Boundary behaviour of nonlocal minimal surfaces

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Geometric and functional inequalities and recent topics in nonlinear PDEs
Nonlocal minimal surfaces

Energy functional dealing with “pointwise interactions” between a given set and its complement

Main idea: the “surface tension” is the byproduct of long-range interactions

Implications: nonlocal phase transitions and nonlocal capillarity theories

New effects due to the long-range interactions

Contributions from “far-away” can have a significant influence on the local structures of these new objects

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The fractional perimeter functional

Given $s \in (0, 1)$ and a bounded open set $\Omega \subset \mathbb{R}^n$ with $C^{1,\gamma}$-boundary, the $s$-perimeter of a (measurable) set $E \subseteq \mathbb{R}^n$ in $\Omega$ is defined as

$$\text{Per}_s(E; \Omega) := L(E \cap \Omega, (CE) \cap \Omega)$$
$$+ L(E \cap \Omega, (CE) \cap (C\Omega)) + L(E \cap (C\Omega), (CE) \cap \Omega),$$

where $CE = \mathbb{R}^n \setminus E$ denotes the complement of $E$, and $L(A, B)$ denotes the following nonlocal interaction term

$$L(A, B) := \int_A \int_B \frac{1}{|x - y|^{n+s}} \, dx \, dy \quad \forall A, B \subseteq \mathbb{R}^n,$$

This notion of $s$-perimeter and the corresponding minimization problem were introduced in [Caffarelli-Roquejoffre-Savin, 2010].
1) **Existence theorem:**
there exists \( E \) s-minimizer for \( \text{Per}_s \) in \( \Omega \) with \( E \setminus \Omega = E_0 \setminus \Omega \).

2) **Maximum principle:**
\( E \) s-minimizer and \((\partial E) \setminus \Omega \subset \{|x_n| \leq a\} \Rightarrow \partial E \subset \{|x_n| \leq a\} \).

3) If \( \partial E \) is an hyperplane, then \( E \) is s-minimizer.

4) If \( E \) is s-minimizer in \( B_1 \), then \( \partial E \) is \( C^{1,\alpha} \) in \( B_{1/2} \) except in a closed set \( \Sigma \), with Hausdorff dimension less or equal than \( n - 2 \).

5) If \( E \) is s-minimizer and \( 0 \in \partial E \), then

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\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{E^c}(y)}{|y|^{a+3}} dy = 0.
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Regularity in dimension 2

[Savin-Valdinoci, 2013]:

Regularity of cones in dimension 2.

If $E$ is $s$-minimizer in $B_1$, then $\partial E$ is $C^{1,\alpha}$ in $B_{1/2}$ except in a closed set $\Sigma$, with Hausdorff dimension less or equal than $n - 3$. 
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Limit as $s \to 1$


$$(1 - s)\text{Per}_s \to \text{Per}, \quad \text{as } s \nearrow 1$$

(up to normalizing multiplicative constants).

\[\sqsubseteq\]

[Caffarelli-Valdinoci, 2013]:
$s$ close to 1: nonlocal minimal surfaces are as regular as classical minimal surfaces.

(If $E$ is $s$-minimizer in $B_1$, then $\partial E$ is $C^{1,\alpha}$ in $B_{1/2}$ except in a closed set $\Sigma$, with Hausdorff dimension less or equal than $n - 8$.)
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Limit as $s \to 0$

[Maz’ya-Shaposhnikova, 2002] and [Dipierro-Figalli-Palatucci-Valdinoci, 2013]: If there exists the limit

$$\alpha(E) := \lim_{s \searrow 0} s \int_{E \cap (C B_1)} \frac{1}{|y|^{n+s}} \, dy,$$

then

$$\lim_{s \searrow 0} s \operatorname{Per}_s(E, \Omega) = (\omega_{n-1} - \alpha(E)) \frac{|E \cap \Omega|}{\omega_{n-1}} + \alpha(E) \frac{|\Omega \setminus E|}{\omega_{n-1}}.$$
Stickiness to half-balls

For any $\delta > 0$,

$$K_\delta := (B_{1+\delta} \setminus B_1) \cap \{x_n < 0\}.$$

We define $E_\delta$ to be the set minimizing the $s$-perimeter among all the sets $E$ such that $E \setminus B_1 = K_\delta$. 
There exists $\delta_o > 0$ such that for any $\delta \in (0, \delta_o]$ we have that

$$E_\delta = K_\delta.$$
Given a large $M > 1$ we consider the $s$-minimal set $E_M$ in $(-1, 1) \times \mathbb{R}$ with datum outside $(-1, 1) \times \mathbb{R}$ given by the jump $J_M := J_M^- \cup J_M^+$, where

$$J_M^- := (-\infty, -1] \times (-\infty, -M)$$

and

$$J_M^+ := [1, +\infty) \times (-\infty, M).$$
There exist $M_o > 0$ and $C_o \geq C'_o > 0$, depending on $s$, such that if $M \geq M_o$ then

$$[-1, 1) \times \left[ C_o M^{\frac{1+s}{2+s}}, M \right] \subseteq E_M^c$$

and

$$(-1, 1] \times \left[ -M, -C_o M^{\frac{1+s}{2+s}} \right] \subseteq E_M.$$

Also, the exponent $\beta := \frac{1+s}{2+s}$ above is optimal.
Stickiness to the sides of a box
Stickiness as $s \to 0^+$

We consider a sector in $\mathbb{R}^2$ outside $B_1$, i.e.

$$\Sigma := \{(x, y) \in \mathbb{R}^2 \setminus B_1 \text{ s.t. } x > 0 \text{ and } y > 0\}.$$ 

Let $E_s$ be the $s$-minimizer of the $s$-perimeter among all the sets $E$ such that $E \setminus B_1 = \Sigma$.

Then, there exists $s_o > 0$ such that for any $s \in (0, s_o]$ we have that $E_s = \Sigma$. 
Stickiness as $s \rightarrow 0^+$
Instability of the flat fractional minimal surfaces

Fix $\epsilon_0 > 0$ arbitrarily small. Then, there exists $\delta_0 > 0$, possibly depending on $\epsilon_0$, such that for any $\delta \in (0, \delta_0]$ the following statement holds true.

Assume that $F \supset H \cup F_- \cup F_+$, where

$$H := \mathbb{R} \times (-\infty, 0),$$

$$F_- := (-3, -2) \times [0, \delta)$$

and

$$F_+ := (2, 3) \times [0, \delta).$$

Let $E$ be the $s$-minimal set in $(-1, 1) \times \mathbb{R}$ among all the sets that coincide with $F$ outside $(-1, 1) \times \mathbb{R}$.

Then

$$E \supset (-1, 1) \times (-\infty, \delta^{\frac{2+\epsilon_0}{1-s}}].$$
Instability of the flat fractional minimal surfaces

\[ \beta := \frac{2 + \epsilon_0}{1 - s} \]
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Three further questions
[Dipierro-Savin-Valdinoci, 2020]

1. How regular are the nonlocal minimal surfaces *coming from inside the domain*?

2. Is the Euler-Lagrange equation satisfied *up to the boundary*?

3. How *typical* is the stickiness phenomenon?
Three further questions
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Three further questions
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3. How typical is the stickiness phenomenon?
“Continuity implies differentiability”

Consider a nonlocal minimal graph in \((0, 1)\), with a smooth external graph \(u_0\).

There is a dichotomy:

- either
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  \lim_{x \searrow 0} u_0(x) \neq \lim_{x \searrow 0} u(x)
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  and
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  \lim_{x \searrow 0} |u'(x)| = +\infty,
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- or
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and \(u\) is \(C^{1, \frac{1+s}{2}}\) at 0.
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Regularity coming from inside

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Some remarks

This dichotomy is a purely nonlinear effect, since the boundary behavior of linear equation is of Hölder type [Serra-Ros Oton].
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Stickiness + dichotomy = butterfly effect

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As a curve, the nonlocal minimal graph turns out to be always $C^{1, \frac{1+s}{2}}$:

it is either the graph of a $C^{1, \frac{1+s}{2}}$-function (when it is continuous at the boundary!), or it is discontinuous and sticks vertically detaching in a $C^{1, \frac{1+s}{2}}$ fashion [Caffarelli-De Silva-Savin] (then the inverse function is a $C^{1, \frac{1+s}{2}}$ function).
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The nonlocal mean curvature can be written in the form

$$\int_{-\infty}^{+\infty} F \left( \frac{u(x+y) - u(x)}{|y|} \right) \frac{dy}{|y|^{1+s}}.$$ 

And this is a “$C^{1,s}$ operator”.

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And this is a “\(C^{1,s}\) operator”.

But \( \frac{1+s}{2} > s \), therefore we can “pass the equation to the limit”...
If $u$ is a nonlocal minimal graph in $(0, 1)$ with smooth datum outside, then

$$\int_{-\infty}^{+\infty} F \left( \frac{u(x+y) - u(x)}{|y|} \right) \frac{dy}{|y|^{1+s}} = 0$$

for all $x \in [0, 1]$. 
Stickiness is generic

Let $\varphi \in C^\infty_0([-2, -1], [0, 1])$, with $\varphi \not\equiv 0$.

Let $u^{(t)}$ be the nonlocal minimal graph in $(0, 1)$ with external datum

$$u_0^{(t)} := u_0 + t\varphi.$$ 

Suppose that

$$\lim_{x \to 0} u_0(x) = \lim_{x \not\to 0} u(x).$$

Then

$$\lim_{x \to 0} u_0^{(t)}(x) < \lim_{x \not\to 0} u^{(t)}(x).$$
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With the Euler-Lagrange equation up to the boundary, we can take any configuration, add an arbitrarily small bump and use the unperturbed configuration as a barrier.

At touching points the additional bump produces an extra-mass violating the Euler-Lagrange equation.

Notice that now also touching at the boundary can be taken into account!
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Proof of dichotomy

Think about the usual suspects (discontinuous, Lipschitz, Hölder, smooth).

Blow-up.

The “worst” cases to understand are the Hölder and the smooth (the Lipschitz produces non-minimal corners).

The smooth case produces flat objects: use a boundary improvement of flatness (combined with a boundary monotonicity formula) to deduce smoothness of the initial minimizer (for this, use new barrier to go beyond the linear theory!).

The Hölder case produces vertical angles: rule them out by proving that close-to-vertical nonlocal minimal graphs are indeed vertical (for this, slide balls).
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We consider nonlocal minimal surfaces in a cylinder with prescribed datum given by the complement of a slab.

\[ \Omega := \{(x', x_n) \text{ with } |x'| < 1\}. \]

\[ E_0 := \{(x', x_n) \text{ with } |x'| > M\}. \]
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Click for video
As in the classical case, when the width of the slab is large the minimizers are disconnected and when the width of the slab is small the minimizers are connected.

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Differently from the classical case, the minimizer contains

$$B_{cM^{-s}}(0, \ldots, 0, -M) \cup B_{cM^{-s}}(0, \ldots, 0, M),$$

so it is not the complement of a slab. Also (at least in dimension 2) it sticks at the boundary.
(Dis)connectedness of nonlocal minimal surfaces

[DiPerro-Onoue-Valdinoci, 2020]

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Yin-Yang Theorems

...con'è difficile trovare l'alba dentro l'imbrunire...
There exists $\vartheta > 1$ such that if $E$ is $s$-minimal in $\Omega \subset \mathbb{R}^n$ and $E \cap (\Omega_{\vartheta \text{diam}(\Omega)} \setminus \Omega) = \emptyset$, then

$$E \cap \Omega = \emptyset.$$
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Stickiness in dimension 3
[Dipierro-Savin-Valdinoci, 2020]

Let $u$ be $s$-minimal in $(-1, 1) \times (0, 1) \times \mathbb{R}$ with $u = 0$
in $(-2, 2) \times (-\frac{1}{100}, 0)$.

Consider the trace of $u$

$$f(x) := \lim_{y \searrow 0} u(x, y).$$

Assume that $f(0) = 0$. Then, near the origin,

$$|u(x, y)| \leq C (x^2 + y^2)^{\frac{1+s}{4}}.$$

In particular, $f'(0) = 0$.  


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[Almeida-Castro, 2020]
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Vanishing of the gradient of the trace at the zero crossing points
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On the one hand, boundary points which attain the flat exterior datum in a continuous way have necessarily horizontal tangency.

On the other hand, boundary points with a jump have necessarily a vertical tangency.

Consequently, points with vertical tangency accumulate to zero crossing points possessing horizontal tangency, preventing a differentiable boundary regularity in a neighborhood of horizontal points!
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Pivotal step: if a homogeneous nonlocal minimal graph in \( \{ x > 0 \} \) vanishes in \( \{ x < 0 \} \) and is continuous at the origin, then it necessarily vanishes at all points:

Let \( u : \mathbb{R}^2 \to \mathbb{R} \) be an \( s \)-minimal graph in \( \{ x > 0 \} \), with \( u = 0 \) in \( \{ x < 0 \} \).

Assume also that \( u \) is positively homogeneous of degree 1, i.e. \( u(tX) = tu(X) \) for all \( X \in \mathbb{R}^2 \) and \( t > 0 \). Suppose that

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\lim_{x \searrow 0} u(x, y) = 0.
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Then \( u \equiv 0 \).
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Open problems [Dipierro-Savin-Valdinoci, 2020]

What happens in dimension $n \geq 4$?

(Dimension 3 was “easier” because the trace is a function from $\mathbb{R}$ to $\mathbb{R}$, so knowing the derivative at a point, together with the 1-homogeneity, determines already half of the trace; in dimension 4 this only determines the trace along a half-line).
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Is it possible to construct examples of nonlocal minimal graphs which are locally flat from outside and whose trace develops vertical tangencies?
What is the behavior of a nonlocal minimal graph and of its trace at the corners of the domain and in their vicinity?

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Nicholas of Cusa
If full knowledge about the very base of our existence could be described as a circle, the best we can do is to arrive at a polygon.
How “nonlinear” is the problem?

The linearization of the trace of a nonlocal minimal graph is given by the fractional normal derivative of a fractional Laplace problem.

Indeed, if $u$ is a nonlocal minimal graph, say in $x \in (0, 1)$, and it is $\varepsilon$-flat near the origin, then $\frac{u}{\varepsilon}$ (the “vertical rescaling”) tends to a function $\bar{u}$ which is a solution of $(-\Delta)^{\frac{1+s}{2}} \bar{u}(x) = 0$ for $x \in (0, 1)$.

By the boundary regularity of linear equation (Serra, Ros-Oton, Grubb, etc.) the first order of $\bar{u}$ is of Hölder type: near the origin $\bar{u} \simeq ax^{\frac{1+s}{2}}$, for some $a \in \mathbb{R}$.

So, one may expect that, near the origin, $u(x) \simeq a\varepsilon x^{\frac{1+s}{2}}$.

But since $|u(x, 0)| \leq Cx^{\frac{1+s}{2}}$, one is tempted to guess that necessarily $a = 0$. 
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Boundary behaviour of nonlocal minimal surfaces

S. Dipierro

Introduction

Limits

Stickiness phenomenon
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But this is **not the case**! The fractional normal derivative of a fractional Laplace problem is not only different than zero in general, but it can be **arbitrarily prescribed**:

Let $n \geq 2$ and $f \in C(\mathbb{R}^{n-1})$. Then, for every $\delta > 0$ there exist $f_\delta$, $u_\delta \in C(\mathbb{R}^{n-1})$ such that

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\begin{align*}
\sup_{|x'| \leq 1} |f_\delta(x') - f(x')| &\leq \delta, \\
(-\Delta)^{\sigma} u_\delta &\equiv 0 \text{ in } B_1 \cap \{x_n > 0\}, \\
u_\delta &\equiv 0 \text{ in } \{x_n < 0\}, \\
\lim_{x_n \searrow 0} \frac{u_\delta(x)}{x_n^{n-1}} &\equiv f_\delta(x') \text{ for all } |x'| < 1.
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[Dipierro-Savin-Valdinoci, 2020]

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and nonlocal minimal surfaces are precisely one of such cases (in which the nonlinearity is the outcome of a complex and nonlocal geometric problem)!
Thank you very much for your attention!