Weak-type $(1,1)$ property for the Riesz transforms

Polymath REU Riesz Transform Group

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Basic Notations

- $\mathbb{R}$ = set of real numbers
- $\mathbb{R}^n = \{x = (x_1, x_2, \ldots, x_n) : x_k \in \mathbb{R}\}$, $\quad |x| = \left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}}$
- $\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$
- $\delta_c(x) \text{``}=\text{''} \begin{cases} 1 & x = c \\ 0 & x \neq c \end{cases}$
- $|E|$ = "measure" of $E \subseteq \mathbb{R}^n$ (length, area, volume, etc.)
- $|\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|$ = the distribution function of f

Hilbert Transform and Riesz Transforms

Definition

The **Hilbert transform**, H, satisfies

$$Hf(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{x-y} f(y)\, dy.$$

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For $j \in \{1, 2, \ldots, n\}$, the j^{th} **Riesz transform**, R_j, satisfies

$$R_j f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy.$$

When n=1, the Riesz transform reduces to the Hilbert transform.

$L^p(\mathbb{R}^n)$ bounds

Definition ($L^p(\mathbb{R}^n)$ space)

For $p \geq 1$, the $\mathbf{L^p(\mathbb{R}^n)}$ **space** is the space of functions where

$$\|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{1/p} < \infty.$$

Theorem

The Riesz transforms are bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ if and only if $1 < p < \infty$. In other words, $\|R_j f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$ for some fixed C. This is known as the **strong type $(\mathbf{p}, \mathbf{p})$ property**.

However, the Riesz transforms fail to be bounded on $L^1(\mathbb{R}^n)$. We instead study the **weak-type $(\mathbf{1}, \mathbf{1})$ property** for the Riesz transforms.

Weak-type $(1,1)$ Property

Definition (Weak-type $(1,1)$ Property)

A linear operator T is said to have the weak-type $(1,1)$ property if there exists $C > 0$ such that

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \leq \frac{C}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}$$

for all $\lambda > 0$ and all $f \in L^1(\mathbb{R}^n)$.
The infimum over all such $C > 0$ is denoted $\|T\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)}$.

Theorem

The Hilbert transform and Riesz transforms have the weak-type $(1,1)$ *property.*

Our Problems

Question 1

Determine $\|R_j\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)}$. Do we have a constant independent of the dimension n?

Question 2

For $\lambda > 0$, $E \subseteq \mathbb{R}^2$ with finite measure, and $\nu = \sum_{k=1}^N a_k \delta_{c_k}$ with each $a_k > 0$, compute or bound

$$|\{x \in \mathbb{R}^2 : |R_j \chi_E(x)| > \lambda\}|$$

and

$$|\{x \in \mathbb{R}^2 : |R_j \nu(x)| > \lambda\}|.$$

Existing Work from Loomis

Notice that

$$H\delta_c(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{x-y}\, d\delta_c(y) = \frac{1}{\pi}\frac{1}{x-c}.$$

More generally, if $\nu = \sum_{k=1}^{N} a_k \delta_{c_k}$, then

$$H\nu(x) = \frac{1}{\pi}\sum_{k=1}^{N} \frac{a_k}{x-c_k}.$$

Theorem (Loomis 1946)

If $\nu = \sum_{k=1}^{N} a_k \delta_{c_k}$ with each $a_k > 0$, then

$$|\{x \in \mathbb{R} : |H\nu(x)| > \lambda\}| = \frac{2}{\pi\lambda}\sum_{k=1}^{N} a_k.$$

Motivation for our Questions

Theorem (Dimension-free $L^p(\mathbb{R}^n)$ Bounds)

For any $n \in \mathbb{Z}^+$, $j \in \{1, ..., n\}$, and $1 < p < \infty$, the smallest constant $C > 0$ such that $\|R_j f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$ for all $f \in L^p(\mathbb{R}^n)$ is given by

$$C = \begin{cases} \tan\left(\frac{\pi}{2p}\right) & 1 < p \leq 2 \\ \cot\left(\frac{\pi}{2p}\right) & 2 \leq p < \infty \end{cases}.$$

In the 1980's, Stein asked if the smallest constant in the weak-type $(1,1)$ property is also dimension-free. Classical arguments for the weak-type $(1,1)$ property use the Calderón-Zygmund decomposition and give that $\|R_j\|_{L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)}$ depends at worst exponentially on n. Janakiraman (2004) and Spector and Stockdale (2020) gave two different proofs which improve this dependence to $\log n$. It is currently unknown whether this dependence on n can be removed completely.

Reduction to Dirac Masses

It turns out that we can reduce our question to studying linear combinations of Dirac masses.

Theorem (Spector-Stockdale 2020)

There exists (dimension-free) $C > 0$ such that

$$|\{x \in \mathbb{R}^n : |R_j f(x)| > \lambda\}| \leq C \left[\frac{1}{\lambda} \|f\|_{L^1(\mathbb{R}^n)} + \sup_{\nu} |\{|R_j \nu| > \lambda\}| \right]$$

for any $\lambda > 0$ and $f \in L^1(\mathbb{R}^n)$, where the supremum is taken over all finite linear combinations of Dirac masses $\sum_{k=1}^N a_k \delta_{c_k}$ with $c_k \in \mathbb{R}^n$ and positive a_k satisfying $\sum_{k=1}^N a_k \leq 16\|f\|_{L^1(R^n)}$

The term involving the supremum is where the dependence on the dimension appears, and is therefore of interest to bound.

How this Simplifies the Problem

The reduction to Dirac masses is useful because R_j applied to Dirac masses is an explicit rational function.

A more detailed description of our goal

Given a finite set of Dirac masses in the Euclidean space $\mathbb{R}^n$ with mass $\{a_1, ..., a_N\}$ and location $\{c_1, ..., c_N\}$, we want to find the optimal $C > 0$ such that

$$|\{x \in \mathbb{R}^n : |R_j\nu(x)| > \lambda\}| = \left|\left\{x \in \mathbb{R}^n : \left|\sum_{k=1}^{N} a_k \frac{(x-c_k)_j}{|x-c_k|^{n+1}}\right| > \lambda\right\}\right|$$

$$\leq \frac{C}{\lambda}\sum_{k=1}^{N} a_k,$$

where ν is the linear combination of the Dirac Masses.

Integer masses are enough

Claim (Polymath REU 2020)

It is enough to consider sums (instead of linear combinations) of Dirac masses, i.e. $\sum_{k=1}^{N} \delta_{c_k}$.

- *Reduce arbitrary linear combinations into positive linear combinations.*
- *Reduce positive linear combinations into positive rational linear combinations. (Density)*
- *Reduce positive rational linear combinations into positive integer linear combinations.*

Lemma (Polymath REU 2020)

If $\nu = \sum a_k \, \delta_{c_k}$ *where each* $a_k > 0$ *and* $\sum a_k = 1$ *and* $C > 0$ *is such that* $|\{x : |R_j \nu(x)\}| > \lambda| \leq \frac{C}{\lambda} \|\nu\|$, *then for any* $t > 0$, *we have* $|\{x : |R_j(t\delta)(x)| > \lambda\}| \leq \frac{C}{\lambda} \|t\nu\|$.

Integer points are enough

Claim (Polymath REU 2020)

It is enough to consider sums of Dirac Masses positioned at integer points, i.e. $\sum_{k=1}^{N} \delta_{c_k}$ *with each* $c_k \in \mathbb{Z}^n$.

- *Reduce arbitrary points to rational points. (Density)*
- *Reduce rational points to integer points. (Dilate)*

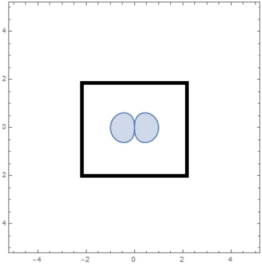
Lemma (Polymath REU 2020)

If $c_k \in \mathbb{R}^n$ *for* $k = 1, \ldots, N$ *and the measure of the resulting region induced by distribution function at* $\lambda > 0$ *is of* $O(\frac{N}{\lambda})$, *then we can dilate the set of points into* $\{t\, c_k\}$ *for any positive* t, *and the measure of region induced by distribution function of* λ *is still of* $O(\frac{N}{\lambda})$.

Covering algorithm [Polymath REU 2020]

We use these reductions to give a new geometric proof of the weak-type $(1, 1)$ inequality for the 2-dimensional Riesz transforms applied to sums of Dirac masses.

- Draw a square whose center is c_1 and side length is $\frac{8}{\sqrt{\lambda}}$, call this square S.



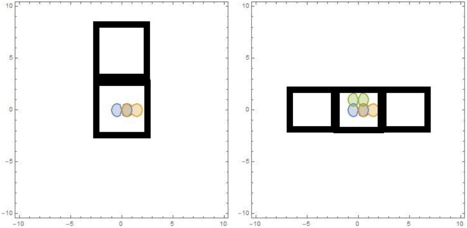
Blue region is the set $\left\{x \in \mathbb{R}^2 : \left|\frac{(x-c_1)_j}{|x-c_1|^3}\right| > \lambda\right\}$

Covering algorithm, cont.

- If there exists $c_2 \in S$, and $d(c_2, \partial S) \geq \frac{3}{\sqrt{\lambda}}$, then we draw a square with side length $\frac{8}{\sqrt{\lambda}}$ next to S.

  ...

- If there exists $c_9 \in S$, and $d(c_9, \partial S) \geq \frac{3}{\sqrt{\lambda}}$, then there are 8 squares with side length $\frac{8}{\sqrt{\lambda}}$ surrounding S.



Covering algorithm, cont.

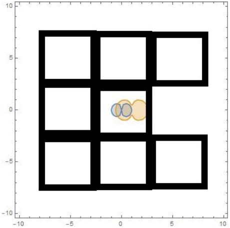
- If there exists $c_k \in S$ with $k < 9$ and $d(c_k, \partial S) < \frac{3}{\sqrt{\lambda}}$, then we draw a square with side length $\frac{8}{\sqrt{\lambda}}$ at the place next to S and closest to c_k.
- If there exists $c_k \in S$ and $k \geq 10$, we draw a square with length $\frac{8}{\sqrt{\lambda}}$ next to the existing squares in any direction.
- Repeat this process until every c_k has a corresponding square with length $\frac{8}{\sqrt{\lambda}}$.

Covering algorithm, cont.

We claim that for any permutation of $c_2,\ldots,c_8$, and for all x outside the union of squares,

$$\left|\sum_{k=1}^{8}\frac{(x-c_k)_j}{|x-c_k|^3}\right|\leq\lambda$$

Extreme Case: $c_1=(0,0)$, $c_2=c_3\ldots=c_8=(\frac{1}{\sqrt{\lambda}},0)$, $x=(\frac{4}{\sqrt{\lambda}},0)$



Covering algorithm, cont.

- From the covering algorithm, each c_k is assigned to a square, hence the total covering area is $N\frac{8^2}{\lambda}$

$$|\{x \in \mathbb{R}^2 : |R_j\nu(x)| > \lambda| \leq \frac{64N}{\lambda} = \frac{64}{\lambda}\|\nu\|$$

- However, we cover the region by squares which makes the constant blow up as dimension increases.

$$|\{x \in \mathbb{R}^n : |R_j\nu(x)| > \lambda| \leq \frac{8^n N}{\lambda} = \frac{8^n}{\lambda}\|\nu\|.$$

Future Works

- Work on a cover by balls rather than squares so that we can find a bound independent of dimension.
- Work on different measure spaces on a similar problem for Hilbert and Riesz Transform.
- Work on a similar problem using different linear operators (like Bergman Projection).

References

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