Analysis of the Speed of an Excited Random Walk

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Overview of Background and Results

An excited random walk (ERW) is a non-Markovian extension of the simple random walk. The qualitative behavior of an ERW is largely determined by a parameter δ that can be explicitly calculated. Despite this, we show that the limiting speed of the model cannot be written as a function of δ.

We also generalize the standard ERW by introducing a "bias" to the right and call this generalization an excited asymmetric random walk (EARW). Under certain initial conditions we are able to compute an explicit formula for the limiting speed of an EARW.

Excited Random Walk (ERW)

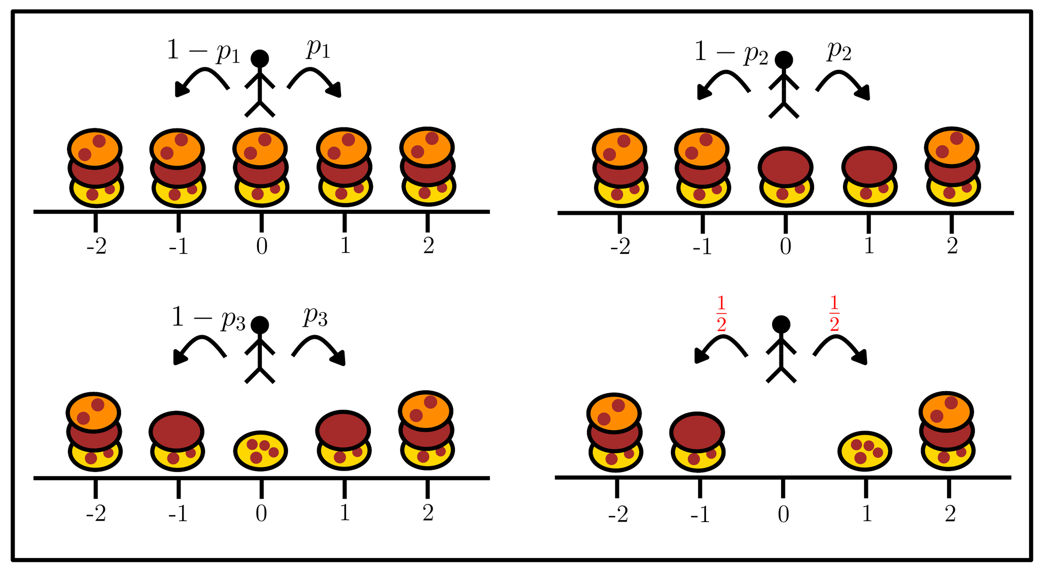

Figure: An excited random walk with 3 cookies

The excited random walk (ERW), also called a cookie random walk, is a variation of a simple random walk which can be informally described as:

- At each site on the number line, we place M cookies.
- The random walker starts at the origin and takes an infinite sequence of steps.
- The probability distribution of each step depends on the number of cookies left at the walker's current location.
- When the walker leaves a site with cookies remaining, he eats a cookie.

Excited Random Walk Definition

We specify the number of cookies M and a vector of cookie strengths $\mathbf{p} \in \mathbb{R}^M$ with $p_i \in (0,1)$. We let $(Y_n)_{n \geq 0}$ be our ERW. At site x, the probability that the next step is to the right is p_i if it is the i^{th} time the walker reaches site x for $i = 1, \dots, M$, and $\frac{1}{2}$ otherwise.

In this model, $(Y_n)_{n \geq 0}$ is neither a Markov chain nor a sum of i.i.d. random variables. This makes analyzing its asymptotic behavior difficult.

Excited Asymmetric Random Walks (EARW)

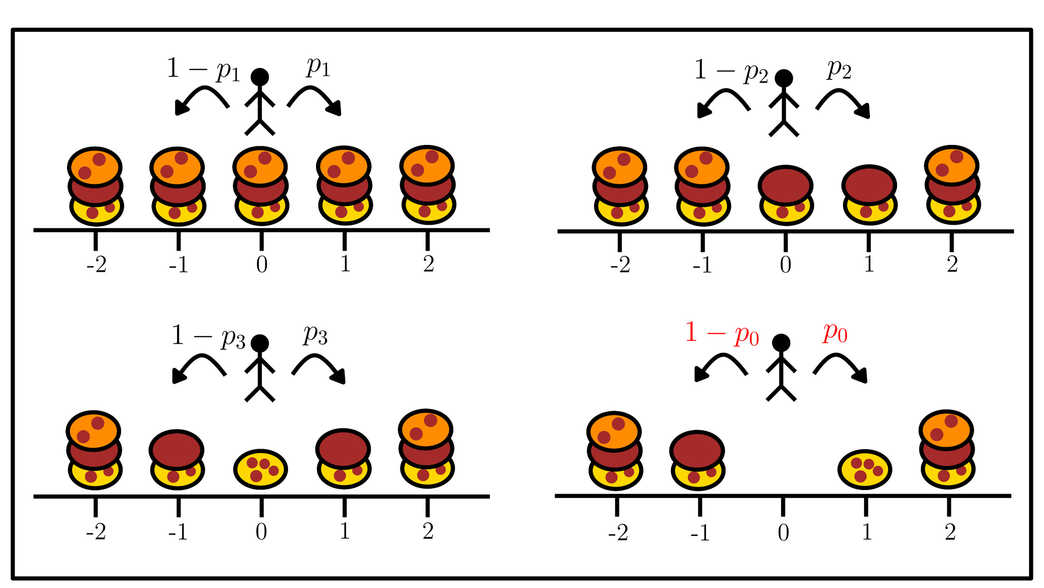

Figure: An excited asymmetric random walk with 3 cookies

An excited asymmetric random walk (EARW) is a generalization of the ERW, in which the probabilities of stepping left or right from a site with no cookies need not be $\frac{1}{2}$ (see figure above).

Excited Asymmetric Random Walk Definition

We specify the number of cookies M and a vector of cookie strengths $\mathbf{p} \in \mathbb{R}^M$ with $p_i \in (0,1)$. We let $(x_n)_{n \geq 0}$ be our EARW. At site x, the probability that the next step is to the right is p_i if it is the i^{th} time the walker reaches site x for $i = 1, \dots, M$, and p_0 otherwise.

We will assume throughout that $p_0 > \frac{1}{2}$; a symmetry argument extends our analysis to the other case.

Background Information

Definitions

- The speed of an ERW is $v(M, \mathbf{p}) = \lim_{n \to \infty} \frac{X_n}{n}$.
- The parameter $\delta(M, \mathbf{p})$ is defined to be $\delta(M, \mathbf{p}) = \sum_{i=1}^{M} (2p_i - 1)$.
- The bias parameter of an EARW is p_0, the probability of stepping right when there are no cookies.

Theorem: Kosygina and Zerner 2008

An ERW:

- is transient to the right when $\delta(M, \mathbf{p}) > 1$, and
- has positive speed when $\delta(M, \mathbf{p}) > 2$.

Why Consider EARWs?

- In a standard ERW, the speed function is either zero if the number of cookies, M, is less than 3, and is either zero or unknown when $M \geq 3$.
- Adding a bias to the ERW makes the speed function nontrivial, i.e. nonzero, even when M is small.
- In the case of $M = 1$, we can compute the limiting speed of an EARW exactly.

An Associated Markov Chain

While ERWs and EARWs are non-Markovian processes, there exists a Markov chain associated with each walk. We let U_x^n be the number of steps from x to $x-1$ before the walk reaches state n for the first time. The figure below gives an example of a random walk and the corresponding values of U_x^4, for $x = 0, 1, 2, 3$.

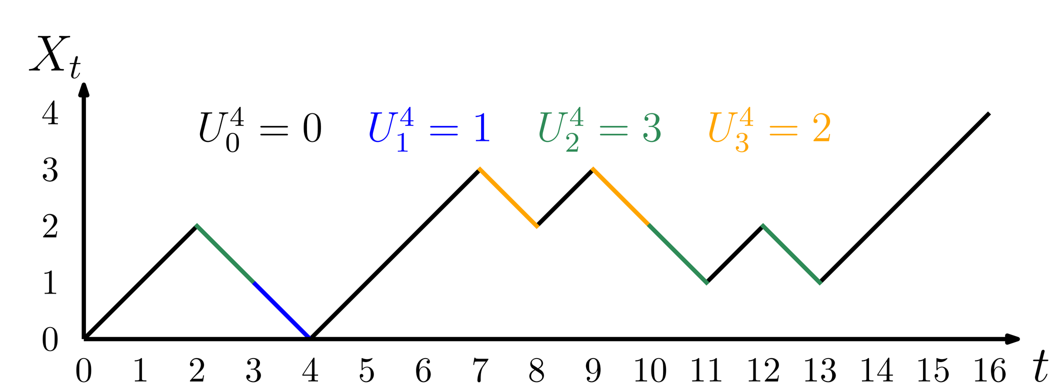

Figure: An example showing U_x^4

The Markovian structure of $(U_n^n, U_{n-1}^n, \dots, U_0^n)$ can be described as follows: for an ERW with cookie stack $\mathbf{p}$, let $(c_i)_{i \geq 1}$ be a sequence of Bernoulli random variables where $\mathrm{P}(c_i = 1) = p_i$. The backwards branching-like process $(Z_n)_{n \geq 0}$ associated with this ERW is a Markov chain with transition probabilities

$$p(j,k) = \mathrm{P}\left(k \text{ failures before } j+1 \text{ successes using } (c_i)_{i \geq 1}\right).$$

If an ERW is recurrent or transient to the right, then for all $n \in \mathbb{N}$, the processes $(Z_0, Z_1, \dots, Z_n)$ and $(U_n^n, U_{n-1}^n, \dots, U_0^n)$ have the same law [1].

Using this associated Markov chain, we can more easily investigate the asymptotic behavior of ERWs.

Theorem: Basdevant and Singh 2008

Given an ERW with backwards branching-like process $(Z_n)_{n \geq 0}$, if the stationary distribution π of $(Z_n)_{n \geq 0}$ exists, the speed of an ERW is

$$v(M, \mathbf{p}) = \frac{1}{1 + 2\mathbb{E}_\pi[Z_0]}. \tag{1}$$

Results on Excited Asymmetric Random Walks

Generally, an explicit formula for the speed of an EARW is too difficult to compute. However, we are able to calculate it when there is a single cookie of strength p_1 and bias parameter $p_0 > \frac{1}{2}$.

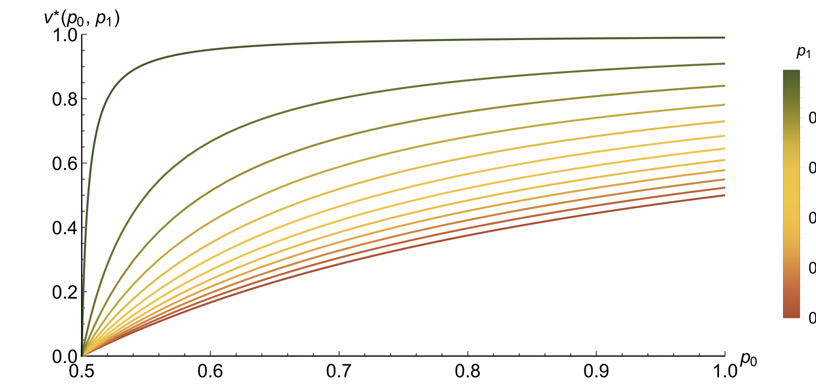

Figure: The speed of an EARW with one p_1 cookie for several values of p_1

Theorem 1: Speed of an EARW

The limiting speed of an EARW with one p_1 cookie and bias parameter $p_0 > 1/2$ is

$$v^*(p_0, p_1) = \frac{2p_0 - 1}{2p_0 - 1 + 2(1 - p_1)}. \tag{2}$$

Probability Generating Function

- Explicitly calculating the stationary distribution π of the backwards branching-like process $(Z_n)_{n \geq 0}$ is a difficult problem.
- Instead we investigated the probability generating function (p.g.f) of π.
- We define
$$G(s) = \mathbb{E}_\pi\left[s^{Z_0}\right] = \sum_{k=0}^{\infty} \pi(k) s^k \tag{3}$$
as the p.g.f. of π.
- Since $G(s)$ is a p.g.f, we know that $G'(1) = \mathbb{E}_\pi[Z_0]$, where $G'(1)$ is the left derivative at 1. This allows us to calculate the speed without explicitly finding π.

Theorem 2: Recursive Formula for the P.G.F.

The p.g.f. for π satisfies the following recursive formula:

$$G(s) = \left(\frac{p_1 + s(p_0 - p_1)}{1 - s(1 - p_0)}\right) G\left(\frac{p_0}{1 - s(1 - p_0)}\right). \tag{4}$$

Sketch of Proof: This is proven using properties of the stationary distribution:

$$G(s) = \mathbb{E}_\pi\left[s^{Z_0}\right] = \mathbb{E}_\pi\left[s^{Z_1}\right] = \sum_{k=1}^{\infty} \pi(k) \mathbb{E}\left[s^{Z_1} \mid Z_0 = k\right]. \tag{5}$$

We can show that

$$\mathbb{E}\left[s^{Z_1} | Z_0 = k\right] = \left(\frac{p_1 + s(p_0 - p_1)}{1 - s(1 - p_0)}\right)\left(\frac{p_0}{1 - s(1 - p_0)}\right)^k.$$

Substituting into (5), we derive the formula given in (4).

Then taking the derivative of the recursive formula at 1 and solving for $G'(1)$, we find that

$$G'(1) = \mathbb{E}_\pi[Z_0] = \frac{1 - p_1}{2p_0 - 1}, \tag{6}$$

and using (1) for the speed we have (2) in Theorem 1 given above.

Unfortunately, for EARWs with more than one cookie, the recursive formula for the speed depends on some terms of the stationary distribution which are too difficult to calculate, so we cannot calculate the speed.

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Results on Excited Random Walks

Although $\delta(M, \mathbf{p})$ determines whether or not an ERW has positive speed, we showed that the speed of an ERW is not a function of δ. We prove this by proving a slightly more general theorem, which loosely states that an ERW with a few strong cookies tends to move faster than an ERW with many weaker cookies.

Theorem 3: δ and v are unrelated when $\delta > 2$

Choose $M \geq 3$ and $\mathbf{p} \in \mathbb{R}^M = (p, p, \dots, p)$ such that $\delta(M, \mathbf{p}) = M(2p - 1) > 2$. Now let $M_i = M + i$ and define

$$p^{(i)} = \frac{1}{2} + \frac{M(2p-1)}{2M_i},$$
$$\mathbf{p_i} \in \mathbb{R}^{M_i} = (p^{(i)}, p^{(i)}, \dots, p^{(i)}),$$

so that $\delta(M_i, \mathbf{p_i}) = \delta(M, \mathbf{p})$ for all i. Then $\lim_{i \to \infty} v(M_i, \mathbf{p_i}) = 0$.

Sketch of a Proof of Theorem 3

- Choose $\mathbf{p} = (p, p, p)$ such that $\delta(3, \mathbf{p}) > 2$.
- For $M \geq 3$, let $p^{(M)}$ be such that $\mathbf{p}_M = (p^{(M)}, p^{(M)}, \cdots, p^{(M)})$ and $\delta(M, \mathbf{p}_M) = \delta(3, \mathbf{p})$.
- For all M, $v(M, \mathbf{p}_M) \leq v^*(p^{(M)}, p)$, where v^* is the speed of an EARW.
- As M increases, $p^{(M)} \to \frac{1}{2}$, and hence $v^*(p^{(M)}, p) \to 0$.
- Thus $\lim_{M \to \infty} v(M, \mathbf{p}_M) = 0$.

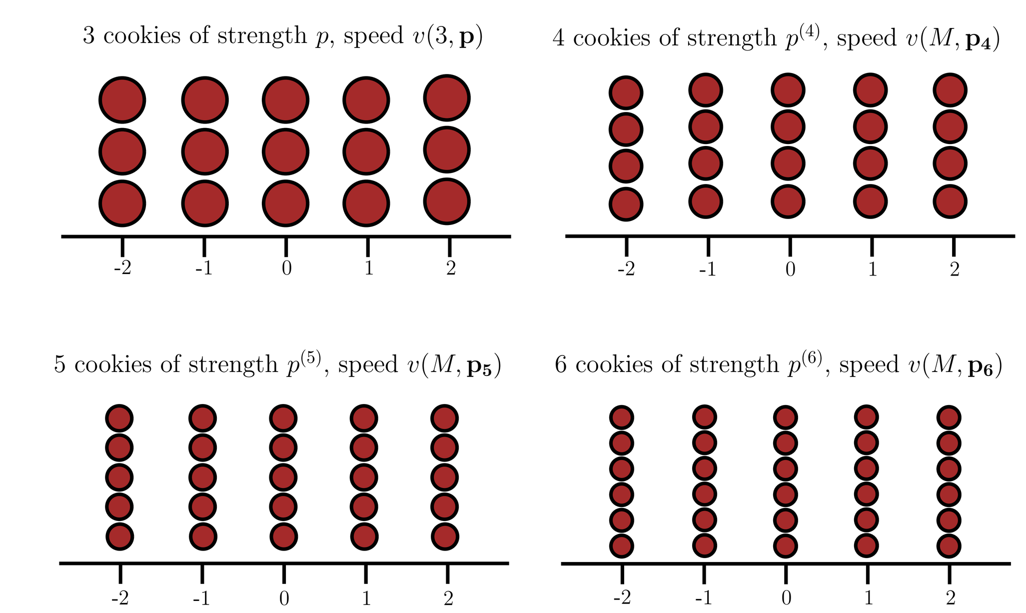

Figure: For a constant δ, $\lim_{M \to \infty} v(M, \mathbf{p}_M) = 0$

Example using Theorem 3

Theorem 3 shows that we can make $v(M, \mathbf{p})$ as small as we like while keeping $\delta(M, \mathbf{p})$ constant.

Here we have an example of another use of the theorem, in which we produce two ERWs with $\delta_1 < \delta_2$ and $v_1 > v_2$:

- Let $M = 3$, $\mathbf{q} = (0.9, 0.9, 0.9)$, and $\mathbf{p} = (0.99, 0.99, 0.99)$.
- Since $\lim_{i \to \infty} v(M_i, \mathbf{p_i}) = 0$, there exists some $i \geq 1$ such that $v(M_i, \mathbf{p_i}) < v(M, \mathbf{q})$.
- However, we have the reverse relationship when comparing $\delta(3, \mathbf{q})$ and $\delta(M_i, \mathbf{p_i})$:
$$\delta(3, \mathbf{q}) = 3(2q - 1) < 3(2p - 1) = \delta(M_i, \mathbf{p_i}).$$
- Thus for i large enough, we have $v(M_i, \mathbf{p_i}) < v(M, \mathbf{q})$ and $\delta(M_i, \mathbf{p_i}) > \delta(M, \mathbf{q})$.

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