty). 
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where \(m > 1\).
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\]

where \(m > 1\).

→ Our main theorem for the associated entropy can be formulated as follows.
Main theorem for the PM type model problem

Theorem (Kwon-M., 2021)

For \( \rho_0 \in \mathcal{P}(\Omega) \) such that \( J(\rho_0) < +\infty \), there exists \( \rho \in L^\beta([0, T] \times \Omega) \) and \( p \in L^2([0, T]; H^1(\Omega)) \cap L^\infty([0, T] \times \Omega) \) with \( \rho^{m-\frac{1}{2}} \in L^2([0, T]; H^1(\Omega)) \) such that \((\rho, p)\) is a weak solution of

\[
\begin{cases}
\partial_t \rho - \Delta \left( [(m-1)\rho^m + 1] \frac{p}{m} \right) - \nabla \cdot (\nabla \Phi \rho) = 0, & \text{in } (0, T) \times \Omega, \\
\rho(0, \cdot) = \rho_0, & \text{in } \Omega, \\
(\nabla \left( [(m-1)\rho^m + 1] \frac{p}{m} \right) + \nabla \Phi \rho) \cdot \mathbf{n} = 0, & \text{in } [0, T] \times \partial \Omega,
\end{cases}
\]  

(3)

in the sense of distribution. Furthermore, \((\rho, p)\) satisfies

\[
\begin{cases}
p(t, x) = \frac{m}{m-1} & \text{a.e. in } \{0 < \rho < 1\}, \\
p(t, x) \in \left[ \frac{m}{m-1}, \frac{2m}{m-1} \right] & \text{a.e. in } \{\rho = 1\}, \\
p(t, x) = \frac{2m}{m-1} & \text{a.e. in } \{\rho > 1\}.
\end{cases}
\]

In addition, if \( \rho_0 \in L^\infty(\Omega) \) and \( \nabla \Phi \cdot \mathbf{n} > 0 \) on \( \partial \Omega \), then \( \rho \in L^\infty([0, T] \times \Omega) \) and \( \rho^m \in L^2([0, T]; H^1(\Omega)) \).
The ‘fully’ general problem

→ Recall that if $S$ is differentiable, then we have

$$
\varphi(\rho) = \rho S'(\rho) - S(\rho) + S(1)
$$

→ Based on the observation and the derivation of $p$, we define the operator $L_S$ pointwisely for functions $(\rho, p) : [0, T] \times \Omega \to \mathbb{R}$ by

$$
L_S(\rho, p)(t, x) := [\rho(t, x)S'(\rho(t, x)) - S(\rho(t, x)) + S(1)] 1_{\{\rho \neq 1\}}(t, x) + p(t, x) 1_{\{\rho = 1\}}(t, x)
$$

→ Recall that for a.e. $(t, x) \in [0, T] \times \Omega$ the pressure variable $p : [0, T] \times \Omega \to \mathbb{R}$ satisfies a.e.

$$
\begin{cases}
  p(t, x) = S'(1-) & \text{if } 0 \leq \rho(t, x) < 1, \\
  p(t, x) \in [S'(1-), S'(1+)] & \text{if } \rho(t, x) = 1, \\
  p(t, x) = S'(1+) & \text{if } \rho(t, x) > 1.
\end{cases} \quad (P)
$$

→ We aim to find a solution to the PDE

$$
\begin{cases}
  \partial_t \rho - \Delta(L_S(\rho, p)) - \nabla \cdot (\nabla \Phi \rho) = 0, & \text{in } (0, T) \times \Omega, \\
  \rho(0, \cdot) = \rho_0, & \text{in } \Omega, \\
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$$L_S(\rho, p)(t, x) := [\rho(t, x)S'(\rho(t, x)) - S(\rho(t, x)) + S(1)] \mathbb{1}_{\{\rho \neq 1\}}(t, x) + p(t, x) \mathbb{1}_{\{\rho = 1\}}(t, x)$$

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→ We aim to find a solution to the PDE

$$\begin{cases} \partial_t \rho - \Delta(L_S(\rho, p)) - \nabla \cdot (\nabla \Phi \rho) = 0, & \text{in } (0, T) \times \Omega, \\ \rho(0, \cdot) = \rho_0, & \text{in } \Omega, \\ (\nabla(L_S(\rho, p)) + \nabla \Phi \rho) \cdot n = 0, & \text{in } [0, T] \times \partial \Omega. \end{cases} \quad (G)$$
Assume that $S \sim \rho^m$ in $(0, 1)$ and $S \sim \rho^r$ in $(1, +\infty)$, for some $m \geq 1$, $r \geq 1$. Set $\beta \geq 1$ as before, i.e.

$$\beta := \begin{cases} 
(2r - 1) \frac{d}{d-2} & \text{if } d \geq 3, \\
[1, \infty) & \text{if } d = 2, \\
+\infty & \text{if } d = 1.
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→ Based on the ‘regularization of $S$’ and the Sobolev embedding theorem, we obtain the uniform bound $L^\beta([0, T] \times \Omega)$ for $(\rho^\tau)_{\tau > 0}$. 
Assume that \( S \sim \rho^m \) in \((0, 1)\) and \( S \sim \rho^r \) in \((1, +\infty)\), for some \( m \geq 1, r \geq 1 \). Set \( \beta \geq 1 \) as before, i.e.

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Based on the ‘regularization of \( S \)’ and the Sobolev embedding theorem, we obtain the uniform bound \( L^\beta([0, T] \times \Omega) \) for \( (\rho^\tau)_{\tau>0} \).

Our main theorem reads as

**Theorem (Kwon-M., 2021)**

Suppose that the above growth conditions are fulfilled and

\[
m < r + \frac{\beta}{2}
\]

holds true. For \( \rho_0 \in \mathcal{P}(\Omega) \) such that \( \mathcal{E}(\rho_0) < +\infty \), there exists \( \rho \in L^\beta([0, T] \times \Omega), \rho^{m - \frac{1}{2}} \in L^2([0, T]; H^1(\Omega)) \) and \( p \in L^2([0, T]; H^1(\Omega)) \cap L^\infty([0, T] \times \Omega) \) such that \((\rho, p)\) is a solution of (G)-(P) in the sense of distributions.
The main idea of the proof \((m = 1, \text{ the less involved case})\)

\[ \rightarrow \text{ The hypothesis } m < r + \frac{\beta}{2} \text{ is always true.} \]
The main idea of the proof ($m = 1$, the less involved case)

→ The hypothesis $m < r + \frac{\beta}{2}$ is always true.
→ We define the auxiliary functions $S_a$ and $S_b : [0, +\infty) \to \mathbb{R}$ by

$$S_a(\rho) := \begin{cases} 
S'(1-)\rho \log \rho, & \text{for } \rho \in [0, 1], \\
S'(1+)\rho \log \rho, & \text{for } \rho \in (1, +\infty), 
\end{cases}$$

and

$$S_b(\rho) := S(\rho) - S_a(\rho).$$
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The main idea of the proof \((m = 1, \text{the less involved case})\)

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and

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\]

→ It turns out that \(S_b\) is differentiable on \((0, +\infty)!\)

→ We obtain that \(\rho_k > 0\) a.e. and the optimality condition,

\[
p_k(1 + \log \rho_k) + S'_b(\rho_k) + \frac{\phi_k}{\tau} + \Phi = C \text{ a.e.}
\]
The main idea of the proof \((m > 1)\)

→ The proof is technical.
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\end{cases}
\]

Lemma

*For all \(k \in \{1, \ldots, N\}\), there exists \(C \in \mathbb{R}\) such that*

\[
\rho_k^{m-1} p_k = \left( C - \frac{\phi_k}{\tau} - \Phi \right)_+ \text{ a.e.}
\]

*In particular, \(p_k\) and \(\rho_k^{m-1}\) are Lipschitz continuous.*
The main idea of the proof \((m > 1)\) (Continued)

→ In order to have the strong convergence, we need spacial Sobolev estimates.
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Lemma

1. If \(r \geq m\), then \((\rho^\tau)^{m-\frac{1}{2}}\) is uniformly bounded in \(L^2([0, T]; H^1(\Omega))\).

2. If \(r < m < r + \frac{\beta}{2}\), then \((\rho^\tau)^{m-\frac{1}{2}}\) is uniformly bounded in \(L^q([0, T]; W^{1,q}(\Omega))\) for some \(q \in (1, 2)\).
The main idea of the proof ($m > 1$) (Continued)

→ In order to have the strong convergence, we need spatial Sobolev estimates.

Lemma

(1) If $r \geq m$, then $((\rho^\tau)^{m-\frac{1}{2}})^{\tau>0}$ is uniformly bounded in $L^2([0, T]; H^1(\Omega))$.

(2) If $r < m < r + \frac{\beta}{2}$, then $((\rho^\tau)^{m-\frac{1}{2}})^{\tau>0}$ is uniformly bounded in $L^q([0, T]; W^{1,q}(\Omega))$ for some $q \in (1, 2)$.

→ Together with the previous estimates, these are enough to pass to the limit, using again a refined version of the Aubin-Lions lemma.
Under suitable additional assumptions, our main equation (G) also reads as

\[
\begin{cases}
\partial_t \rho - \nabla \cdot \left( \rho \nabla \left( S'(\rho) \mathbb{1}_{\rho \neq 1} + p \mathbb{1}_{\rho = 1} \right) \right) - \nabla \cdot \left( \rho \nabla \Phi \right) = 0, & \text{in } (0, T) \times \Omega, \\
\rho(0, \cdot) = \rho_0, & \text{in } \Omega, \\
\rho \left[ \nabla \left( S'(\rho) \mathbb{1}_{\rho \neq 1} + p \mathbb{1}_{\rho = 1} \right) + \nabla \Phi \right] \cdot \mathbf{n} = 0, & \text{in } [0, T] \times \partial \Omega.
\end{cases}
\]

(4)
Representation as continuity equations

Under suitable additional assumptions, our main equation (G) also reads as

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\rho(0, \cdot) &= \rho_0, & \text{in } \Omega, \\
\rho \left[ \nabla \left( S'(\rho) \mathbb{1}_{\rho \neq 1} + p \mathbb{1}_{\rho = 1} \right) + \nabla \Phi \right] \cdot n &= 0, & \text{in } [0, T] \times \partial \Omega.
\end{aligned}
\]  

(4)

Corollary

If

\[ m < r + \frac{1}{2}, \beta > 2 \text{ and } m < \frac{\beta}{2} + \frac{1}{2}, \]

then \((\rho, p)\) is a weak solution of (4) in the sense of distribution.

We underline that additional assumptions are needed to guarantee Sobolev estimates on \(S'(\rho)\).
The emergence of the region \{\rho = 1\}

The phenomenon observed in [Bántay-Jánosi, 1992] (they use Dirichlet boundary conditions):

Figure: Time evolution of \(\rho\)

Figure: The growth of the critical region on a log-log scale
The model problem via GF in \((\mathcal{P}(\Omega), W_2)\)

## Confirming such a phenomenon

Our results support such phenomena by the simple reasoning below.

**Lemma**

*If* \(t \in (0, T)\) *is a Lebesgue point both for* \(t \mapsto \rho_t\) *and* \(t \mapsto p_t\) *with*

\[ \mathcal{L}^1(\{\rho_t < 1\}) > 0 \text{ and } \mathcal{L}^1(\{\rho_t > 1\}) > 0 \text{ then } \mathcal{L}^1(\{\rho_t = 1\}) > 0. \]

→ The proof is based on \(p(t, \cdot) \in C^{0, \frac{1}{2}}(\Omega)\) (coming from the \(H^1\) spacial regularity in 1D) for all Lebesgue point \(t\) for \(t \mapsto \rho_t\) and \(t \mapsto p_t\).
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If \( t \in (0, T) \) is a Lebesgue point both for \( t \mapsto \rho_t \) and \( t \mapsto p_t \) with
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\[ \rightarrow \] The proof is based on \( p(t, \cdot) \in C^{0, \frac{1}{2}}(\Omega) \) (coming from the \( H^1 \) spacial regularity in 1D) for all Lebesgue point \( t \) for \( t \mapsto \rho_t \) and \( t \mapsto p_t \).
\[ \rightarrow \] \( p = S'(1-) \) a.e. on \( \{\rho < 1\} \), \( p = S'(1+) \) a.e. on \( \{\rho > 1\} \) and \( S'(1-) < S'(1+) \).
The fact that \( \mathcal{L}^d(\{\rho_k = 1\}) > 0 \), is supported by our numerical experiments as well.
The fact that $\mathcal{L}^d(\{\rho_k = 1\}) > 0$, is supported by our numerical experiments as well.

We computed one minimizing movement step in 1D, for $\Phi(x) = 2x$, $\Omega = [0, 1]$ and $S$ in the logarithmic entropy.

$$\rho_k := \arg\min_{\rho \in \mathcal{P}(\Omega)} \left\{ \int_{\Omega} S(\rho(x)) \, dx + \int_{\Omega} 2x \, d\rho(x) + \frac{1}{2\tau} W^2_2(\rho, \rho_{k-1}) \right\},$$

$$\begin{cases} p_k(x) = 1 & \text{a.e. in } \{0 < \rho_k(x) < 1\}, \\
p_k(x) \in [1, 2] & \text{a.e. in } \{\rho_k(x) = 1\}, \\
p_k(x) = 2 & \text{a.e. in } \{\rho_k(x) > 1\}. \end{cases}$$

Figure: $\rho_0$

Figure: $\rho_1$
Theorem

Let $(\rho_1, p_1), (\rho_2, p_2)$ be solutions to $(G)-(P)$ with initial conditions $\rho_{i0}, \rho_{20} \in P(\Omega)$ such that $J(\rho_i_0) < +\infty, i = 1, 2$. Suppose that $LS(\rho_i, p_i) \in L^2([0, T] \times \Omega), i = 1, 2$. Then we have

$$k |\rho_1(t) - \rho_2(t)|_{L^1(\Omega)} \leq k |\rho_1(0) - \rho_2(0)|_{L^1(\Omega)},$$

$L^1-a.e. t \in [0, T].$

→ The assumption $LS(\rho, p) \in L^2([0, T] \times \Omega)$ seems natural in the context of the PME equation. This is not needed if $\rho_{i0} \in L^\infty(\Omega)$.

Because of the $L^\beta([0, T] \times \Omega)$ estimates on $\rho_i$, this assumption is fulfilled already if $\beta \geq 2 + r$.

Open question: can one obtain $W^2(\rho_1(t), \rho_2(t)) \leq C(t) W^2(\rho_1(0), \rho_2(0))$? (cf. [Bolley-Carrillo, CPDE, 2014]).

→ By an involved analysis, carefully combining ideas from [Vázquez, OSP, 2007] and [Di Marino-M., M3AS, 2016] we obtain an $L^1$ contraction result.
Uniqueness of solutions

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**Theorem**

Let $(\rho^1, p^1), (\rho^2, p^2)$ be solutions to (G)-(P) with initial conditions $\rho_0^1, \rho_0^2 \in \mathcal{P}(\Omega)$ such that $J(\rho_0^i) < +\infty, i = 1, 2$. Suppose that $L_S(\rho^i, p^i) \in L^2([0, T] \times \Omega), i = 1, 2$. Then we have

$$\|\rho_t^1 - \rho_t^2\|_{L^1(\Omega)} \leq \|\rho_0^1 - \rho_0^2\|_{L^1(\Omega)}, \mathcal{L}^1 - \text{a.e. } t \in [0, T].$$

The assumption $L_S(\rho, p) \in L^2([0, T] \times \Omega)$ seems natural in the context of the PME equation.

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Theorem

Let $(\rho^1, p^1), (\rho^2, p^2)$ be solutions to (G)-(P) with initial conditions $\rho^1_0, \rho^2_0 \in \mathcal{P}(\Omega)$ such that $\mathcal{J}(\rho^i_0) < +\infty$, $i = 1, 2$. Suppose that $L_S(\rho^i, p^i) \in L^2([0, T] \times \Omega)$, $i = 1, 2$. Then we have

$$\|\rho^1_t - \rho^2_t\|_{L^1(\Omega)} \leq \|\rho^1_0 - \rho^2_0\|_{L^1(\Omega)}, \mathcal{L}^1 - \text{a.e. } t \in [0, T].$$

→ The assumption $L_S(\rho, p) \in L^2([0, T] \times \Omega)$ seems natural in the context of the PME equation.

→ This is not needed if $\rho^i_0 \in L^\infty(\Omega)$.

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Open question: can one obtain $W_2(\rho^1_t, \rho^2_t) \leq C(t)W_2(\rho^1_0, \rho^2_0)$? (cf. [Bolley-Carrillo, CPDE, 2014]).
Singular limits

For \( \varepsilon_1, \varepsilon_2 > 0 \), consider \( \mathcal{E}_{\varepsilon_1, \varepsilon_2} : \mathcal{P}(\Omega) \to \mathbb{R} \cup \{+\infty\} \), defined as

\[
\mathcal{E}_{\varepsilon_1, \varepsilon_2}(\rho) := \begin{cases} 
\int_{\Omega} S_{\varepsilon_1, \varepsilon_2}(\rho(x)) \, dx, & \text{if } S_{\varepsilon_1, \varepsilon_2}(\rho) \in L^1(\Omega), \\
+\infty, & \text{otherwise},
\end{cases}
\]

where \( S_{\varepsilon_1, \varepsilon_2} : \mathbb{R} \to \mathbb{R} \) is convex and has the form

\[
S_{\varepsilon_1, \varepsilon_2}(s) = \begin{cases} 
\varepsilon_1 S_1(s), & \text{if } s \in (0, 1), \\
\varepsilon_2 S_2(s), & \text{if } s \geq 1, \\
+\infty, & \text{otherwise}.
\end{cases}
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Singular limits

For $\varepsilon_1, \varepsilon_2 > 0$, consider $\mathcal{E}_{\varepsilon_1, \varepsilon_2} : \mathcal{P}(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$, defined as

$$
\mathcal{E}_{\varepsilon_1, \varepsilon_2}(\rho) := \begin{cases} 
\int_{\Omega} S_{\varepsilon_1, \varepsilon_2}(\rho(x)) \, dx, & \text{if } S_{\varepsilon_1, \varepsilon_2}(\rho) \in L^1(\Omega), \\
+\infty, & \text{otherwise,}
\end{cases}
$$

where $S_{\varepsilon_1, \varepsilon_2} : \mathbb{R} \rightarrow \mathbb{R}$ is convex and has the form

$$
S_{\varepsilon_1, \varepsilon_2}(s) = \begin{cases} 
\varepsilon_1 S_1(s), & \text{if } s \in (0, 1), \\
\varepsilon_2 S_2(s), & \text{if } s \geq 1, \\
+\infty, & \text{otherwise.}
\end{cases}
$$

It turns out that we have uniform estimates w.r.t $\varepsilon_1, \varepsilon_2 > 0$.

One can take $\varepsilon_1 \downarrow 0$ (and $\varepsilon_2$ fixed) to obtain the well-posedness of the original sandpile model.

One can take $\varepsilon_2 \rightarrow +\infty$ (and $\varepsilon_1$ fixed) to obtain well-posedness results for (parabolic) problems under density constraints $\rho \leq 1$. 
Open question #1

→ Can we obtain the higher regularity of $\rho$ and $p$?

→ More properties of the critical region $\{\rho = 1\}$?

→ Can we obtain the regularity of the interface $\partial\{\rho = 1\}$?
Free boundary approach

Figure: Two phases

Figure: Three phases
Free boundary approach

→ Formally, we can write the *three phase free boundary problem*

\[
\Delta p = -\Delta \Phi, \text{ in } \{\rho = 1\}, \quad p = S'(1-) \text{ in } \{\rho < 1\} \text{ and } p = S'(1+) \text{ in } \{\rho > 1\},
\]
Free boundary approach

Formally, we can write the **three phase free boundary problem**

\[
\Delta p = -\Delta \Phi, \text{ in } \{\rho = 1\}, \quad p = S'(1-) \text{ in } \{\rho < 1\} \text{ and } p = S'(1+) \text{ in } \{\rho > 1\},
\]

or more in details for our first example as

\[
\left\{
\begin{array}{ll}
\partial_t \rho = \Delta \rho + \nabla \cdot (\nabla \Phi \rho) & \text{in } \{p \rho < 1\}, \\
-\Delta p = \Delta \Phi, & \text{in } \{1 < p \rho < 2\}, \\
\partial_t \rho = 2\Delta \rho + \nabla \cdot (\nabla \Phi \rho) & \text{in } \{p \rho > 2\},
\end{array}
\right.
\]

with boundary conditions

\[
\left\{
\begin{array}{l}
|D(p \rho)^{1+}| - |D(p \rho)^{1-}| = 0 \text{ on } \{p \rho = 1\}. \\
|D(p \rho)^{2+}| - |D(p \rho)^{2-}| = 0 \text{ on } \{p \rho = 2\}
\end{array}
\right.
\]

and

\[
\left\{
\begin{array}{ll}
p = 1 & \text{in } \{p \rho < 1\}, \\
\rho = 1, & \text{in } \{1 < p \rho < 2\}, \\
p = 2 & \text{in } \{p \rho > 2\},
\end{array}
\right.
\]
Open questions #2

Recall the growth of $S$: $S \sim \rho^m$ in $(0, 1)$ and $S \sim \rho^r$ in $(1, +\infty)$.

→ What happens if $m \gg r$?
→ Can we obtain Sobolev estimates?
→ If not, can we observe some singular phenomena as below?

Figure: $t = 0$

Figure: $t = t^* > 0$
Thank you for your attention!