From the Peierls-Nabarro model to the equation of motion of the dislocation continuum

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- S. Patrizi and T. Sangsawang, From the Peierls-Nabarro model to the equation of motion of the dislocation continuum, *Nonlinear Analysis*, **202** (2021).

Main problem

We study the limit as $\epsilon \to 0$ of the solution u^ϵ of the following fractional reaction-diffusion PDE:

$$\begin{cases} \delta \partial_t u^\epsilon = -(-\Delta)^{\frac{1}{2}} u^\epsilon - \frac{1}{\delta} W'\left(\frac{u^\epsilon}{\epsilon}\right) & \text{in } \mathbb{R}^+ \times \mathbb{R} \\ u^\epsilon(0,\cdot) = u_0(\cdot) & \text{on } \mathbb{R} \end{cases} \tag{1}$$

where $\epsilon, \delta > 0$ are small scale parameters and $\delta = \delta_\epsilon \to 0$ as $\epsilon \to 0$, W is a multi-well potential with nondegenerate minima at integer points and u_0 is non-decreasing.

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where ϵ, $\delta > 0$ are small scale parameters and $\delta = \delta_\epsilon \to 0$ as $\epsilon \to 0$, W is a multi-well potential with nondegenerate minima at integer points and u_0 is non-decreasing.

- If $\epsilon = 1$, (1) is a fractional Allen-Cahn problem (González-Monneau);
- If $\delta = 1$, (1) is a homogenization problem (Monneau-P.);
- We do not assume any assumption about how δ goes to 0 when $\epsilon \to 0$.

Allen-Cahn equations

- Classical Allen-Cahn equation (Chen): for $n \geq 2$,

$$\partial_t u^\delta = \Delta u^\delta - \frac{1}{\delta} W'(u^\delta) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^n$$

  with a suitable initial condition, $u^\delta(0,x) = u_0(x)$, $0 < u_0 < 1$, where W is a double well potential with minima at 0 and 1.
- n=1, works by Fife and co.
- The stationary case previously studied by Modica and Mortola.

Fractional Allen-Cahn equations

- When Δ is replaced by $-(-\Delta)^s u$, $s \in (0,1)$, the motion of forming interphases in dimension $n \geq 2$ studied by Imbert, Souganidis;
- Stationary case, $n \geq 2$: Savin, Valdinoci (non-local version of Modica-Mortola);
- In dimension 1, Gonzalez and Monneau studied

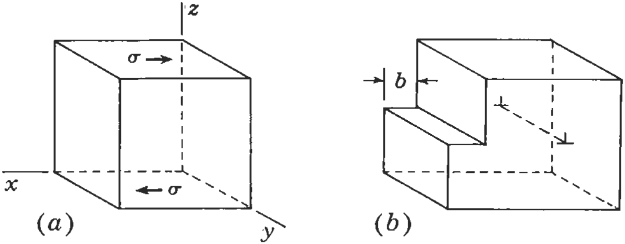
$$\delta \partial_t v^\delta = -(-\Delta)^{\frac{1}{2}} v^\delta - \frac{1}{\delta} W'(v^\delta) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}$$

with a well-prepared initial condition. Here W is a multi-well potential.

Dislocations

Dislocations are defect lines in crystalline solids whose motion is directly responsible for the plastic deformation of these materials. Their typical length is of order of $10^{-6}m$ with thickness of order of $10^{-9}m$.

Geometry of an edge dislocation



Dislocations can be described at several scales by different models:

1. atomic scale (Frenkel-Kontorova model)
2. microscopic scale (Peierls-Nabarro model)
3. mesoscopic scale (Discrete dislocation dynamics)
4. macroscopic scale (elasto-visco-plasticity with density of dislocations)

The Peierls-Nabarro model

We consider a straight dislocation line parallel to e_3.

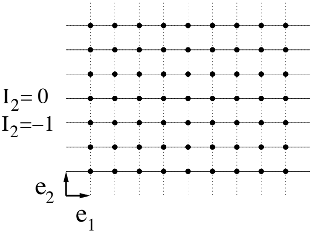


Figure 1: Perfect crystal

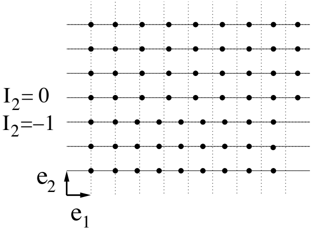


Figure 2: Schematic view of a edge dislocation in the crystal

Assumptions

- the dislocation defects are described by the mismatch between the two planes $I_2 = 0$ and $I_2 = -1$
- the displacement of the crystal is antysimmetric wrt the plane $e_1 e_3$
- any atoms move only in the direction e_1
- the displacement is independent of e_3

The Peierls-Nabarro model

The P-N model is a *continuous* model where a dislocation is described by means of a scalar phase field defined over the slip plane.

The medium will be $\mathbb{R}^2$, endowed with coordinates (x, y).

The disregistry of the upper half crystal $\{y > 0\}$ relative to the lower half $\{y < 0\}$ is given by $\phi(x)$, which is a transition between 0 and 1:

$$\begin{cases} \phi(-\infty) = 0, \quad \phi(+\infty) = 1 \\ \phi' > 0. \end{cases}$$

The Peierls-Nabarro model

The total energy is given by

$$\mathcal{E} = \underbrace{\frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}^+} |\nabla U(x,y)|^2 dxdy}_{\text{elastic energy}} + \underbrace{\int_{\mathbb{R}} W(U(x,0))dx}_{\text{misfit energy}}$$

where $U : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ represents (twice) the (scalar) displacement and it is such that

$$U(x,0) = \phi(x).$$

The potential W satisfies

- $W(u+1) = W(u) \quad \forall u \in \mathbb{R}$ (periodicity)
- $W(\mathbb{Z}) = 0 < W(u) \quad \forall u \in \mathbb{R} \setminus \mathbb{Z}$ (minimum property)

The Peierls-Nabarro model

A critical point of the energy satisfies

$$\begin{cases} \Delta U(x,y) = 0 & (x,y) \in \mathbb{R} \times \mathbb{R}^+ \\ \partial_y U(x,0) = W'(U(x,0)) & x \in \mathbb{R} \end{cases}$$

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The system can be rewritten for

$$\phi(x) = U(x,0)$$

as follows

$$-(-\Delta)^{\frac{1}{2}}\phi = W'(\phi) \quad \text{in } \mathbb{R}$$

where

$$(-\Delta)^{\frac{1}{2}} v = \mathcal{F}^{-1}(|\xi|\mathcal{F}(v)) \quad \text{for any } v \in S(\mathbb{R}^n)$$

and $\mathcal{F}$ is the Fourier transform. If $v \in C^{1,1}_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $n = 1$,

$$-(-\Delta)^{\frac{1}{2}} v = PV \frac{1}{\pi} \int_{\mathbb{R}} \frac{v(y) - v(x)}{(y-x)^2}\, dy$$

The Peierls-Nabarro model

The phase transition ϕ (also called layer solution) therefore satisfies

$$\begin{cases} -(-\Delta)^{\frac{1}{2}}\phi = W'(\phi) \quad \text{in } \mathbb{R} \\ \phi' > 0 \\ \phi(-\infty) = 0, \quad \phi(+\infty) = 1, \quad \phi(0) = \frac{1}{2} \end{cases}$$

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In the original PN model:

$$W(u) = \frac{1}{4\pi^2}(1 - \cos(2\pi u))$$

and

$$\phi(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(2x)$$

The Peierls-Nabarro model

$$\begin{cases} -(-\Delta)^{\frac{1}{2}}\phi = W'(\phi) & \text{in } \mathbb{R} \\ \phi' > 0 \\ \phi(-\infty) = 0, \quad \phi(+\infty) = 1, \quad \phi(0) = \frac{1}{2} \end{cases}$$

- Existence, uniqueness by Cabré, Sòla-Morales. Asymptotic estimates by González, Monneau;
- When $-(-\Delta)^{\frac{1}{2}}$ is replaced by $-(-\Delta)^s$, $s \in (0,1)$, existence, uniqueness and asymptotic estimates are proven in as series of paper by Cabré, Sire, Dipierro, Figalli, Palatucci, Savin, Valdinoci.

Evolutive PN-model

Suppose that there are N *straight edge dislocations lines all lying in the same plane*:

After a cross section:

Evolutive PN-model

The dynamics for an ensemble of N straight dislocations lines with the same Burgers' vector and all contained in a single slip plane, moving with self-interactions (no exterior forces) is described by the evolutive version of the Peierls-Nabarro model:

$$\partial_t u = -(-\Delta)^{\frac{1}{2}} u - W'(u) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}.$$

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$$\partial_t u = -(-\Delta)^{\frac{1}{2}} u - W'(u) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}.$$

with the following initial condition

$$u(0,x) = \sum_{i=1}^{N} \phi\left(x - \frac{y_i^0}{\delta}\right),$$

where ϕ is the transition layer introduced before and $0 \leq y_{i+1}^0 - y_i^0 \simeq 1$.

Consider the following rescaling

$$v^\delta(t,x) = u\left(\frac{t}{\delta^2}, \frac{x}{\delta}\right)$$

Then, v^δ is solution of the fraction fractional Allen-Cahn type equation:

$$\delta \partial_t v^\delta = -(-\Delta)^{\frac{1}{2}} v - \frac{1}{\delta} W'(v) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}$$

associated to the *well-prepared* initial condition:

$$v^\delta(0,x) = \sum_{i=1}^{N} \phi\left(\frac{x - y_i^0}{\delta}\right).$$

González and Monneau proved that the solution v^δ converges, as $\delta \to 0$ to the stable minima of W, i.e. integers. More precisely,

$$v^\delta(t,x) \to \sum_{i=1}^{N} H(x - y_i(t)),$$

where H is the Heaviside function and the interface points $y_i(t)$, $i = 1, \dots, N$ evolve in time driven by the following system of ODE's:

$$\begin{cases} \dot{y}_i = \dfrac{c_0}{\pi} \displaystyle\sum_{j \neq i} \dfrac{1}{y_i - y_j} & \text{in } (0, +\infty) \\ y_i(0) = y_i^0, \end{cases} \tag{2}$$

where $c_0 = \left(\int_{\mathbb{R}} (\phi')^2\right)^{-1}$. System (2) corresponds to the classical discrete dislocation dynamics (DDD).

In our paper we consider the case $N \to +\infty$. Precisely,

$$N = N_\epsilon \simeq \frac{1}{\epsilon}.$$

that is

$$\partial_t u = -(-\Delta)^{\frac{1}{2}} u - W'(u) \quad \text{in } \mathbb{R} \times \mathbb{R}^+,$$

$$u(0, x) = \sum_{i=1}^{N_\epsilon} \phi\left(x - \frac{y_i^0}{\delta}\right),$$

We want to identify at large (macroscopic) scale the evolution model for the dynamics of a density of dislocations.

We consider the following rescaling

$$u^\epsilon(t,x) = \epsilon u\left(\frac{t}{\epsilon\delta^2}, \frac{x}{\epsilon\delta}\right),$$

then we see that u^ϵ is solution of

$$\delta\partial_t u^\epsilon = -(-\Delta)^{\frac{1}{2}} u^\epsilon - \frac{1}{\delta} W'\left(\frac{u^\epsilon}{\epsilon}\right) \quad \text{in } (0,+\infty)\times\mathbb{R}$$

with initial datum

$$u^\epsilon(0,x) = \sum_{i=1}^{N_\epsilon} \epsilon\phi\left(\frac{x-\epsilon y_i}{\epsilon\delta}\right).$$

More in general, we consider

$$\begin{cases} \delta \partial_t u^\epsilon = -(-\Delta)^{\frac{1}{2}} u^\epsilon - \frac{1}{\delta} W'\left(\frac{u^\epsilon}{\epsilon}\right) & \text{in } \mathbb{R}^+ \times \mathbb{R} \\ u^\epsilon(0,\cdot) = u_0(\cdot) & \text{on } \mathbb{R} \end{cases}$$

where $\epsilon, \delta > 0$ are small scale parameters and $\delta = \delta_\epsilon \to 0$ as $\epsilon \to 0$,

$$\begin{cases} W \in C^{2,\beta}(\mathbb{R}) & \text{for some } 0 < \beta < 1 \\ W(u+1) = W(u) & \text{for any } u \in \mathbb{R} \\ W = 0 & \text{on } \mathbb{Z} \\ W > 0 & \text{on } \mathbb{R} \setminus \mathbb{Z} \\ W''(0) > 0. \end{cases}$$

On the function u_0 we assume

$$\begin{cases} u_0 \in C^{1,1}(\mathbb{R}) \\ u_0 \text{ non-decreasing.} \end{cases}$$

Main result

$$
\begin{cases}
\delta \partial_t u^\epsilon = -(-\Delta)^{\frac{1}{2}} u^\epsilon - \frac{1}{\delta} W'\left(\frac{u^\epsilon}{\epsilon}\right) & \text{in } \mathbb{R}^+ \times \mathbb{R} \\
u^\epsilon(0,\cdot) = u_0(\cdot) & \text{on } \mathbb{R}
\end{cases}
\tag{3}
$$

Theorem

Let u^ϵ be the viscosity solution of (3). Then, as $\epsilon \to 0$, u^ϵ converges locally uniformly in $(0,+\infty) \times \mathbb{R}$ to the non-decreasing viscosity solution of

$$
\begin{cases}
\partial_t u = -c_0 \partial_x u (-\Delta)^{\frac{1}{2}} u & \text{in } \mathbb{R}^+ \times \mathbb{R} \\
u(0,\cdot) = u_0 & \text{on } \mathbb{R}
\end{cases}
\tag{4}
$$

where $c_0 = \left(\int_{\mathbb{R}} (\phi')^2\right)^{-1}$.

Mechanical interpretation of the convergence result

The limit equation

$$\begin{cases} \partial_t u = -c_0 \partial_x u (-\Delta)^{\frac{1}{2}} u & \text{in } \mathbb{R}^+ \times \mathbb{R} \\ u(0, \cdot) = u_0 & \text{on } \mathbb{R} \end{cases}$$

represents the plastic flow rule for the macroscopic crystal plasticity with density of dislocations.

- u is the plastic strain
- $\partial_t u$ is the plastic strain velocity;
- $\partial_x u$ is the dislocation density;
- $-(-\Delta)^{\frac{1}{2}} u$ is the internal stress created by the density of dislocations contained in a slip plane.

The theorem says that in this regime, the plastic strain velocity $\partial_t u$ is proportional to the dislocation density u_x times the effective stress $-(-\Delta)^{\frac{1}{2}} u$. This physical law is known as Orowan's equation.

Equation

$$\partial_t u = -c_0 \partial_x u(-\Delta)^{\frac{1}{2}} u \tag{5}$$

is an integrated form of a model studied by Head for the self-dynamics of a dislocation density represented by u_x

- A. K. HEAD, Dislocation group dynamics III. Similarity solutions of the continuum approximation, *Phil. Magazine*, **26**, (1972), 65-72.

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Let $f = u_x$, differentiating (5), we get

$$\partial_t f = c_0 \partial_x (f \mathcal{H}[f])$$

where $\mathcal{H}$ is Hilbert transform defined in Fourier variables by

$$\mathcal{F}(\mathcal{H}[v])\,(\xi) = i\,\mathrm{sgn}(\xi)\,\mathcal{F}(v)(\xi),$$

for $v \in \mathcal{S}(\mathbb{R})$. The Hilbert transform has the representation formula

$$\mathcal{H}[v](x) = \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{v(y)}{y - x} dy$$

and if $u \in C^{1,\alpha}(\mathbb{R})$ and $u_x \in L^p(\mathbb{R})$ with $1 < p < +\infty$, then

$$-(-\Delta)^{\frac{1}{2}} u = \mathcal{H}[u_x]. \tag{6}$$

Equation

$$\partial_t f = c_0 \partial_x (f \mathcal{H}[f]) \tag{7}$$

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- Existence of a smooth solution of (7) is proven by Castro and Còrdoba under the assumption that the initial datum is strictly positive and in $C^\alpha(\mathbb{R}) \cap L^2(\mathbb{R})$
- Carrillo, Ferreira and Precioso apply transportation methods and show that the solution can be obtained as a gradient flow in the space of probability measures with bounded second moment.

More literature

- $\delta = 1$, homogenization problem studied by R. Monneau and S.P in any dimension.
  Limit equation $\partial_t u = \overline{H}(\nabla u, -(-\Delta)^{\frac{1}{2}} u)$, where the effective Hamiltonian $\overline{H}$ is defined through a cell problem.
- When $n = 1$, $\overline{H}(p, L) \simeq c_o |p| L$.
- $\delta = 0$, corresponds to the (DDD). The passage from the discrete model (DDD) to continuum models has been studied by Forcadel, Imbert and Monneau and more recently by van Meurs, Peletier, Pozar.

Let $v \in C^{1,1}(\mathbb{R})$. Assume for simplicity that v is strictly increasing. Let $\epsilon > 0$ be a small parameter. Let us define the points x_i as follows,

$$v(x_i) = \epsilon i, \quad i = M_\epsilon, \dots, N_\epsilon$$

where $M_\epsilon := \left\lceil \frac{\inf_{\mathbb{R}} v + \epsilon}{\epsilon} \right\rceil$ and $N_\epsilon = \left\lfloor \frac{\sup_{\mathbb{R}} v - \epsilon}{\epsilon} \right\rfloor$. By the monotonicity of v the points x_i are ordered,

$$x_i < x_{i+1} \quad \text{for all } i.$$

Then, we show that

$$-(-\Delta)^{\frac{1}{2}} v(x_i) \simeq -\frac{1}{\pi} \sum_{j \neq i} \frac{\epsilon}{x_i - x_j},$$

where the error goes to 0 when $\epsilon \to 0$.

To show it, we consider a small radius $r = r_\epsilon \to 0$ as $\epsilon \to 0$ and we split

$$\sum_{i \neq i_0} \frac{\epsilon}{x_i - x_{i_0}} = \sum_{\substack{i \neq i_0 \\ |x_i - x_{i_0}| \leq r}} \left(\frac{\epsilon}{x_i - x_{i_0}} + \sum_{|x_i - x_{i_0}| > r} \frac{\epsilon}{x_i - x_{i_0}} \right.$$

Heuristics. Approximation of $-(-\Delta)^{\frac{1}{2}}$

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Then, we have

$$\begin{aligned}
\frac{1}{\pi}\sum_{|x_i-x_{i_0}|>r}\frac{\epsilon}{x_i-x_{i_0}} &= \frac{1}{\pi}\sum_{|x_i-x_{i_0}|>r}\frac{v(x_{i+1})-v(x_i)}{x_i-x_{i_0}}\\
&\simeq \frac{1}{\pi}\sum_{|x_i-x_{i_0}|>r}\frac{v_x(x_i)(x_{i+1}-x_i)}{x_i-x_{i_0}}\\
&\simeq \frac{1}{\pi}\int_{|x-x_{i_0}|>r}\frac{v_x(x)}{x-x_{i_0}}dx\\
&= \frac{1}{\pi}\int_{|x-x_{i_0}|>r}\frac{v(x)-v(x_{i_0})}{(x-x_{i_0})^2}dx - \frac{1}{\pi}\frac{v(x_{i_0}+r)+v(x_{i_0}-r)-2v(x_{i_0})}{r}\\
&\simeq -(-\Delta)^{\frac{1}{2}}[v](x_{i_0}).
\end{aligned}$$

We can control the error produced in the approximation by choosing r not too small (r such that $\epsilon/r \to 0$ as $\epsilon \to 0$).

On the other hand, for $i \neq i_0$,

$$\epsilon(i - i_0) = v(x_i) - v(x_{i_0}) \simeq v_x(x_{i_0})(x_i - x_{i_0})$$

from which

$$\begin{aligned}
\sum_{\substack{i \neq i_0 \\ |x_i - x_{i_0}| \leq r}} \frac{\epsilon}{x_i - x_{i_0}} &\simeq v_x(x_{i_0}) \sum_{\substack{i \neq i_0 \\ |i - i_0| \leq v_x(x_{i_0}) \frac{r}{\epsilon}}} \frac{1}{(i - i_0)} \\
&\simeq v_x(x_{i_0}) \left(\sum_{i \leq i_0 - 1} \frac{1}{(i - i_0)} + \sum_{i \geq i_0 + 1} \frac{1}{(i - i_0)} \right) \\
&= v_x(x_{i_0}) \left(- \sum_{k \geq 1} \frac{1}{k} + \sum_{k \geq 1} \frac{1}{k} \right) \\
&= 0.
\end{aligned}$$

We can control the error produced by choosing r sufficiently small ($r \leq \epsilon^{\frac{1}{2}}$).

Heuristics. Any function is well-prepared

Let ϕ be the transition layer. If $H(x)$ is the Heaviside function, then

$$\phi(x) \simeq H(x) - \frac{1}{\alpha \pi x}, \quad \text{if } |x| >> 1,$$

where $\alpha = W''(0)$. Then, if $v \in C^{1,1}(\mathbb{R})$ is non-decreasing

$$v(x) \simeq \sum_{i=M_\epsilon}^{N_\epsilon} \epsilon\phi\left(\frac{x - x_i}{\epsilon\delta}\right) + \epsilon M_\epsilon,$$

where $\epsilon M_\epsilon \simeq \inf_{\mathbb{R}} v$. Indeed, assume $x = x_{i_0}$ for some i_0. Then,

$$\begin{aligned}
\sum_{i=M_\epsilon}^{N_\epsilon} \epsilon\phi\left(\frac{x_{i_0} - x_i}{\epsilon\delta}\right) + \epsilon M_\epsilon &= \sum_{i=M_\epsilon}^{i_0-1} \epsilon\phi\left(\frac{x_{i_0} - x_i}{\epsilon\delta}\right) + \epsilon\phi(0) + \sum_{i=i_0+1}^{N_\epsilon} \epsilon\phi\left(\frac{x_{i_0} - x_i}{\epsilon\delta}\right) + \epsilon M_\epsilon \\
&\simeq \sum_{i=M_\epsilon}^{i_0-1} \epsilon \left(1 + \frac{\epsilon\delta}{\alpha\pi(x_i - x_{i_0})}\right) + \frac{\epsilon\delta}{\alpha\pi} \sum_{i=i_0+1}^{N_\epsilon} \frac{\epsilon}{x_i - x_{i_0}} + \epsilon M_\epsilon \\
&= \frac{\epsilon\delta}{\alpha\pi} \sum_{i \neq i_0} \frac{\epsilon}{x_i - x_{i_0}} + \epsilon i_0 \\
&\simeq \frac{\epsilon\delta}{\alpha} \left(-(-\Delta)^{\frac{1}{2}}[v](x_{i_0})\right) + \epsilon i_0 \\
&\simeq \epsilon i_0 \\
&= v(x_{i_0}).
\end{aligned}$$

Heuristics. Proof of convergence

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- Differentiate,

$$\partial_t u(t, x_i(t)) + \partial_x u(t, x_i(t))\dot{x}_i(t) = 0,$$

from which

$$\dot{x}_i(t) = -\frac{\partial_t u(t, x_i(t))}{\partial_x u(t, x_i(t))}.$$

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- Next we consider as ansatz for u^ϵ the approximation of u given by

$$\Phi^\epsilon(t, x) := \sum_{i=M_\epsilon}^{N_\epsilon} \epsilon\phi\left(\frac{x - x_i(t)}{\epsilon\delta}\right) + \epsilon M_\epsilon.$$

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$$\Phi^\epsilon(t, x) := \sum_{i=M_\epsilon}^{N_\epsilon} \epsilon\phi\left(\frac{x - x_i(t)}{\epsilon\delta}\right) + \epsilon M_\epsilon.$$

- Plugging the anzatz into the PDE $\delta\partial_t u^\epsilon = -(-\Delta)^{\frac{1}{2}} u^\epsilon - \frac{1}{\delta} W'\left(\frac{u^\epsilon}{\epsilon}\right)$,

$$\dot{x}_i \simeq \frac{c_0}{\pi} \sum_{j \neq i} \frac{\epsilon}{x_i - x_j} \simeq -c_0(-\Delta)u(t, \cdot)(x_i)$$

Heuristics. Proof of convergence

- Therefore,

$$\partial_t u(t, x_i(t)) = -c_0 \partial_x u(t, x_i(t))(-\Delta)^{\frac{1}{2}} u(t, x_i(t)).$$

Passing to the limit as $\epsilon \to 0$ we see that u solves

$$\partial_t u = -c_0 \partial_x u(-\Delta)^{\frac{1}{2}} u.$$

Heuristics. Proof of convergence

- Therefore,

$$\partial_t u(t, x_i(t)) = -c_0 \partial_x u(t, x_i(t))(-\Delta)^{\frac{1}{2}} u(t, x_i(t)).$$

Passing to the limit as $\epsilon \to 0$ we see that u solves

$$\partial_t u = -c_0 \partial_x u(-\Delta)^{\frac{1}{2}} u.$$

- Notice that if we define

$$y_i(\tau) := \frac{x_i(\epsilon\tau)}{\epsilon}$$

then the y_i's solve

$$\dot{y}_i(\tau) = \dot{x}_i(\epsilon\tau) \simeq \frac{c_0}{\pi} \sum_{j \neq i} \frac{\epsilon}{x_i - x_j} = \frac{c_0}{\pi} \sum_{j \neq i} \frac{1}{y_i - y_j},$$

which is the (DDD).

Proof of convergence

In the formal proof we prove that:

- The limit function is u is viscosity solution of the limit equation when testing with test functions with derivative in x different than 0;
- For all $t \geq 0$, $\lim_{x\to-\infty} u(t,x) = \inf_{\mathbb{R}} u_0$ and $\lim_{x\to+\infty} u(t,x) = \sup_{\mathbb{R}} u_0$, that is the mass of the non-negative function $\partial_x u(t,x)$ is conserved: for all $t \geq 0$,
$$\|\partial_x u(t,\cdot)\|_{L^1(\mathbb{R})} = \|\partial_x u_0\|_{L^1(\mathbb{R})}.$$
- By a comparison argument, we conclude that u is the non-decreasing viscosity solution of the limit equation.