Instability of the soliton for the focusing, mass-critical generalized KdV equation

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The focusing gKdV

Consider

$$u_t = -(u_{xx} + u^5)_x, \qquad u(0,x) = u_0 \in L^2(\mathbb{R}).$$

The sign $-$ makes it focusing.

This equation is called mass-critical because the scaling leaving the equation invariant, i.e.

$$u(t,x) \mapsto \lambda^{\frac{1}{2}} u\left(\lambda^3 t, \lambda x\right)$$

leaves the L^2 norm, or mass, invariant. The mass of a solution, defined by

$$M(u(t)) := \int_{\mathbb{R}} |u(t,x)|^2 dx$$

is conserved.

Ben Dodson proved that in the defocusing case, there does not exist a nonzero, almost periodic solution, which implies scattering for the defocusing equation. The proof uses interaction Morawetz estimates.

Definition 1 (Almost periodic solution)

Suppose v is a strong solution to gKdV on the maximal interval of existence I. Such a solution v is said to be almost periodic (modulo symmetries) if there exist continuous functions $N(t) : I \to (0, \infty)$ and $x(t) : I \to \mathbb{R}$, such that

$$\{v(t,x) = N(t)^{-1/2} u(t, N(t)^{-1} x + x(t)) : t \in I\}$$

is contained in a compact subset of $L^2(\mathbb{R})$.

The soliton

For the focusing gKdV, there exists the soliton $u(t,x) = Q(x-t)$, where

$$Q(x) = \frac{3^{1/4}}{\cosh^{1/2}(2x)} > 0.$$

The function $Q(x)$ solves the elliptic equation

$$Q_{xx} + Q^5 = Q,$$

Therefore, $Q(x-t)$ solves gKdV and $Q(x-t)$ is an almost periodic solution.

In a remarkable series of works, Martel and Merle proved, among many nice results, the instability of the soliton in an H^1 sense, for initial data with mass greater than or equal to the soliton. In fact, they proved something more, that there initial data arbitrarily close to the soliton in H^1-norm, which eventually move away from the soliton in an L^2-sense.

Conjecture:
If $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then the solution is globally well-posed and scattering.

Main result

We show that there are no solutions which are uniformly close to $Q(x)$ in L^2_x modulo symmetries.

Definition 2

If a maximal-lifespan strong solution u to gKdV on I satisfies

$$\sup_{t \in I} \inf_{\lambda_0, x_0} \| u(t,x) - \frac{1}{\lambda_0^{1/2}} Q(\frac{x - x_0}{\lambda_0}) \|_{L^2(\mathbb{R})} \leq \delta \tag{1}$$

then we say u is δ-close to Q.

The main result is

Theorem 3 (G-Dodson)

There exists $\delta > 0$ sufficiently small such that there does not exist a maximal-lifespan solution to gKdV with $\|u_0\|_{L^2} < \|Q\|_{L^2}$ satisfying (1).

We say that u scatters forward/backwards in time if there exists a unique $u_\pm \in L^2_x(\mathbb{R})$ such that

$$\lim_{t\to\pm\infty} \left\| u(t) - e^{-t\partial_x^3} u_\pm \right\|_{L^2_x(\mathbb{R})} = 0.$$

The symmetry group G is defined as

$$G = \{ g_{x_0,\lambda} : L^2_x(\mathbb{R}) \to L^2_x(\mathbb{R}) |\, (x_0, \lambda) \in \mathbb{R}\times(0,\infty),\; g_{x_0,\lambda} f(x) := \lambda^{-\frac{1}{2}} f\left(\lambda^{-1}(x - x_0)\right) \}.$$

For $u : I \times \mathbb{R} \to \mathbb{R}$

$$T_{g_{x_0,\lambda}} u(t,x) := \lambda^{-\frac{1}{2}} u\left(\lambda^{-3} t, \lambda^{-1}(x - x_0)\right).$$

Local theory

The L^2 local well-posedness of gKdV was established by Kenig, Ponce, Vega in 1993:

Theorem 4 (Local well-posedness)

For any $u_0 \in L_x^2(\mathbb{R})$ and $t_0 \in \mathbb{R}$, there exists a unique solution u with $u(t_0) = u_0$ which has maximal lifespan. Let I denote the lifespan of u. Then:

1. *I is an open neighborhood of t_0.*
2. *If $\sup(I)/\inf(I)$ is finite then u blows up forward / backward in time.*
3. *If $\sup(I) = +\infty$ and u does not blow up forward, then u scatters.*
4. *If $M(u_0)$ is small then u is global solution and does not blow up either forward or backward in time.*

Short time stability also holds.

The embedding of NLS into gKdV

A tool is an approximation of solutions to gKdV by certain modulated, rescaled versions of solutions to NLS originating in work of Christ, Colliander, T. Tao, and developed further by Killip, Kwon, Shao, Visan.

$$\tilde{u}_n^T(t,x) := \begin{cases} \operatorname{Re}\left[e^{ix\xi_n\lambda_n + it(\xi_n\lambda_n)^3} V_n\left(3\xi_n\lambda_n t, x + 3(\xi_n\lambda_n)^2 t\right)\right], & \text{when } |t| \leq \frac{T}{3\xi_n\lambda_n} \\ \exp\left\{-\left(t - \frac{T}{3\xi_n\lambda_n}\right)\partial_x^3\right\}\tilde{u}_n\left(\frac{T}{3\xi_n\lambda_n}\right), & \text{when } t > \frac{T}{3\xi_n\lambda_n} \\ \exp\left\{-\left(t + \frac{T}{3\xi_n\lambda_n}\right)\partial_x^3\right\}\tilde{u}_n\left(-\frac{T}{3\xi_n\lambda_n}\right), & \text{when } t < -\frac{T}{3\xi_n\lambda_n} \end{cases}$$

is defined in terms of certain frequency-localized solutions V_n to NLS.

Airy linear profile decomposition of Shao

Let $v_n : \mathbb{R} \to \mathbb{R}$ be a sequence of functions bounded in $L^2_x(\mathbb{R})$. Then, after passing to a subsequence, there exist functions $\phi^j : \mathbb{R} \to \mathbb{C}$ in $L^2_x(\mathbb{R})$, $g^j_n := g_{x^j_n,\lambda^j_n} \in G$, $\xi^j_n \in [0,\infty)$ and $t^j_n \in \mathbb{R}$

$$v_n = \sum_{1\le j\le J} g^j_n e^{-t^j_n\partial_x^3}\mathrm{Re}[e^{ix\xi^j_n\lambda^j_n}\phi^j] + w^J_n, \qquad \forall J \ge 1$$

for some real-valued sequence w^J_n in $L^2_x(\mathbb{R})$ with

$$\lim_{J\to\infty}\limsup_{n\to\infty} \|e^{-t\partial_x^3} w^J_n\|_{L^5_x L^{10}_t(\mathbb{R}\times\mathbb{R})} = 0.$$

One has

$$\|v_n\|^2_{L^2} - \sum_{1\le j\le J} \|\mathrm{Re}[e^{ix\xi^j_n\lambda^j_n}\phi^j]\|^2_{L^2} - \|w^J_n\|^2_{L^2} \xrightarrow{n} 0.$$

The family of sequences $\Gamma^j_n = (\lambda^j_n, \xi^j_n, x^j_n, t^j_n) \in (0,\infty)\times\mathbb{R}^3$ are pair-wise asymptotically orthogonal.

The proof of the theorem splits into two parts.

The first part reduces the study to the existence of almost-periodic solutions.

1. If $u : I \times \mathbb{R} \to \mathbb{R}$ is solution with $\|u_0\|_{L^2} < \|Q\|_{L^2}$ which is δ-close to Q. Then there exists an almost periodic such solution.

   Once we have this reduction, we prove that such solutions cannot exist.

2. There are no almost periodic solutions with mass less than Q which are δ-close to Q.

The proof of the first part consists of the following Palais-Smale -type proposition, inspired from Killip, Kwon, Shao, Visan, which is used to extract subsequences convergent in L^2.

Lemma 5

Let $u_n : I_n \times \mathbb{R} \to \mathbb{R}$ be solutions δ-close to Q, i.e. for some continuous $g_n : I_n \to G$ one has

$$\|g_n(t)u_n(t) - Q\|_{L^2} \leq \delta \qquad \forall\ t \in I_n,\ \ n \geq 1. \tag{2}$$

Suppose $M(u_n) \searrow m_0 =$ and let $t_n \in I_n$ be a sequence of times. Then the sequence $g_n(t_n)u_n(t_n)$ has a subsequence which converges in L^2 to a function ϕ with $M(\phi) = m_0$.

Here m_0 is the infimum of the masses of solutions δ -close to Q.
We apply this twice.

Step 1 - Decomposing the sequence

Assume all $t_n = 0$, $g_n(0)$ are the identity.

By Banach-Alaoglu, we obtain a function $\phi^1 \in L^2$ such that

$$u_n(0) \rightharpoonup \phi^1 \text{ weakly in } L^2.$$

Assuming $\|\phi^1\|_{L^2}^2 < m_0$ and we will obtain a contradiction.

Using the profile decomposition, write for any $J \geq 2$

$$u_n(0) - \phi^1 = \sum_{2 \leq j \leq J} g_n^j e^{-t_n^j \partial_x^3} \mathrm{Re}[e^{ix\xi_n^j \lambda_n^j} \phi^j] + w_n^J.$$

The terms $j \geq 2$ satisfy a smallness condition.

Step 2 - Construct nonlinear profiles

Let $v^1 : I \times \mathbb{R} \to \mathbb{R}$ be the solution with data $v^1(0) = \phi$.

A) For $j \in \overline{2, J_0}$ one has $\xi_n^j \equiv 0$. Can reduce to either:

- When $t_n^j \equiv 0$, let v^j be the solution with $v^j(0) = \mathrm{Re}\phi^j$.
- If $t_n^j \to \pm\infty$, let v^j be the solution which scatters to $e^{-t\partial_x^3}\mathrm{Re}\phi^j$.

B) For $j \in \overline{J_0 + 1, J}$ $\xi_n^j \lambda_n^j \to \infty$. Take the solution with data

$$\tilde{v}_n^j(t_n^j) = e^{-t_n^j\partial_x^3}\mathrm{Re}[e^{ix\xi_n^j\lambda_n^j}\phi^j]$$

For both A) and B) obtain nonlinear profiles

$$v_n^j(t) := T_{g_n^j}[v^j(\cdot + t_n^j)](t), \qquad j \in \overline{2, J_0},\ n \geq 1,$$

the decoupling property holds

$$\lim_{n\to\infty} \|v_n^j v_n^k\|_{L_x^{\frac{5}{2}} L_t^5(I\times\mathbb{R})} = 0 \qquad\qquad \forall\ 1 \leq j < k$$

Step 3 - Construct approximate solutions and bound the difference

Construct the approximate solution and the remainders r_n^J by

$$\tilde{u}_n^J(t) := v^1(t) + \sum_{j=2}^{J} v_n^j(t) + e^{-t\partial_x^3} w_n^J.$$

$$u_n(t) = \tilde{u}_n^J(t) + r_n^J(t).$$

Divide $[0, t]$ into small intervals $[t_k, t_{k+1}]$ and by induction prove

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|r_n^J(t)\|_{L^2} = 0.$$

This follows from stability if one checks:

$$\limsup_{n\to\infty} \|\tilde{u}_n^J\|_{L_x^5 L_t^{10}([t_k,t_{k+1}]\times\mathbb{R})} \leq \frac{\varepsilon_0}{2}$$

$$\lim_{J\to\infty} \limsup_{n\to\infty} \| \, |\partial_x|^{-1} \left[(\partial_t + \partial_x^3)\tilde{u}_n^J - \partial_x(\tilde{u}_n^J)^5\right] \|_{L_x^1 L_t^2([t_k,t_{k+1}]\times\mathbb{R})} = 0.$$

Step 4 - $v_n^j(t)$ converges weakly to 0

A) $\xi_n^j \equiv 0$. Then $v_n^j(t) = g_n^j v^j\left(t_n^j + \frac{t}{(\lambda_n^j)^3}\right)$. By passing to a subsequence, assume

$$t_n^j + \frac{t}{(\lambda_n^j)^3} \to T_j \in [-\infty, \infty].$$

If T_j the claim reduces to $g_n^j v^j(T_j) \rightharpoonup 0$. If $T_j \to \pm\infty$ we use scattering and apply the dispersive estimate.

B) $\xi_n^j \lambda_n^j \to \infty$. Using the approximation involving NLS solutions (in Killip, Kwon, Shao, Visan), one can reduce the claim to

$$e^{\pm i\theta_n} g_{z_n,\lambda_n^j} e^{-s_n\partial_x^3}[e^{\pm ix\xi_n\lambda_n} W(T_1)] \rightharpoonup 0,$$

These limits are proved in Shao's paper.

From A) and B) we conclude $v_n^j(t) \rightharpoonup 0, \qquad \forall\ t \in \mathbb{R},\ j \geq 2.$

Step 5 - Prove the first profile is δ -close to Q

$$u_n(t) = v^1(t) + \sum_{j=2}^{J} v_n^j(t) + e^{-t\partial_x^3} w_n^J + r_n^J(t).$$

Expand

$$\delta^2 \geq \|u_n(t) - g_n(t)^{-1}Q\|_{L^2}^2 = \|v^1(t) - g_n(t)^{-1}Q\|_{L^2}^2 + A_n^J(t) + B_n^J(t)$$

$$A_n^J(t) := \|\sum_{j=2}^{J} v_n^j(t) + e^{-t\partial_x^3} w_n^J + r_n^J(t)\|_{L^2}^2$$

$$B_n^J(t) := 2\langle v^1(t) - g_n(t)^{-1}Q \ , \sum_{j=2}^{J} v_n^j(t) + e^{-t\partial_x^3} w_n^J + r_n^J(t)\rangle.$$

Extract a subsequence such that $g_n(t)$ converges to some $g(t)$.
Note $A_n^J(t) \geq 0$ and $\lim_{J\to\infty} \limsup_{n\to\infty} B_n^J(t) = 0$. In the limit

$$\|g(t)v^1(t) - Q\|_{L^2} \leq \delta \qquad \forall t \in I.$$

This means v^1 is $\delta-$close to Q with $M(v^1) < m_0$, a contradiction.

The main result in now reduced to the case of almost periodic solutions. Now combine ideas from the following results:

- B. Dodson - defocusing case. Scattering reduced to 3 scenarios: a self-similar solution, a double rapid cascade solution, and a quasisoliton solution.
- Merle (energy space)
- Martel-Merle (Liouville theorem for gKdV).
- Martel-Merle (instability of soliton)
- Martel-Merle (blow-up).

Rely on Morawetz arguments from these papers.

Step 6

Begin with studying $N(t)$. Show can be reduced to $N(t) \geq 1$ and

$$\int_I N(t)^2 dt = \infty.$$

1) Scenario $N(t) \sim 1$ for any $t \in [0, \infty)$. Taking

$$u(t_n, x - x(t_n)) \to u_0$$

generates a solution with

$$\{u(t, x - x(t)) : t \in \mathbb{R}\} \text{ is precompact.} \tag{3}$$

2) Scenario

$$\lim_{T \to \infty} \inf_{t \in [0,T]} N(t) = 0.$$

2A)

$$\limsup_{T\to\sup(I)} \frac{\sup_{t\in[t_0(T),T]} N(t)}{N(t_0(T))} < \infty$$

where

$$t_0(T) = \inf\left\{t \in [0,T] : N(t) = \inf_{t\in[0,T]} N(t)\right.$$

Can choose a sequence such that

$$2^{k/2} u(t_k', 2^k(x - x(t_k'))) \to u_0, \qquad \text{in} \qquad L^2(\mathbb{R}),$$

generates a solution satisfying (3).

2B)

$$\limsup_{T\to\sup(I)} \frac{\sup_{t\in[t_0(T),T]} N(t)}{N(t_0(T))} = \infty.$$

This case is handled by Morawetz estimates and a partition of intervals.

Step 7 - Decomposition near a soliton

Recall

$$\|u - \lambda_0(t)^{-1/2} Q(\frac{x - x_0(t)}{\lambda_0(t)})\|_{L^2} < 2\delta,$$

Lemma 6

There exist $x(t)$ and $\lambda(t)$ such that

$$\epsilon(t, y) := \lambda(t)^{1/2} u(t, \lambda(t) y + x(t)) - Q(y)$$

satisfies

$$(yQ_y, \epsilon) = (y(\frac{Q}{2} + yQ_y), \epsilon) = 0. \tag{4}$$

Moreover,

$$|\frac{\lambda_0(t)}{\lambda(t)} - 1| + |\frac{x_0(t) - x(t)}{\lambda(t)}| + \|\epsilon\|_{L^2} \lesssim \delta. \tag{5}$$

This lemma was proved when u was close to Q in H^1 norm by Martel-Merle. It is based on the implicit function theorem.

Step 8 - Exponential decay

Lemma 7 (Exponential decay to the left of the soliton)

$$\|u(t, x + x(t))\|^2_{L^2(x \leq -x_0)} \leq 10 c_1 e^{-\frac{x_0}{6}}, \qquad x_0 \gg 1.$$

The proof is based on the fact that u is close to a soliton, and the soliton moves to the right while a dispersive solution moves to the left.

Idea: consider

$$I(t) = \int u(t,x)^2 \psi(x - \tilde{x}(0) + x_0 - \frac{1}{4}(\tilde{x}(t) - \tilde{x}(0)))dx.$$

and show

$$I'(t) \leq C e^{\frac{-x_0}{K}} e^{-\frac{3}{4K}(\tilde{x}(t) - \tilde{x}(0))} \dot{\tilde{x}}(t) \int \lambda(t)^2 u(t,x)^6 dx.$$

This will lead to a contradiction.

Lemma 8 (Exponential decay to the right of the soliton)

$$\|u(t, x + x(t))\|^2_{L^2(x \geq x_0)} \leq 10 c_1 e^{-\frac{x_0}{6}}.$$

As a consequence, computing with the functional

$$M(t) = \int \chi(\frac{x}{x_0}) u(t, x + x(0))^2 dx,$$

and with $\frac{d}{dt}M(t)$ one will obtain that there does not exist a solution satisfying

$$\int_0^{\sup(I)} N(t)^2 dt < \infty, \qquad \int_{\inf(I)}^0 N(t)^2 dt = \infty,$$

or vice-versa.

Step 9 - Virial identities

Working with

$$J(s) = \lambda(s)^{1/2} \int \epsilon(s,x) \int_{-\infty}^{x} (\frac{Q}{2} + zQ_z) dz dx - \lambda(s)^{1/2} \kappa$$

and with

$$\frac{d}{ds} J(s) = 2\lambda(s)^{1/2} \int Q(y)\epsilon(s,y) dy + O(\lambda(s)\|\epsilon\|_{L^2}^2) + O(\lambda(s)\|\epsilon\|_{L^2}\|\epsilon\|_{L^8}^4)$$

one proves

$$|\int_0^T \lambda(s)^{1/2} \int \epsilon(s,x) Q(x) dx ds| \lesssim C(u) + \int_0^T \lambda(s)^{1/2} \|\epsilon(s)\|_{L^2}^2 ds.$$

To finish the proof, one uses a second virial identity using

$$M(s) = \frac{1}{2}\lambda(s) \int y\epsilon(s,y)^2 dy.$$

After computations one obtains

$$\frac{\delta_1}{8}\int_0^T \lambda(s)\|\epsilon\|_{H^1}^2 ds \lesssim \frac{MK}{R\delta_1}\int_0^T \lambda(s)^{1/2}\|\epsilon\|_{L^2}^2 ds + C(u) + \frac{MK}{R\delta_1}C(u).$$

If $\lambda(s) = 1$ one obtains a contradiction as in Martel-Merle.

In the general case, the proof will make use of the fact that $\lambda(s) \leq 1$ along with the fact that conservation of energy gives a lower bound on $\lambda(s)$.

Thank you for listening!