Rankin-Selberg Integrals, Langlands Functoriality and Automorphic Descent

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February 11, 2020
Outline

1. Adeles
2. Groups
3. Representations
4. Unramified representations of $G(\mathbb{Q}_p)$
5. Automorphic Representations
6. Langlands Functoriality
7. Eisenstein Series (special cases)
8. Generalized Doubling Integrals and Automorphic Descent-General Case
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\( \mathbb{A} \) - the ring of adeles of \( \mathbb{Q} \);

\[
\mathbb{A} = \mathbb{Q}_\infty \times \prod'_{p \text{ prime}} \mathbb{Q}_p, \text{ (restricted product)}
\]

\( \mathbb{Q}_\infty = \mathbb{R} \),
\( \mathbb{Q}_p \) - the field of \( p \)-adic numbers.

An element of \( \mathbb{A} \) is a sequence \( \{x_\infty, x_2, x_3, x_5, \ldots\} \), such that, for almost all \( p \), \( x_p \) lies in

\[
\mathbb{Z}_p = \{z \in \mathbb{Q}_p \mid |z|_p \leq 1\}.
\]

This is the ring of \( p \)-adic integers. It is compact and open.
\( \mathbb{A} \) - the ring of **adeles** of \( \mathbb{Q} \);

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\( \mathbb{A} \) is a locally compact ring; \( \mathbb{Q} \) embeds in \( \mathbb{A} \) diagonally with discrete image; \( \mathbb{Q} \backslash \mathbb{A} \) is compact.
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Beside $GL_n$, we will consider (as linear algebraic groups defined over $\mathbb{Q}$)

**symplectic groups,**

$$Sp_{2n} = \{ g \in GL_{2n} \mid {}^t g J_{2n} g = J_{2n} \},$$

$$J_{2n} = \begin{pmatrix} -w_n & w_n \\ w_n & -w_n \end{pmatrix}, \quad w_n = \begin{pmatrix} 1 \\ & \ddots \\ & & 1 \end{pmatrix};$$

and **special (split) orthogonal groups,**

$$SO_n = \{ g \in SL_n \mid {}^t g w_n g = w_n \}.$$
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For a group $G$, as above, let

$B = B_G$ - the subgroup of upper triangular matrices in $G$ (the **standard Borel subgroup** of $G$).
Levi decomposition:

\[ B = T \ltimes N, \]

\( T = T_G \) - the subgroup of diagonal matrices in \( G \),
\( N = N_G \) - the subgroup of upper unipotent matrices in \( G \).
For \( G = GL_n \), we denote \( B = B_n, \ T = T_n, \ N = N_n \).
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**Example:** For \( G = Sp_{2n} \),

\[ T = \left\{ \begin{pmatrix} t & \ast \\ \ast & t^* \end{pmatrix} \mid t = \begin{pmatrix} t_1 & \cdots \\ & \ddots & \ast \\ & & t_n \end{pmatrix} \right\}, \]

\[ N = \left\{ \begin{pmatrix} Z & \ast \\ \ast & Z^* \end{pmatrix} \begin{pmatrix} I_n & \ast \\ \ast & I_n \end{pmatrix} \mid z \in N_n, t(w_n x) = w_n x \right\}. \]

(For \( a \in GL_n, a^* = w_n t a^{-1} w_n \).)
The **standard parabolic subgroups** of $G$ are those which are (Zariski) closed and contain $B_G$. 
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Intersect these with our orthogonal groups or symplectic groups to obtain their standard maximal parabolic subgroups.
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**Example:** For $r \leq \frac{n}{2}$,

$$P_r = \{ \begin{pmatrix} a & b \\ b & a^* \end{pmatrix} \begin{pmatrix} l_r & x & y \\ l_{n-2r} & x' & \end{pmatrix} \in SO_n \mid a \in GL_r, b \in SO_{n-2r}, \ldots \}$$

$$= M_r \rtimes U_r, \text{ (Levi decomposition).}$$
We will consider the groups of points of the groups above over the fields \( \mathbb{Q}, \mathbb{Q}_v \) \((v = \infty, p \text{ prime})\), and also over \( \mathbb{A} \):

\[
Sp_{2n}(\mathbb{Q}_v), \ N_G(\mathbb{Q}_v), \ GL_n(\mathbb{A}), \ T_G(\mathbb{A}), \text{ etc.}
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The group $K_p = K_{G,p} = G(\mathbb{Z}_p)$ is the **standard maximal compact subgroup** of $G(\mathbb{Q}_p)$. 
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The group $K_p = K_{G,p} = G(\mathbb{Z}_p)$ is the standard maximal compact subgroup of $G(\mathbb{Q}_p)$.

The group $G(\mathbb{A})$ is locally compact. It is the restricted product

$$G(\mathbb{A}) = \prod' G(\mathbb{Q}_v).$$

A sequence $(g_\infty, g_2, g_3, g_5, \ldots)$ lies in $G(\mathbb{A})$, if, for almost all $p$, $g_p \in K_p$. 
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The irreducible representations of $G(\mathbb{A})$ (over $\mathbb{C}$), that we are interested in (admissible), decompose as a restricted tensor product

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of irreducible representations of the groups \( G(\mathbb{Q}_v) \).

For almost all primes \( p \), \( \pi_p \) is unramified, which means that \( V_{\pi_p}^{K_p} \neq 0 \). In this case,

\[
dim V_{\pi_p}^{K_p} = 1.
\]

Fix \( 0 \neq \xi^0_{\pi_p} \in V_{\pi_p}^{K_p} \), for all such \( p \). Then \( V_{\pi} \) is spanned by

\[
(\bigotimes_{v \in S} V_{\pi_v}) \otimes (\bigotimes_{p \notin S} \xi^0_{\pi_p}),
\]

as \( S \) varies over all finite sets of places containing \( \infty \), so that \( \pi_p \) is unramified whenever \( p \notin S \).
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Such an unramified representation $\pi_p$ appears as an irreducible subquotient of a representation of the form

$$\text{Ind}_{B_G(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_p,$$

where $\chi_p$ is an \textbf{unramified character} of $B_G(\mathbb{Q}_p)$; it is trivial on $N_G(\mathbb{Q}_p)$ (automatic), and on $T_G(\mathbb{Z}_p)$.
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**Example:** For $G = Sp_{2n},$

$$\chi_p \left( \begin{pmatrix} t_1 & \ast & \cdots & \ast \\ \vdots & \ddots & \ddots & \vdots \\ \ast & \ddots & t_n & \ast \\ \ast & \cdots & \ddots & t_1^{-1} \end{pmatrix} \right) = |t_1|_p^{a_1,p} \cdots |t_n|_p^{a_n,p} =$$

$$\chi_{1,p}(t_1) \cdots \chi_{n,p}(t_n).$$
The following \textbf{(semisimple) conjugacy class}, inside $SO_{2n+1}(\mathbb{C})$, is uniquely determined by $\pi_p$:

\[
\begin{pmatrix}
\chi_{1,p}(p) \\
\vdots \\
\chi_{n,p}(p) \\
\end{pmatrix}
\begin{pmatrix}
1 \\
\chi_{n,p}(p) \\
\vdots \\
\chi_{1,p}(p) \\
\end{pmatrix}
\]

\[
c_{\pi_p} = \langle \begin{pmatrix}
\chi_{1,p}(p) \\
\vdots \\
\chi_{n,p}(p) \\
\end{pmatrix} \rangle
\]

\[
\subset SO_{2n+1}(\mathbb{C}) = \mathcal{L}Sp_{2n}.
\]

Similarly, we have the semisimple conjugacy classes $c_{\pi_p} \subset \mathcal{L}G$ \textbf{(L-group)}.
Given an unramified representation \( \tau_p \) of \( \text{GL}_r(\mathbb{Q}_p) \), we define the local \( L \)-function, in the complex variable \( s \),

\[
L(\pi_p \times \tau_p, s) = (\det(I - p^{-s} \langle \pi_p \otimes \tau_p \rangle))^{-1}.
\]

Example: If \( G = SO_{2n+1} \), \( \pi_p \) corresponds to \( \chi_p = \chi_1 \otimes \cdots \otimes \chi_n \), and \( \tau_p \) to \( \mu_p = \mu_1 \otimes \cdots \otimes \mu_r \), then

\[
L(\pi_p \times \tau_p, s) = \prod_{i \leq n, j \leq r} \frac{1 - \chi_i \otimes \mu_j - p^{-s}}{1 - \chi_i \otimes \mu_j - p^{-s}}.
\]
Given an unramified representation $\tau_p$ of $GL_r(\mathbb{Q}_p)$, we define the **local $L$-function**, in the complex variable $s$,

$$L(\pi_p \times \tau_p, s) = (\det(I - p^{-s}(c_{\pi_p} \otimes c_{\tau_p})))^{-1}.$$
Unramified representations of $G(\mathbb{Q}_p)$

$$L^G = \begin{cases} 
GL_n(\mathbb{C}), & G = GL_n \\
Sp_{2n}(\mathbb{C}), & G = SO_{2n+1} \\
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SO_{2n+1}(\mathbb{C}), & G = Sp_{2n}.
\end{cases}$$

Given an unramified representation $\tau_p$ of $GL_r(\mathbb{Q}_p)$, we define the **local $L$-function**, in the complex variable $s$,

$$L(\pi_p \times \tau_p, s) = (\det(l - p^{-s}(c_{\pi_p} \otimes c_{\tau_p})))^{-1}.$$

**Example:** If $G = SO_{2n+1}$, $\pi_p$ corresponds to $\chi_p = \chi_{1,p} \otimes \cdots \otimes \chi_{n,p}$, and $\tau_p$ to $\mu_p = \mu_{1,p} \otimes \cdots \otimes \mu_{r,p}$, then

$$L(\pi_p \times \tau_p, s) = \prod_{i \leq n, j \leq r} \left(1 - \chi_{i,p}(p)\mu_{j,p}(p)p^{-s}\right)^{-1}\left(1 - \chi_{i,p}^{-1}(p)\mu_{j,p}(p)p^{-s}\right)^{-1}.$$

$$= \prod_{i \leq n, j \leq r} L(\chi_{i,p}\mu_{j,p}, s)L(\chi_{i,p}^{-1}\mu_{j,p}, s).$$
We will also consider

\[ L(\tau_p, \wedge^2, s) = (\det(I - p^{-s} \wedge^2 (c_{\tau_p})))^{-1} = \prod_{i<j\leq r} L(\mu_{i,p} \mu_{j,p}, s); \]

\[ L(\tau_p, \text{sym}^2, s) = (\det(I - p^{-s} \text{sym}^2 (c_{\tau_p})))^{-1} = \prod_{i\leq j\leq r} L(\mu_{i,p} \mu_{j,p}, s). \]
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The irreducible representations $\pi$ of $G(\mathbb{A})$ that we will consider are **automorphic**.

This means that $\pi$ is realized inside a subspace $V_\pi$ of (smooth) functions on $G(\mathbb{Q}) \backslash G(\mathbb{A})$, with respect to the action of $G(\mathbb{A})$ by right translations.
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We say that \( \pi \) is also **cuspidal** when, for all \( f \in V_\pi \),

\[
\int_{U(\mathbb{Q}) \setminus U(\mathbb{A})} f(u) du = 0,
\]

\( U \) - a unipotent radical of any parabolic subgroup of \( G \).
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For such irreducible, automorphic, cuspidal representations $\pi$ of $G(\mathbb{A})$ and $\tau$ of $GL_r(\mathbb{A})$, unramified outside a finite set $S_0$ of places containing $\infty$, define, for a finite set of places $S_0 \subset S$, the **partial $L$-function**

$$L^S(\pi \times \tau, s) = \prod_{p \notin S} L(\pi_p \times \tau_p, s).$$
The product converges absolutely, uniformly in vertical strips, in a right half plane. It extends to a meromorphic function in \( \mathbb{C} \). Similarly with
\[
L^S(\tau, \wedge^2, s), \quad L^S(\tau, \text{sym}^2, s).
\]
This was proved by **Langlands**, using Eisenstein series and their constant terms along unipotent radicals of parabolic subgroups.
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It is possible to define the local \( L \)-functions above at all places. The complete \( L \)-functions satisfy functional equations, relating \( s \) and \( 1 - s \), involving \( \epsilon \)-factors.
Theorem (Jacquet, Shalika): Let $\tau, \sigma$ be irreducible, unitary, automorphic, cuspidal representations of $GL_r(\mathbb{A}), GL_m(\mathbb{A})$, respectively. Then

1. $L^S(\tau \times \sigma, s)$ is entire when $r \neq m$, 
Theorem (Jacquet, Shalika): Let $\tau$, $\sigma$ be irreducible, unitary, automorphic, cuspidal representations of $GL_r(\mathbb{A})$, $GL_m(\mathbb{A})$, respectively. Then
1. $L^S(\tau \times \sigma, s)$ is entire when $r \neq m$,
2. When $r = m$, the only possible pole of $L^S(\tau \times \sigma, s)$ appears on the line $Re(s) = 1$, and then it is simple. In this case, $L^S(\tau \times \sigma, s)$ has a pole at $s = 1$ iff $\tau \cong \hat{\sigma}$. 
**Theorem** (Jacquet, Shalika): Let $\tau$, $\sigma$ be irreducible, unitary, automorphic, cuspidal representations of $GL_r(\A)$, $GL_m(\A)$, respectively. Then

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**Theorem** (Shalika): Irreducible, automorphic, cuspidal representations $\pi$ of $GL_m(\A)$, appear with multiplicity one.
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**Theorem** (Shalika): Irreducible, automorphic, cuspidal representations $\pi$ of $GL_m(\mathbb{A})$, appear with **multiplicity one**.

**Theorem** (Jacquet, Shalika): Let $\tau, \sigma$ be irreducible, unitary, automorphic, cuspidal representations of $GL_n(\mathbb{A})$. Assume that, for almost all primes $p$,

$$\tau_p \cong \sigma_p.$$

Then $\tau \cong \sigma$. (**Strong multiplicity one**)
Given a collection of irreducible (admissible) representations \( \{\pi_v\}_v \) of the groups \( G(\mathbb{Q}_v) \), for each place \( v \), unramified at almost all primes, the representation (of \( G(\mathbb{A}) \))

\[
\pi = \bigotimes'_v \pi_v
\]

will generally be **not automorphic** (in the sense that it embeds into an invariant subspace of smooth functions on \( G(\mathbb{Q}) \backslash G(\mathbb{A}) \), or even is isomorphic to a subquotient of such a space.)

A finite number of changes of the \( \pi_v \) cannot guarantee this.
For $G = GL_n$, we have The Converse Theorem of Cogdell and Piatetski-Shapiro.

It gives a criterion for $\pi$ to be automorphic.

The criterion is given in terms of the functions

$$L(\pi \times \tau, s) = \prod_v L(\pi_v \times \tau_v, s),$$

for a certain family of irreducible, automorphic, cuspidal representations $\tau$ of $GL_{n-1}(\mathbb{A})$. 
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It gives a criterion for $\pi$ to be **automorphic**.

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for a certain family of irreducible, automorphic, cuspidal representations $\tau$ of $GL_{n-1}(\mathbb{A})$.

If, for all $\tau$ in the family above,
- the product converges absolutely in a right half plane,
- extends to an entire function in $\mathbb{C}$, bounded in vertical strips,
- satisfying the functional equation,
then $\pi$ is **automorphic**.
Let $G = \text{SO}_{2n+1}$, and let $\pi$ be an irreducible, automorphic, cuspidal representation of $G(\mathbb{A})$. For almost all primes $p$, we have the semisimple conjugacy classes

$$c_{\pi_p} \subset L^G = \text{Sp}_{2n}(\mathbb{C}) \subset \text{GL}_{2n}(\mathbb{C}) = L^G \text{GL}_{2n}.$$
Let $G = SO_{2n+1}$, and let $\pi$ be an irreducible, automorphic, cuspidal representation of $G(\mathbb{A})$. For almost all primes $p$, we have the semisimple conjugacy classes

$$c_{\pi_p} \subset {}^LG = Sp_{2n}(\mathbb{C}) \subset GL_{2n}(\mathbb{C}) = {}^LGL_{2n}.$$

Consider, for all such $p$, the **unramified representation** $\Pi_p$ of $GL_{2n}(\mathbb{Q}_p)$, such that $c_{\pi_p} \subset c_{\Pi_p}$. 
Let $G = SO_{2n+1}$, and let $\pi$ be an irreducible, automorphic, cuspidal representation of $G(\mathbb{A})$. For almost all primes $p$, we have the semisimple conjugacy classes

$$c_\pi \subset {}^L G = Sp_{2n}(\mathbb{C}) \subset GL_{2n}(\mathbb{C}) = {}^L GL_{2n}.$$

Consider, for all such $p$, the **unramified representation** $\Pi_p$ of $GL_{2n}(\mathbb{Q}_p)$, such that $c_\pi \subset c_{\Pi_p}$.

**Question:** Can we find irreducible representations $\Pi_v$ of $GL_{2n}(\mathbb{Q}_v)$ at the remaining places, such that $\otimes' v \Pi_v$ is automorphic? cuspidal? (on $GL_{2n}(\mathbb{A})$)
Let $G = SO_{2n+1}$, and let $\pi$ be an irreducible, automorphic, cuspidal representation of $G(\mathbb{A})$. For almost all primes $p$, we have the semisimple conjugacy classes 

$$c_{\pi_p} \subset {}^LG = Sp_{2n}(\mathbb{C}) \subset GL_{2n}(\mathbb{C}) = {}^LGL_{2n}.$$ 

Consider, for all such $p$, the unramified representation $\Pi_p$ of $GL_{2n}(\mathbb{Q}_p)$, such that $c_{\pi_p} \subset c_{\Pi_p}$.

**Question:** Can we find irreducible representations $\Pi_v$ of $GL_{2n}(\mathbb{Q}_v)$ at the remaining places, such that $\otimes'_v \Pi_v$ is automorphic? cuspidal? (on $GL_{2n}(\mathbb{A})$)

If yes, note that we must have 

$$L^S(\pi \times \Pi, s) = L^S(\hat{\Pi} \times \Pi, s)$$

($\hat{\Pi} \cong \Pi$). If $\Pi$ is also cuspidal, we get that $L^S(\pi \times \Pi, s)$ must have a simple pole at $s = 1$. 


The **Langlands Functoriality Conjecture** says, in general, that when we have a homomorphism

$$\varphi : L^1 G_1 \to L^1 G_2$$

(say, for split reductive groups $G_1, G_2$ over $\mathbb{Q}$,) then there is a correspondence of irreducible, automorphic representations

$$\pi \mapsto \varphi(\pi)$$

($\pi$ on $G_1(\mathbb{A})$, $\varphi(\pi)$ on $G_2(\mathbb{A})$), such that, for almost all $p$,

$$\varphi(\mathfrak{C}_\pi)_p \subset \mathfrak{C}_{\varphi(\pi)_p}.$$
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($\pi$ on $G_1(\mathbb{A})$, $\varphi(\pi)$ on $G_2(\mathbb{A})$), such that, for almost all $p$,

$$\varphi(c_{\pi|p}) \subset c_{\varphi(\pi)|p}.$$ 

The example we gave is for $G_1 = SO_{2n+1}$, $G_2 = GL_{2n}$. The functorial lifts

$$SO_{2n+1} \quad \rightarrow \quad GL_{2n}$$

$$SO_{2n} \quad \rightarrow \quad GL_{2n}$$

$$Sp_{2n} \quad \rightarrow \quad GL_{2n+1}$$

are now known thanks to the work of Arthur and many others on the trace formula. (It took about 50 years.)
Recently, the functorial lifts above had been established by different methods, namely the study of the $L$-functions $L(\pi \times \tau, s)$, via integrals of Rankin-Selberg type, and the Converse Theorem. (Cai, Friedberg, Ginzburg, Kaplan).
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We say that $\pi$ is (globally) generic if the (Whittaker coefficient)

$$W^\psi(f) = \int_{N_G(\mathbb{Q}) \backslash N_G(\mathbb{A})} f(u)\psi^{-1}_{N_G}(u)du$$

is not identically zero as $f \in V_{\pi}$; $\psi_{N_G}$ is a non-degenerate character of $N_G(\mathbb{A})$, trivial on $N_G(\mathbb{Q})$ ($\psi_{N_G}$ is nontrivial on every simple root subgroup.)
We (GRS) characterized the image of the functorial lift of irreducible, automorphic, cuspidal, generic representations from quasi-split classical groups to $GL_n$. 

Example: $G = \text{SO}_{2n+1}$

An irreducible, automorphic, cuspidal representation $\tau$ of $GL_{2n}(A)$ is in the image of the functorial lift above (generic representations) iff $L(\tau, \wedge^2, s)$ has a pole at $s = 1$.

We constructed an explicit irreducible, automorphic, cuspidal, generic representation $D(\tau)(\text{descent})$ of $\text{SO}_{2n}(A)$, which lifts to $\tau$. (Similar results for any quasi-split classical group.)

Now, we can construct the descent to get all the irreducible, cuspidal representations of $\text{SO}_{2n+1}(A)$, which lift $\tau$. 
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Outline

1. Adeles
2. Groups
3. Representations
4. Unramified representations of $G(\mathbb{Q}_p)$
5. Automorphic Representations
6. Langlands Functoriality
7. Eisenstein Series (special cases)
8. Generalized Doubling Integrals and Automorphic Descent-General Case
1. $GL_n$:

Let $\tau$ be an irreducible, automorphic, cuspidal, unitary representation of $GL_n(\mathbb{A})$. Consider, for $m \geq 2$,

$$\text{Ind}_{P_{nm}(\mathbb{A})}^{GL_{nm}(\mathbb{A})} \tau|\det|^{s_1} \times \cdots \times \tau|\det|^{s_m}, \ s_j \in \mathbb{C},$$

where

$$P_{nm} = \left\{ \begin{pmatrix} a_1 & \ast \\ \vdots & \ddots \\ a_m \end{pmatrix} \mid a_j \in GL_n \right\},$$

$$\begin{pmatrix} a_1 & \ast \\ \vdots & \ddots \\ a_m \end{pmatrix} \mapsto |\det(a_1)|^{s_1} \cdots |\det(a_m)|^{s_m} \tau(a_1) \otimes \cdots \otimes \tau(a_m).$$
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We take smooth, holomorphic sections $f_{\tau, \bar{s}}$ of the above parabolic induction. The (scalar valued) function $f_{\tau, \bar{s}}(g)(1)$ is left $P_{nm}(\mathbb{Q})$-invariant. We make it $GL_{nm}(\mathbb{Q})$-invariant by averaging,
$E(f_\tau, \tilde{s}, g) = \sum_{\gamma \in \mathcal{P}_{nm}(\mathbb{Q}) \backslash \mathbb{Q}} f_{\tau, \tilde{s}}(\gamma g)(1)$. 
\[ E(f, \bar{s}, g) = \sum_{\gamma \in P_{nm}(\mathbb{Q}) \backslash GL_{nm}(\mathbb{Q})} f(\gamma g)(1). \]

This series converges absolutely, and uniformly on compacts, in an appropriate domain, and extends to a meromorphic function on \( \mathbb{C}^m \). This is an example of an \textbf{Eisenstein series}. It has a simple pole at

\[ \left( \frac{m-1}{2}, \frac{m-3}{2}, \ldots, \frac{1-m}{2} \right). \]
$E(f_{\tau, \vec{s}}, g) = \sum_{\gamma \in P_{nm}(\mathbb{Q}) \backslash GL_{nm}(\mathbb{Q})} f_{\tau, \vec{s}}(\gamma g)(1)$. 

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The residual representation, $\Delta(\tau, m)$, is irreducible and lies inside $L^2(C_{\mathbb{A}} GL_{nm}(\mathbb{Q}) \backslash GL_{nm}(\mathbb{A}), \omega_\tau)$. Such representations are called Speh Representations. They exhaust all the irreducible subspaces of $L^2(C_{\mathbb{A}} GL_{nm}(\mathbb{Q}) \backslash GL_{nm}(\mathbb{A}), \omega_\tau)$, which are not cuspidal. They are orthogonal to all cuspidal subspaces, and appear with multiplicity one (Theorem of Moeglin, Waldspurger).
2. $G = \text{Sp}_{2k}, \text{SO}_{2k}, \text{SO}_{2k+1}$

Let $P_k \subset G$ be the standard parabolic subgroup with Levi part $M_k \cong \text{GL}_k$. Assume that $k = nm \ (m \geq 1)$. Let $\Delta(\tau, m)$ be a Speh representation of $\text{GL}_k(\mathbb{A})$. We will consider Eisenstein series corresponding to

$$\text{Ind}_{P_{nm}(\mathbb{A})}^{\text{G}(\mathbb{A})} \Delta(\tau, m) \mid \text{det} \cdot |^s.$$
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They are given by

$$E(f_{\Delta(\tau,m),s}, g) = \sum_{\gamma \in P_k(\mathbb{Q}) \backslash G(\mathbb{Q})} f_{\Delta(\tau,m),s}(\gamma g)(1);$$

The series converges nicely in a right half plane, and extends to a meromorphic function in $\mathbb{C}$. 
We show the example $G = Sp_m$, $m = 2m'$. 
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Let $\tau$ be an irreducible, automorphic, cuspidal representation of $GL_n(\mathbb{A})$. Consider an Eisenstein series on $Sp_{2nm}(\mathbb{A})$, corresponding to

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We show the example \( G = \text{Sp}_m, \, m = 2m'. \)
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\[
\text{Ind}_{\text{Sp}_{2nm}(\mathbb{A})}^{\text{Sp}_{2nm}(\mathbb{A})} \Delta(\tau, m)|\det \cdot |^s.
\]

We consider a certain Fourier coefficient

\[
E^\psi_{\tau,m,s,h}(f_{\Delta(\tau,m),s,h}) = \int_{U_{mn-1}(\mathbb{Q}) \backslash U_{mn-1}(\mathbb{A})} E(f_{\Delta(\tau,m),s,uh}) \psi^{-1}_{U_{mn-1}(u)}(u) du.
\]
We show the example $G = Sp_m, m = 2m'$. Let $\tau$ be an irreducible, automorphic, cuspidal representation of $GL_n(\mathbb{A})$. Consider an Eisenstein series on $Sp_{2nm}(\mathbb{A})$, corresponding to

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We consider a certain Fourier coefficient

$$E^\psi(f_\Delta(\tau,m), s, h) = \int_{U_{mn-1}(\mathbb{Q}) \backslash U_{mn-1}(\mathbb{A})} E(f_\Delta(\tau,m), s, uh)\psi_{U_{mn-1}}^{-1}(u)du.$$

The character $\psi_{U_{mn-1}}$ is preserved by the conjugation action of $Sp_m(\mathbb{A}) \times Sp_m(\mathbb{A})$, where $Sp_m \times Sp_m$ is embedded in $Sp_{2nm}$ as
Let $\pi$ be an irreducible, automorphic, cuspidal representation of $G(A)$ (not necessarily generic). Then, with a certain normalization of the Eisenstein series, the global integrals of the generalized doubling method are

$$L(\varphi_{\pi}, \xi_{\pi}, f_{\Delta(\tau, m)}, s) = \int_{G(\mathbb{Q}) \times G(\mathbb{Q}) \setminus G(A) \times G(A)} \varphi_{\pi}(g) \overline{\xi_{\pi}(h)} (E^*) \psi(f_{\Delta(\tau, m)}, s, i(g, h)) \, dg \, dh.$$  

(Cai, Friedberg, Ginzburg, Kaplan)

$$i : (g, h) \mapsto \text{diag}(g, ..., g, \begin{pmatrix} a & b \\ c & d \end{pmatrix}, g^*, ..., g^*)$$

($g, g^*$ repeated $n - 1$ times; $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ($m' \times m'$ blocks).)
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(Cai, Friedberg, Ginzburg, Kaplan).
\[ \mathcal{L}(\varphi_\pi, \xi_\pi, f_\Delta(\tau, m), s) = \prod_{v \in S} \mathcal{L}(\alpha_{\pi v}, \beta_{\pi v}, f_\Delta(\tau, m)_v, s) L^S(\pi \times \tau, s + \frac{1}{2}). \]

This generalizes the well known doubling integrals of Piatetski-Shapiro, Rallis \((m = 1)\).
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If \(\pi\) is an irreducible, automorphic, cuspidal representation of \(Sp_m(\mathbb{A})\), which lifts to \(\tau\) above (on \(GL_{m+1}(\mathbb{A})\)), then \(L^S(\pi \times \tau, s)\) has a simple pole at \(s = 1\), and \(\pi \otimes \bar{\pi}^t\) pairs into

\[ \mathcal{D}_\psi(\tau) = \text{Span}\{(\text{Res}_{s=\frac{1}{2}} E^*)^\psi(f_{\Delta(\tau,m),s}, \cdot)|i(G(\mathbb{A}) \times G(\mathbb{A}))\}. \]
Since we proved that there is such a generic $\pi$ (this is $D\psi(\tau)$), we conclude that, for any given self-dual, cuspidal $\tau$,

$$D\psi(\tau) \neq 0 \text{ (Double Descent)}$$

This is an automorphic representation of $G(\mathbb{A}) \times G(\mathbb{A})$. 
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Our main theorem is
Theorem (Ginzburg, Soudry)

1. $\mathcal{DD}_\psi(\tau)$ is nontrivial.
**Theorem** (Ginzburg, Soudry)

1. $\mathcal{D}_\psi(\tau)$ is nontrivial.

2. $\mathcal{D}_\psi(\tau)$ is a cuspidal $G(\mathbb{A}) \times G(\mathbb{A})$ - module.

3. 

   $\mathcal{D}_\psi(\tau) = \bigoplus (\pi \otimes \bar{\pi}^\tau)$,

   where $\pi$ varies over the **full** set of irreducible, automorphic, cuspidal representations of $Sp_{2m'}(\mathbb{A})$, which **lift** to $\tau$, (full $L$-packet).