FUNCTORIAL RESOLUTION EXCEPT OF TOROIDAL LOCUS. TOROIDAL COMPACTIFICATION

JAROSŁAW WŁODARCZYK

ABSTRACT. Let X be any variety in characteristic zero. Let $V \subset X$ be an open subset which has toroidal singularities. We show existence of a canonical desingularization $f : Y \to X$ except of V which does not modify the subset V , and transforms X into a toroidal variety Y , with singularities extending those on V . Moreover the exceptional divisor has simple normal crossings on Y .

The theorem naturally generalizes the Hironaka canonical desingularization which does not modify the nonsingular locus V and transforms X into a nonsingular variety Y.

The proof uses, in particular, the canonical desingularization of logarithmic varieties recently proved by Abramovich -Temkin-Włodarczyk, and the proven here canonical desingularization of locally toric varieties with an unmodified open toroidal subset. As an application we show existence of a toroidal equisingular compactification of toroidal varieties.

CONTENTS

1. INTRODUCTION

Several general questions related to good resolutions and good compactifications were raised, in particular, by Kollar in his book [Kol07]. In a smooth case or in the

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case of isolated singularities it is certainly possible, using Hironaka desingularization theorem , to compactify the variety $X \subset \overline{X}$, such that $\overline{X}$ is smooth outside of the singularities on X, and the complement $\overline{X} \setminus X$ is a simple normal crossing divisor. ([Hir64],[BM97],[Vil89],[Wło05],[Kol07])

If X admits some mild nonisolated singularities, we would like to still to have a good compactification with the boundary divisor having simple intersections.

Ths question is closely related to another problem of a good partial resolution, which does not modify a given open subset with a certain type of singularities.

In other words, a variety with an open subset with allowed certain type of singularities shall be resolved by a birational projective modification in such a way that the open subset will be unmodified, and no new type of singularities will be introduced, with the boundary divisor having some simple intersections.

This problem generalizes the Hironaka desingularization theorem which does not modify the smooth locus of the scheme, and the exceptional locus is a SNC divisor.

A particular question of existence of partial desingularization except of normal crossing (NC) locus was also posed by Kollar in [Kol07].

Some results in this direction were proven by Bierstone-Milman in [BM12a], [BM12b]. They show that such a (partial) resolution except of SNC locus exists for any reduced and reducible scheme of finite type over a field of characteristic zero. Moreover the SNC locus on the resolved (reducible) variety is the closure of the SNC locus on the given variety.

They also observed in the example of the "pinch point" or "Whitney umbrella", that the partial resolution except of NC locus, should allow some more general singularities.

In their paper(s) they give a complete list of possible singularities in a low dimension and a low codimension- "more general pinch points" which need to be introduced in order to resolve the schemes except for NC locus.

The present paper addresses these problems in a much more general situation where the unmodified set is defined by a toroidal embedding.

One shall mention, that Bierstone and Milman in their approach use their singularity invariant and alter some steps of their proof of Hironaka desinularization to prevent modification of the NC and SNC locus.

Unfortunately, in general, the Bierstone-Milman invariant gives a very little geometric information, as it is specifically designed for the inductive structure of the resolution process.

An alternative resolution tool giving a more precise, more geometric and more efficient control over the resolution process was introduced by Mumford and others in [KKMSD73]. They consider the language of toroidal embeddings, which allows to translate the resolution problems into the language of simpler combinatorial objects - conical complexes.

The main disadvantage of this approach is that it can be applied to very special toroidal singularietes defined by the binomial equations or alternatively by the monoids of monomials (represented by cones). In practice, it means that in order to apply the method one needs to transform singularities to toroidal ones first.

The method was initially introduced in [KKMSD73] to solve the problem of semistable reduction of a dominant morphism to a curve, and proved successful for solving many fundamental problems in birational geometry, weak semistable reduction [AK00], weak factorization theorem[Wło03],[AKMW02],[Wło00], , and many others.

The structures on toroidal varieties defined by the divisor or, equivalently, monomials were further generalized in the language of Fontaine-Illusie logarithmic schemes founded in the papers of Kato [Kat89b]. This gives a more general view-point, where toroidal varieties, called logarithmically smooth, form a class of objects similar to the smooth (resolved) varieties in category of arbitrary reduced schemes of finite type over the field. Similarly to smooth case the logarithmically smooth varieties have a relatively simple structure of the completions of local rings. They are generated by free parameters and algebraically independent monomial part which forms a monoid. Moreover, similar to the smooth case, the module of the logarithmic differentials is free of the rank equal to the dimension of the ring. The latter is the direct sum of the free parameters part and a free monomial part corresponding to the groupification of the monoid.

In the recent papers by Abramovich-Temkin-Włodarczyk [ATW16], [ATW17] the authors prove the canonical desingularization of logarithmic varieties. The resolution is functorial with respect to arbitrary logarithmically smooth morphisms. The resulting resolved object is, as dictated by the strong functoriality properties, a quasi-log smooth variety as in [ATW17], or a toroidal orbifold with a locally toric coarse moduli space, as in [ATW16]. The result is thus a counterpart of the Hironaka desingularization in the logarithmic category. The functoriality properties imply that the the log-smooth (toroidal) locus is unmodified in the process. Quasi log smooth varieties are log smooth in, so called Kummer étale topology (which allows to extract roots from the monomials). They are locally toric varieties which are very similar to log smooth varieties (toroidal embeddings). In particular their logarithmic structure is defined by the smooth open subset which a complement of a certain locally toric divisor.

In the paper we give a proof of the canonical (partial) desingularization of varieties with unmodified open toroidal subset. In the process the open subset is untouched , and the resulting variety is a toroidal embedding with the singularities identical as on the open subset. This means that the (irreducible) equisingular strata of the resolved toroidal variety extend the strata on the open unmodified set. Moreover the exceptional boundary divisor has *relatively simple normal crossings*. By a *relative simple normal crossing divisor* we mean here a divisor whose components are locally described by a part of the coordinate system of free parameters. (see Definition 2.1.14)

We obtain several results in this direction. First, we prove the result for locally binomial varieties with locally toric Weil divisors over a field of any characteristic (Theorem 7.19.1) [1]

Then, using the canonical desingularization of logarithmic varieties combined with the desingularization of locally binomial varieties with an unmodified open toroidal subset we show the canonical desingularization of logarithmic varieties except of open toroidal subset. (Theorem 2.1.23) To further generalize the result for arbitrary varieties with Weil divisors we canonically extend the logarithmic structure from the toroidal subset to the whole variety. (Lemma 2.2.9).

[1] Note that our definition of toroidal embeddings differs slightly from the definition of Abramovich-Denef-Karu [ADK13, Section 2.2]. Both definitions agree over a perfect field.

This can be done for arbitrary varieties with Weil divisor possessing an open *extendable toroidal subset*. The extendable toroidal embeddings satisfy certain conditions on restrictions of local Cartier divisors, like for example, toroidal varieties with quotient singularities, toroidal varieties with a single closed stratum or with extendable local Cartier divisors). (Definition 2.2.2)

As a consequence we show existence of a functorial partial resolution except of open extendable toroidal subsets for arbitrary varieties with a given Weil divisor. (Theorem 2.2.11).

A particular version of this result, shows that any variety with a locally toric singularity at a given point can be modified in such a way that a neighborhood of the point will remain unchanged, and the resulting variety will be a toroidal embedding with a unique closed stratum passing through the point (Theorem 2.2.18).

Using the desingualarization theorem we prove existence of equisingular toroidal compactification of extendable toroidal embeddings (Theorem 2.2.15).

The canonical desingularization of logarithmic varieties from [ATW16], [ATW17] reduces the problem to the quasi-toroidal embeddings. The latter are, in particular, locally toric, with some locally toric divisors. This defines a natural (non smooth) stratification induced by the given divisor and the sigularity type. The resulting variety is not a toroidal embedding but it is a stratified toroidal variety. In order to deal with it we use the theory of stratified toroidal varieties. The theory was developed as a tool in the proof of the Weak factorization theorem [Wło03]. It associates with the variety a semicomplex and allows to run certain (sufficiently functorial) algorithms. In particular, only very special centers of the modifications (star subdivisions) can be used. In section 7 we give a crash course on the theory of stratified toroidal varieties, recalling and reproving a few most basic results used in the proof.

In order to resolve locally toric singularities with unmodified toroidal subset we develop a functorial desingularization combinatorial algorithm, and its relative version. It can be applied to conical complexes and more general semicomplexes. The functoriality properties are critical for gluing the algorithm on the more general objects.

The method gives a functorial resolution of locally toric or locally binomial varieties with stratification over a field in any characteristic and its relative version with unmodified toroidal subset (Theorems 7.18.1, 7.19.1).

One shall mention that the problem of the functorial resolution of locally binomial or locally toric varieties was open in positive characteristic. Existence of noncanonical (weak) resolution of locally toric varieties over a algebraically closed fields was proven in [Wło03, Theorem 8.3.2]. On the other hand the Hironaka approach of the embedded resolution works well only in the case of toroidal embeddings but fails in a locally binomial or a locally toric situation due to lack of, so called, maximal contact. (see Example 9.0.1). Note that a version of the "combinatorial maximal contact" considered for "combinatorial blow-ups", was constructed by Bierstone- Milman in the case of toroidal embeddings ([BM06]).

The functorial desingularization algorithm of locally toric (or locally binomial) varieties is much simpler, more efficient and more geometric than Hironaka resolution in characteristic zero. When combining directly with the logarithmic desingularization of [ATW16] ([ATW17]) it gives also a more efficient algorithm of canonical desingularization of arbitrary varieties in characteristic zero. Moreover the method gives a very good control over resolved singularities and allows to alter the process to avoid undesired modifications.

Another and perhaps most straightforaward application of the combinatorial algorithm is to the toroidal embeddings. In this case we obtain a very efficient method of functorial resolution, and its relative version. (Theorems 6.4.1, 6.5.1, and 6.6.1). Unlike the other desingularizations, in particular Hironaka's toroidal desingularization in positive characteristic, ours does not depend upon the toroidal structure and is controlled by simple geometric invariants without additional bulk. When forgetting about the divisors defining the the toroidal (or log smooth) structure we are left with locally toric varieties, which reduces the language to the previous situation of stratified toroidal varieties (Theorems 7.18.1) without changing the algorithm. This also explains why the algorithm works in a locally toric case in positive characteristic. On the other hand the presentation of the algorithm in the paper from toroidal embeddings to locally binomial varieties, illustrates the main feature of the theory of stratified toroidal varieties which studies the combinatorial modifications independent of the locally toric coordinates.

Recall that the embedded functorial desingularization of the (not necessarily normal) toroidal embeddings over perfect field was proven, as it was mentioned earlier) by Bierstone-Milman by extending methods developed in characteristic zero in [BM06]. Similar results were shown by Nizioł. She was using a combinatorial interpretation of the simplified Hironaka algorithm. [Niz06, Theorem 5.10]. The non-embedded resolution in characteristic 0, preserving a simple normal crossing locus was also proven by Illusie-Temkin [IT14, Theorem 3.3.16], Gillam-Molcho[GM15, Theorem 9.4.5]. Another simple combinatorial method was provided in [ACMW14, Theorem 4.4.2] by Abramovich-Chen-Marcus-Wise. It combines barycentric subdivisions with the lattice reduction algorithm of [KKMSD73, Theorem 11*], is not functorial and it modifies the normal crossings locus.

The paper is organized as follows. In Chapter 2 we formulate and prove the main theorems using the desingularization theorems from Chapter 7. In Chapter 3 we introduce the basic definitions and results on toroidal embeddings, and conical complexes. Chapter 4 is entirely devoted to developing tools and giving a proof for canonical desingularization of conical complexes.

In Chapter 5 we introduce the language of relative conical complexes, and prove the relative version of the canonical desingularization of the conical complexes The algorithm in the relative version is nearly identical, and the introduced notions are perfectly analogous to the standard nonrelative situation. However the language of relative complexes is somewhat more involved and thus perhaps less intuitive than the language of complexes. That is why we deal separately with the nonrelative and relative case, although the first one is the particular case of the second, with the trivial relative structure.

Chapter 6 contains the proof of the functorial desingularization of toroidal embeddings, and its relative version with unmodified open subset. The results are quite immediate consequences of the canonical desingularization of complexes in Chapters 4, 5.

Finally, Chapter 7 contains the proof of the functorial desingularization of locally toric varieties over the fields, and its relative version. The main tool in the proof is a theory of the stratified toroidal varieties. One constructs a stratification defined by the singularity type and by a given divisor (in the relative situation). These data define the associated conical semicomplex, which is , roughly, a collection of the cones defined up to automorphisms groups associated with strata and some rather sparse face relation defined by the generization of the strata

The functoriality of the algorithm developed in Chapters 4, 5 for complexes and the canonicity of the centers allow to run it on semicomplexes to give rise to the desingularization of stratified toroidal varieties.

main

2. Main results

2.1. **Desingularization of logarithmic varieties except of log smooth locus.**

2.1.1. *Logarithmic varieties.* The logarithmic structures are used in this paper only in the formulation of Theorem 2.1.23, and in the Extension Lemma 2.2.9, in Chapter 2.

Recall that a *logarithimic structure* on a scheme X of finite type is given by the sheaf of monoids $\mathcal{M}_X$, containing subsheaf of monoids $\mathcal{O}^*_X$ (of the invertible regular functions), and admitting the map of the monoids $\mathcal{M}_X \to \mathcal{O}_X$ under multiplication. A logarithimic structure is called *coherent* if étale locally induced (or generated) by a map of monoids $P \to \mathcal{O}_X$, called *chart*, where P is a finitely generated monoid. [Kat89b, Sections 1.1, 1.2] [Kat89a, Sections 1.2] A coherent logarithimic structure is *fine* (fs) if P is fine and saturated, which means P is finitely generated, has no zero divisors, so injects in its groupification P^{gp}. It is fines and saturated (fs) if it is fine in P^{gp}, so if $a \in P^{gp}$, such that $a^n \in P$ implies $a \in P$. In particular, the monoids defined by the intersections of rationally generated cones with lattices is an example of fine and saturated monoids. [Kat89b, Sections 1.1, 1.2]

Any variety or a scheme, with a coherent logarithmic structure can be made canonically into a variety or a scheme with a fine and saturated structure by the natural canonical procedure, called *saturation* [Kat89a, Proposition 1.2.9], [Ogu16, Proposition 2.1.5]. It is locally described by

$$X^{\rm sat} := X \times_{\operatorname{Spec} K[P]} \operatorname{Spec} K[P^{\rm sat}],$$

with

$$P^{\rm sat} := \{a \in P^{gp} \mid a^n \in i(P)\},$$

where $i : P \to P^{gp}$ is the natural map defined by the groupification. [Kat89a, Proposition 1.2.9] [Mau00, Page 1-2].

Logarithmic structure are usually defined in the étale topology (for functoriality properties). Note that, when working over nonclosed fields or $\mathbb{Z}$ one needs to pass to étale neighborhoods to have a nice properties of local rings and the corresponding monoids.

A coherent or fs logarithmic structure will be called *strict* if the charts are defined in the Zariski topology.

By a *logarithmic variety* (respectively a *strict logarithmic variety*) in this paper we mean a variety equipped with a fine and saturated logarithmic structure (respectively a strict fs logarithmic structure).

2.1.2. *Logarithmic structure on toroidal embeddings.* Toroidal and strict toroidal embeddings were introduced in [KKMSD73] by Mumford and others (initially over an algebraically closed field). They are reviewed in Section 3.4 over nonclosed fields. Strict toroidal and toroidal embeddings are defined by an open subset $U \subset X$, and are locally (respectively locally in étale topology) étale isomorphic to toric varieties (X_σ, T) with open torus T corresponding to open subset U. The open subset U is the complement of a Weil divisor $D := X \setminus U$, and in the sequel we shall often use the divisor D to describe the toroidal structure. The logarithmic structure $\mathcal{M}$ on a toroidal embedding (X, U) is defined as

$$\mathcal{M} = (\mathcal{O}_X)_{\text{ét}} \cap j_*(\mathcal{O}_U^*)_{\text{ét}},$$

where $j : U \to X$ is the open immersion. The logarithmic structure on a strict toroidal embedding is strict, so it is defined on the Zariski topology

$$\mathcal{M} = (\mathcal{O}_X) \cap j_*(\mathcal{O}_U^*),$$

Equivalently, in such a case, the sheaf of monoids $\mathcal{M}$ is generated at any point $x \in X$ by the group of the effective Cartier divisors $\mathrm{Cart}_x(X, D)$ supported on D in a neighborhood (respectively étale neighborhood of x), so we can write

$$\mathcal{M}_x = \mathrm{Cart}_x(X, D) \cdot \mathcal{O}_{X,x}^* \subset \mathcal{O}_X.$$

2.1.3. *Stratifications on toroidal varieties and the associated conical complex.* Any toric variety admits a natural stratification by the orbits. This stratification induces locally the canonical stratification S on strict toroidal and toroidal embeddings. The the closed strata of the stratification are defined by the irreducible components of the intersections of the components of the divisor D. (see Section 3.5)

As it was observed in [KKMSD73] there exists a conical complex Σ associated with a strict toroidal (X, D), with faces $\sigma \in \Sigma$ in bijective correspondence with strata $s = s(\sigma) \in S$ (see Section 3.5, and Theorem 3.8.16). The stratification is equisingular it means the completion of local rings at geometric points are isomorphic. Each stratum s defines an open subset, called the star of s

$$\mathrm{Star}(s, S) := \bigcup_{\{s' \mid s \subset \overline{s'}\}} s',$$

which is an open (saturated) neighborhood.

One can associate with s the group $\mathrm{Cart}(s)$ of the Cartier divisors on $\mathrm{Star}(s, S)$ supported on $\mathrm{Star}(s, S) \cap D$. This group is isomorphic to $\mathrm{Cart}_x(X, D)$ for any $x \in s$, and it generates the logarithmic structure $\mathcal{M}_{\mathrm{Star}(s,S)}$ on $\mathrm{Star}(s, S)$:

$$\mathcal{M}_{\mathrm{Star}(s,S)} = \mathrm{Cart}(s) \cdot \mathcal{O}^*_{\mathrm{Star}(s,S)} \subset \mathcal{O}_{\mathrm{Star}(s,S)}.$$

2.1.4. *Saturated toroidal subsets.*

`tor sub` **Definition 2.1.5.** By a *toroidal subset* of a logarithmic variety $(X, \mathcal{M})$ we mean an open subset $V \subset X$, such that (V, D_V) is a strict toroidal embedding , where D_V is the Weil divisor which defines the restricted log structure $\mathcal{M}_{|V}$.)

We extend this notion to varieties with Weil divisor. Note that the logarithmic structure defined by the Weil divisors is not coherent, and the example of the "pinch point" shows that the operation of the restriction may behave quite badly. This is reflected in the definition below.

Definition 2.1.6. By a *toroidal subset* of a variety with a Weil divisor (X, D) we mean an open subset $V \subset X$, with the induced divisor $D_V := V \cap D$ such that (V, D_V) is a strict toroidal embedding.

saturation **Definition 2.1.7.** Let (X, D) be a variety with a Weil divisor D (respectively a logarithmic structure $\mathcal{M}$). Let S_D be the induced stratification with closed strata defined by the components of the intersections of divisors (respectively defined by the rank of monoids).

By the *saturation* of an open toroidal subset

we mean the maximal open toroidal subset $(V^0, D^0) \subset (X, D)$ with $D^0 := V^0 \cap D$, containing V, such that the strata on (V^0, D^0) intersect the subset V (so extend the strata in (V, D_V)).

A toroidal subset (V, D_V) of (X, D_X) is called *saturated* if it is equal to its saturation.

Example 2.1.8. In the case of the strict toroidal embedding any open subset is toroidal. The saturated subsets are just those which are the unions of some stars.

locus **Definition 2.1.9.** The largest saturated toroidal subset of (X, D) or $(X, \mathcal{M})$ is called the *toroidal locus* (or the *log smooth locus*). It is the set of all points of X where (X, D) or $(X, \mathcal{M})$ is a strict toroidal embedding (a strict logarithmically smooth variety).

Remark 2.1.10. Observe that since the strata on strict toroidal embeddings are equisingular, any toroidal subset $V \subset X$ has the same singularities as its toroidal saturation V^0. So the saturated subsets, in particular, represent the sets of all the points with certain given types of toroidal singularities (including information on the divisor). For example, the smallest nonempty saturated toroidal subset on X is the set of all nonsingular points $(X^0)^{ns}$ of the set X^0 of the points where the logarithmic structure is trivial $\mathcal{M} = \mathcal{O}^*$ (resp. $X^0 := X \setminus D$). The largest saturated toroidal subset on X is the set X^{tor} is its toroidal locus. There are finitely many saturated toroidal subsets on X, as these are exactly the unions of the stars of strata on X^{tor}.

extension **Lemma 2.1.11.** *Let (X, D) be a toroidal embedding (which is not necessarily strict). Let (V, D_V) be its toroidal subset intersecting all the strata of (X, D). Then (X, D) is strict toroidal and it is the toroidal saturation of (V, D_V).*

Proof. Let E be the Weil divisor consisting of the components of D, passing through $x \in E$ be any point in a certain stratum s on X . Let $\pi : (U, D_U) \to (X, D)$ be an étale neighborhood of x which is strictly toroidal. It defines an étale map between the relevant strict toroidal embeddings $\pi_{|U} : (U_V, D_{U_V}) \to (V, D_V)$, where $U_V := \pi^{-1}(V)$.

There exists a bijective correspondence between the components of the Weil divisor of E at x and the components of D_U at a point $\overline{x}$ over x, since such a correspondence exists at the local rings of generic point of the stratum s, where it is defined by $\pi_{|U}$. at their isomorphic completions. So the natural surjection between the components of Weil divisor D_U and E defined by $\pi : \mathrm{Spec}(\mathcal{O}_{\overline{x},U}) \to \mathrm{Spec}(\mathcal{O}_{x,X})$ is a bijection. Moreover each such a component of E itself is a toroidal embedding (strict toroidal in étale topology). So, in particular, each component of E is normal. This implies, by Lemma 3.8.8, that (X, D) is strictly toroidal.

♣

2.1.12. *Relative SNC divisors on toroidal varieties.* In the classical Hironaka desingularization the exceptional locus has SNC. In the relative desingularization the exceptional divisor has similar properties.

nc **Definition 2.1.13.** Let (X, D) be a toroidal embedding. By *free coordinates* we mean parameters $u_1, \ldots, u_k$ on X such that there is a monoid $P \to \mathcal{O}_{U,x}$ on an étale neighborhood U of x and an étale morphism $U \to \operatorname{Spec} K[x_1, \ldots, x_k, P]$ defined by $x_i \mapsto u_i$, and $P \to \mathcal{O}_{U,x}$.

nc **Definition 2.1.14.** Let (X, D) be a toroidal embedding and E be a Weil divisor. Then we say that E has *normal crossings* (NC) (resp. *simple normal crossings* (SNC)) on (X, D) (or with D), if it is étale locally (resp. locally) defined by a part of the free coordinate system. Equivalently we shall call E a *relative NC divisor* (respectively a *relative SNC divisor*) on X.

One can rephrase this definition

1 **Lemma 2.1.15.** *E has normal crossings (NC) on (X, D) iff $(X, D \cup E)$ is a toroidal embedding and any point $x \in X$ admits an étale neighborhood such that $(X, D \cup E)$ is étale isomorphic to*

$$X_\sigma \times X_\tau = X_\sigma \times \mathbb{A}^n,$$

where $X_\tau = \mathbb{A}^n$, with a toric divisor D_1 on X_σ, and E_1 is an SNC toric divisor on $X_\tau = \mathbb{A}^n$, such that D is étale isomorphic $D_1 \times X_\tau = D_1 \times \mathbb{A}^n$, and E is étale isomorphic to $X_\sigma \times E_1$.

2 **Lemma 2.1.16.** *With the preceding notation and the assumptions. The divisor E has simple normal crossings (SNC) on (X, D) if it has NC and its components are normal. If additionally (X, D) is strict then $(X, D \cup E)$ is a strict toroidal embedding, locally étale isomorphic to*

$$X_\sigma \times X_\tau = X_\sigma \times \mathbb{A}^n,$$

with D is étale isomorphic $D_1 \times X_\tau = D_1 \times \mathbb{A}^n$, and E is étale isomorphic to $X_\sigma \times D_2$.

Proof. If (X, D) is a toroidal embedding and the components of E are locally defined by free parameters on a toroidal embedding, then they are toroidal embeddings and hence normal. Conversely, if E has NC on (X, D) and its components are normal. Then, since E has NC the components are étale locally defined by a local parameter. Since the components are normal on X the irreducible components in étale neigborhood of X are the inverse image of the components on X. Thus, by Lemma 3.8.7 they are Cartier divisors defined locally by a single function which is necessarily a local parameter.

Now if additionally (X, D) is strict then all the components of $D \cup E$ are normal and there is an étale chart to $X_\sigma \times X_\tau$ by Lemma 3.8.8.

♣

Remark 2.1.17. If E is relative NC (respectively SNC) then it has (NC) SNC with the strata of (X, D). Moreover the restriction $E \cap (X \setminus D)$ is an (NC) SNC divisor on a smooth subset.

Lemma 2.1.18. *Let (X^0, D_{X^0}) be a strict toroidal embedding, and E has NC with D^0. Set $X := X^0 \setminus E$, $D_X := D_{X^0}$. Then (X^0, D_{X^0}) is the saturation of (X, D_X).*

Proof. We verify this property in an strict toroidal étale enighborhood , where it reduces to the obvious fact for toric variety that $X_\sigma \times \mathbb{A}^n$ is the saturation of $X_\sigma \times (\mathbb{A}^n \smallsetminus E)$.

♣

2.1.19. *Divisors with locally ordered components.*

order

Definition 2.1.20. Let (X, D) be a strict toroidal embedding. We say that a Weil divisor D on X has *locally ordered components* if there is given a partial order on the set of components, which is total for any subset of the components passing through a common point.

Remark 2.1.21. The condition of ordering components is unavoidable in view of Example 5.15.1!

2.1.22. *Desingularization of logarithmic varieties except of log smooth locus.*

th: resolution5

Theorem 2.1.23. *Let $(X, \mathcal{M})$ be a logarithmic variety over a field K of characteristic zero. Let V be an open toroidal subset of X [2]. Assume that the divisor D_V on V defining the smooth logarithmic (toroidal) structure on V has locally ordered components.*[3]

There exists a canonical resolution of singularities of $(X, \mathcal{M})$ except of V i.e. a birational projective morphism $f : Y \to X$ such that

1. *f is an isomorphism over the open set V .*
2. *The variety (Y, D_Y) is a strict toroidal embedding, where $D_Y := \overline{D_V}$ is the closure of the divisor D_V in Y. Moreover (Y, D_Y) is the saturation*[4] *of the toroidal subset (V, D_V) in Y. (In particular (Y, D_Y) has the same singularities as (V, D_V))*
3. *The complement $E_0 := Y \smallsetminus V$ of V in Y is a divisor with simple normal crossings (SNC) with D_Y [5]. So is the exceptional divisor $E \subseteq E_0$.*
4. *If V is the toroidal (i.e. log smooth) locus of $(X, \mathcal{M})$ then $E = E_0$.*
5. *In particular, if V is smooth and D_V is SNC divisor on V then Y is smooth, and $D_Y \cup E_0$ is an SNC divisor.*
6. *f is a composition of a sequence of the blow-ups at functorial centers.*
7. *f commutes with field extensions and smooth morphisms respecting the logarithmic structure, the subset V and the order of the components of D_V, in the sense that the centers of the blow-ups are transformed functorially, and the trivial blow-ups are omitted.*
8. *In particular, if G is an algebraic group acting on $(X, \mathcal{M})$ and preserving the logarithmic structure $\mathcal{M}$ and the subset V, and the components of D_V on (a G-stable) V then the action of G on X lifts to Y, and $f : Y \to X$ is G-equivariant.*

Proof. Let X be a logarithmic variety. We can assume that the the complement $X \smallsetminus V$ is the support of a Cartier divisor E_X. To this end consider the blow-up of the ideal of the complement $X \smallsetminus V$, and let E_X be the exceptional divisor. We can also assume that the logarithmic structure $\mathcal{M}$ is not trivial anywhere outside of V,

[2] Definition 2.1.5
[3] Definition 2.1.20
[4] Definition 2.1.7
[5] Definition 2.1.14

by replacing $\mathcal{M}$ with the saturation of the log-structure generated by $\mathcal{M}$ and E_X. Then the components of E_X are the closed irreducible strata of $\mathcal{M}$.

By [ATW16], and [ATW17] we can canonically desingularize X that is transform birationally X to a *quasi- toroidal variety* $\overline{X}$. The process is functorial with respect to logarithmically smooth morphisms, and thus it preserves a subset V.

The quasi-toroidal variety $\overline{X}$, is in particular, étale locally toric, with the singular locus defined by a locally toric divisor on $\overline{X}$. By Theorem 7.19.1, applied with respect to the open subset V, the variety $(\overline{X}, \overline{D})$ can be transformed into the strict toroidal embedding (Y^0, D^0_Y), where D^0_Y is the inverse image of $\overline{D}$. The complement of V is an SNC divisor.

♣

Remark 2.1.24. Consider the "pinch point" D defined by $y^2 = x^2 z$ in $\mathbb{A}^3$. The natural logarithmic structure on $\mathbb{A}^3$ defined by the complement of D is not fine and saturated (not even coherent). Its restriction to $V := \mathbb{A}^3 \setminus \{0\}$ defines a toroidal embedding with NC singularities. The "pinch point" singularity however cannot be resolved without modifying the NC locus V. The subset V is not a toroidal subset of $(\mathbb{A}^3, D)$. It is not a strict toroidal embedding.

2.2. Desingularization of varieties except of log smooth locus.

2.2.1. *Extendable toroidal embeddings.*

trivia **Definition 2.2.2.** Let (X, D) be a strict toroidal embedding and S be the induced stratification. For for any stratum $s \in S$ consider the monoid of the effective Cartier divisors $\mathrm{Cart}(s, S)^+$ on an open neighborhood

$$U_s := Star(s, S) = \bigcup_{\{s' \mid s \subset \overline{s'}\}} s'$$

supported on $D_s := D \cap U_s$. Then (X, D) will be called *extendable* if there exists a *Cartier system* $\Phi = \Phi(X)$, that is, a collection $\Phi = \{\Phi_s\}_{s \in S}$ of finite subsets $\Phi_s \subset \mathrm{Cart}(s, S)^+$ of Cartier divisors on U_s for $s \in S$, such that

(1) $1 \in \Phi_s$[6].
(2) $\Phi_s \subset \mathrm{Cart}(s, S)^+$ generates $\mathrm{Cart}(s, S)^+$ (as monoid).
(3) If $s \subset \overline{s'}$ then $U_{s'} \subset U_s$, and the restriction of the Cartier divisors defines a surjective map $\Phi_s \to \Phi_{s'}$ of the sets.

Example 2.2.3. The SNC divisors on smooth varieties are extendable, with the Cartier systems Φ_s defined by the components of the Weil divisors through s.

Lemma 2.2.4. *Strict toroidal embeddings with quotient singularities are extendable.*

Proof. For any strict toroidal étale neighborhood $U \to X$, define the Cartier system $\Phi(U) := \{\Phi_s(U)\}$, with $\Phi_s(U) := \mathrm{Cart}(s, U_s)^+_{\leq n}$ to be the set all the effective Cartier divisors on $U_s \subset U$ with coefficients $\leq n$. This collection is compatible in the strict étale topology.

Moreover it defines a Cartier system on each such U for sufficiently large n.

[6] The trivial Cartier divisor

Indeed, for any irreducible Weil divisor D on U let n_D denote the smallest integer for which $n_D \cdot D$ is Cartier. Let n be the integer $\geq n_D$, and such that each $\Phi_s = \mathrm{Cart}(s,S)^+_{\leq n}$ generates $\mathrm{Cart}(s,S)^+$.

If $s \subset \overline{s'}$ then any Cartier divisor $E_s = \sum m_d D$ in Φ_s extends to a Cartier divisor

$$E_{s'} = \sum m_d D + \sum s_{D'} D'$$

in $\mathrm{Cart}(s', S)^+$ by an elementary fact on the cones (See also Example 2.2.7). Then for each Weil component D' in $U_s \setminus U_{s'}$ of $E_{s'}$, the multiple of $k \cdot n_{D'} D'$ can be subtracted from $E_{s'}$ such that the coefficient $s_{D'}$ can be adjusted to a new coefficient $s'_{D'} < n_D \leq n$ in the presentation of the modified Cartier divisor

$$E'_{s'} = \sum m_d D + \sum s'_{D'} D' \in \Phi_{s'}.$$

♣

tr1 **Example 2.2.5.** If (X, D_X) is a strict toroidal embedding for which any effective Cartier divisor on $U_s \subset X$ supported on $D_X \cap U_s$ extends to an effective Cartier divisor on X supported on D_X then (X, D_X) is extendable. We can set Ψ_s to be the set of the extensions of generators of $\mathrm{Cart}(s,S)^+$ to X and put $\Psi = \bigcup_{s \in S} \Psi_s$. Then we put Φ_s to be the set of restrictions of Ψ to U_s.

In particular, any strict toroidal embedding (X, D) with a single minimal (closed) stratum is extendable.

The conditions for strict extendable toroidal embeddings can be translated into the language of associated conical complexes.

Lemma 2.2.6. *A strict toroidal embedding (X, D) is extendable if the associated complex Σ [7] satisfies the condition: There exists a functor $\mathcal{L}$ associating with each face $\sigma \in \Sigma$ a finite set of integral linear functions $\mathcal{L}_\sigma$ from N_σ to $\mathbb{Z}$, such that*

(1) $0 \in \mathcal{L}_\sigma$
(2) $\mathcal{L}_\sigma$ generates the monoid $\sigma^\vee \cap M_\sigma$ of the integral linear functions which are nonnegative on $\sigma \cap N_\sigma$.
(3) The restriction defined by a face inclusion $i_{\tau\sigma} : \tau \to \sigma$ yields the surjective map of the sets $\mathcal{L}_\sigma \to \mathcal{L}_\tau$.

We shall call such a complex *extendable.*

tr2 **Example 2.2.7.** If Σ is a complex with a single maximal face σ then it is extendable. (See also Example 2.2.5)

Proof. Follows from a, well know fact, that any nonnegative integral linear function on a face τ of σ extends to a nonnegative integral linear function on σ. (See for instance [Ful93])

♣

2.2.8. *Extension of log smooth logarithmic structures.*

extension **Lemma 2.2.9.** *Let (V, D_V) be a extendable toroidal subset of a variety (X, D). Assume that D_V has étale locally ordered components. Then there is a canonical projective birational modification $\tilde{X}$ of X which is an isomorphism on V, and a canonical extension $\mathcal{M}$ of the logarithmic structure on (V, D_V) such that*

[7] Definition 3.5.1, Theorem 3.8.16

(1) $(\tilde{X}, \mathcal{M})$ is a logarithmic variety.
(2) $\tilde{X} \setminus V$ is the support of an effective Cartier divisor.
(3) (V, D) is saturated in $(\tilde{X}, \mathcal{M})$.
(4) The extension is functorial with respect to smooth morphisms respecting V and D, dominant on strata of D_V, and respecting their order.

Proof. We can assume that $X \setminus V$ is the support of an effective Cartier divisor E by blowing up the ideal of the complement of $X \setminus V$. Let $D_X := \overline{D_V}$ be the closure of the Weil divisor D_V. Consider the natural (irreducible) toroidal stratification S) on (V, D). For any $s \in S$ consider the group $\mathrm{Cart}(s)$ of effective Cartier divisors on V. The elements in $\mathrm{Cart}(s)$ can be thought as Weil divisors on X . To get the functorial properties of the Cartier system we can choose $\Phi = \{\Phi_s\}$ to be the minimal with respect to, some a priori chosen, lexicographic order on the components.

For any $s \in S$ consider the corresponding locally closed subset $t = t(s) \in T$ defined as

$$t := \overline{s} \setminus (\bigcup_{s' < s} \overline{s'})$$

Then $t \cap V = s$, and the order $\leq$ on S defines the order on T: $t(s) \geq t(s')$ if $s \leq s'$. So $t := \bar{t} \setminus (\bigcup_{t' < t} \overline{t'})$.

For any $t = t(s) \in T$, let

$$U_t := \mathrm{Star}(t, T) = \bigcup_{t \leq t} t'$$

be an open subset of X. Then $U_t \cap V = U_s$.

Denote by Φ_t, with $t = t(s) \in T$ the set of closures of divisors in Φ_s in U_t which are Weil divisors on U_t.

If $t_1 \leq t_2$ then, $U_{t_1} \subset U_{t_2}$, and by the assumption on Φ, there is a natural surjective map of the sets of divisors on U_{t_1}: $\Phi_{t_2|U_{t_1}} \to \Phi_{t_1}$.

Note that for any point $x \in V$ there is a single stratum t passing through x, but, in general, two distinct strata $t_1, t_2 \in T$ may intersect outside of V.

For any $x \in X$ let Φ_x be the set of the Weil divisors

$$\Phi_x := \bigcup_{x \in t} \Phi_t$$

on the open subset $U_x := \bigcap_{x \in t} U_t$. Then consider the ideal sheaf

$$\mathcal{I}_x := \prod_{D \in \Phi_x} \mathcal{O}_X(-D)$$

on the open subset $U_x := \bigcap_{x \in t} U_t$.

Note that I_x is invertible on $V \cap U_x$ so the blow-up of I_x on U_x defines an isomorphism on $V \cap U_x$

We define the modification $\overline{X} \to X$ to be the blow-up of the ideal sheaf I_x on U_x. This defines the canonical birational transformations $\sigma_x : \tilde{U}_x \to U_x$, which glues coherently to $\sigma : \tilde{X} \to X$.

Indeed, let $x \in t_1, \ldots, t_k$. Then for any $z \in U_x$, the closures of the strata $t_{k+1}, \ldots, t_m$ through z contain t_i $i \leq k$. So the ideals $\mathcal{O}_X(-D)$ in the product $\mathcal{I}_z$ are the restrictions of the ideals in $\mathcal{I}_x$, by the assumption on surjectivity of $\Phi_{t'} \to \Phi_t$.

The blow ups of the ideals I_x on U_x glue to define the transformation $\sigma : \tilde{X} \to X$ which is an isomorphism on V

Moreover, any divisor $\sigma^{-1}(D)$, where $D \in \Phi_x$ is now Cartier on $\widetilde{U}_x$. For any such $\widetilde{U}_x$ consider the monoid $P(\widetilde{U}_x)$ of effective Cartier divisors on $\widetilde{U}_x$ generated by the inverse image $\sigma^{-1}(\Phi_x)$ and the divisor E. The restriction of the Cartier diviors in $P(\widetilde{U})_x$ to $V \subset \widetilde{U}$ defines surjective map to $\mathrm{Cart}(V, D_V)$. This induces the canonical coherent logarithmic structure

$$\mathcal{M}_{\widetilde{U}} = P(\widetilde{U}) \cdot \mathcal{O}^*_{\widetilde{U}} \subset \mathcal{O}_{\widetilde{U}}$$

on each $\widetilde{U}$ which glues to the coherent logarithmic structure $\mathcal{M}_{\widetilde{X}} \subset \mathcal{O}_{\widetilde{X}}$ on $\widetilde{X}$. This structure extends the toroidal logarithmic structure on V induced by D_V. Then the saturation functor *sat* transforms the variety $(\widetilde{X}, \mathcal{M}_{\widetilde{X}})$ with the coherent logarithmic structure into the variety $(\widetilde{X}^{\mathrm{sat}}, \mathcal{M}^{\mathrm{sat}}_{\widetilde{X}})$ with fine and saturated logarithmic structure.

The fact that the toroidal subset V is saturated in $(\tilde{X}, \mathcal{M})$ follows from the property that the support of $E = \tilde{X} \setminus V$ is a closed stratum (thus the union of strata) on $\tilde{X}$. ♣

2.2.10. *Desingularization of varieties except of a toroidal subset.*

th: resolution6

Theorem 2.2.11. *Let X be a variety over a field K of characteristic zero, and D be any Weil divisor on X. Let (V, D_V) be an open extendable toroidal subset of (X, D_X)*[8]*. Assume that D_V has locally ordered components* [9] *.*

There exists a canonical resolution of singularities of (X, D_X) except of the toroidal subset V i.e. a projective birational morphism $f : Y \to X$ such that

1. *f is an isomorphism over the open set V .*
2. *The variety (Y, D_Y) is toroidal embedding, where $D_Y := \overline{D_V}$ is the closure of the divisor D_V in Y.*
3. *(Y, D_Y) is the saturation of (V, D_V)*[10]*. In particular (Y, D_Y) is an extendable toroidal embedding.*
4. *The complement $Y \setminus V$ is a divisor which has simple normal crossings with D_Y* [11]*. So is the exceptional divisor E.*
5. *If D is the closure of D_V in X and (V, D_V) is saturated in (X, D) then V is the set where f is an isomorphism.*
6. *If V is the toroidal (log smooth) locus of (X, D_X) then the complement $Y \setminus V = E$ is the exceptional divisor.*
7. *$(Y, D_Y \cup E)$ is a strict toroidal embedding.*
8. *In particular, if V is smooth and $D \cap V$ is an SNC divisor on a smooth subset $V \subseteq X$ then Y is smooth and D_Y is a SNC divisor E is SNC and $D_Y \cup E$ is an SNC divisor.*
9. *f commutes with field extensions and smooth morphisms respecting the saturated toroidal subset, dominant on the strata and preserving the order of the components D_V, in the sense that the centers are transformed functorially, and the trivial blow-ups are omitted.*

[8]Definitions 2.1.5, 2.2.2
[9]Definition 2.1.20
[10]Definition 2.1.7
[11]Definition 2.1.14

(10) In particular if G is an algebraic group acting on (X, D) and preserving the components of the divisor D then the action of G on X lifts to Y, and $f : Y \to X$ is G-equivariant.

Proof. By Lemma 2.2.9 we extend canonically the logarithmic structure on (V, D_V) to X so that V is saturated in $(X, \mathcal{M})$. To finish the proof we apply Theorem 2.1.23. ♣

Remark 2.2.12. The theorem extends naturally the nonembedded Hironaka desingularization. We consider the zero divisor $D = 0$ on a variety X, and a nonemtpy nonsingular subset. Then $V = X^{ns}$ is the toroidal locus of $(X, 0)$ with a single stratum V. Let $Y \to X$ be the desingularization except of V. Since Y is the saturation of V it is nonsingular with one big smooth stratum Y which is an extension of the unique smooth stratum V on $(V, 0)$. Consequently the resolution $Y \to X$ except of V is simply a canonical Hironaka desingularization which modifies the singular locus $X \setminus X^{ns}$. Moreover the exceptional locus $E = Y \setminus X^{ns}$ is a simple normal crossing divisor.

The above theorem, in particular, implies a variant of Bierstone-Milman desingularization theorem except of SNC locus, while removing the assumption on equidimensionality. ([BM12a]).

Theorem 2.2.13. *Let X be a nonsingular variety and $Z \subset X$ be its reduced subscheme (without any restrictions on the dimension). Denote by $Z^{snc} \subset Z$ the set of the points where Z is an SNC divisor or Z is smooth.*

Then there exists a resolution of Z except of SNC locus, that is the projective birational such that

(1) ϕ is an isomorphism over $X \setminus (Z \setminus Z^{snc}) = (X \setminus Z) \cup Z^{snc}$.
(2) The strict transform of $Z_Y \subset Y$ of $Z \subset X$ is birational to Z and is the disjoint union of the distinct smooth components and the SNC divisor $\overline{Z^{snc}}$, which is the closure of Z^{snc}.
(3) The exceptional divisor E of ϕ is an SNC divisor having SNC crossings with Z_Y.

Proof. Let D be the closure of Z^{snc} in X Let $V \subset X$ be a maximal open (nonsingular) subset such that (V, D_V) is an SNC divisor. Thus, in particular such that $Z^{snc} \subseteq D \cap V$. Then the resolution except of SNC transforms X into a nonsingular variety Y with an SNC divisor D_Y which is the closure of Z^{snc}. So D_Y is the strict transform of the closure $\overline{D^{snc}}$ on X. In the process we only affected the points in $\overline{Z^{snc}} \setminus Z^{snc}$. So no smooth points of Z were affected. Moreover we created the exceptional SNC divisor $Z \setminus V$, and SNC divisor E.

♣

2.2.14. *Toroidal compactifications of toroidal varieties.* As an immediate corollary from the above we obtain the following result.

compactification

Theorem 2.2.15. *Let (X, D) be a extendable toroidal embedding over a field of characteristic zero [12]. There exists a toroidal compactification $(\overline{X}, \overline{D})$ of (X, D) such that*

[12] Definitions 2.1.5, 2.2.2

(1) $(\overline{X},\overline{D})$ *is a complete strict toroidal embedding, where* $\overline{D}$ *is the closure of* D *in* $\overline{X}$*.*
(2) $(\overline{X},\overline{D})$ *is the saturation of* (X,D) *in* $(\overline{X},\overline{D})$[13]*. In particular it is extendable.*
(3) The complement $E:=\overline{X}\smallsetminus X$ *of* X *is a divisor, which has SNC with* $\overline{D}$*.* [14]
(4) If X *is quasi-projective then* $\overline{X}$ *is projective.*
(5) Moreover if an algebraic group G *acts on* (X,D) *preserving the components then there exists a* G*-equivariant compactification* $(\overline{X},\overline{D})$ *satisfying the above properties.*

Proof. We consider the Nagata ([Nag]) or projective completion X_0 of X, and let D_0 be the closure of D in X_0. Let $(V,D_V)\subseteq(X_0,D_0)$ be the saturation of X_0. To finish the proof we apply Theorem 2.2.11 to the saturated toroidal subset $(V,D_V)\subset(X_0,D_0)$. If G acts on X we use the Sumihiro compactification theorem ([Sum]) to construct X_0. The action preserves the saturation V and lifts to $\overline{X}$. ♣

The above theorem is a natural extension of a much simpler fact of the compactification of toric varieties due to Sumihiro ([Sum]). We give here a strenghtenning of the Sumihiro result

Theorem 2.2.16. *Any (normal) toric variety* (X,D_X)*, admits an equivarinat toric compactification* $\overline{X}$ *such that* $E:=\overline{X}\smallsetminus X$ *is a (toric) divisor having SNC with the closure* $\overline{D_X}$ *of* D_X*.*

Proof. To prove the second part it suffices to apply partial (T-equivariant) toroidal desingularization to the toric compactification X_0 and its saturated subset $V=X$. (Theorem 6.6.1). ♣

2.2.17. *Desingularization except of locally toric singularities.*

compactification2

Theorem 2.2.18. *Let* X *be a variety over a field of characteristic zero. Let* $x\in X$ *be a point where* X *has a locally toric singularity. There exists a projective birational transformation* $Y\to X$*, which is an isomorphism over a certain neighborhood* U *of* $x\in X$*, and such that* (Y,D_Y) *is a strict toroidal variety for a certain Weil divisor* D_Y *on* Y*, and such that* (Y,D_Y) *has a single closed stratum, and all its strata pass through* $x\in X$*.*

Proof. Consider the structure of a toroidal embedding on a neighborhood (U,D_U) of x. We can assume (by shrinking U if necessary that the there exists a unique closed stratum on U which passes through x. This implies that (U,D_U) is extendable, by Examples 2.2.5, 2.2.7. Let $D_X=\overline{D_U}$ be the closure of D_U, and $(V,D_V)\subset(X,D_X)$ be the saturation. It suffice to apply Theorem 2.2.11 to $(V,D_V)\subset(X,D_X)$. ♣

[13]Definition 2.1.7
[14]Definition 2.1.14

complexes

3. Toroidal embeddings

3.1. **Toric varieties.** Let K be a field and $T = \mathrm{Spec}(K[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}])$ be an n-dimensional torus. Denote by

$$M := \mathrm{Hom}_{alg.gr.}(T, K^*)$$

the lattice of group homomorphisms to K^*, i.e. characters of T. Then the dual lattice $N = \mathrm{Hom}_{alg.gr.}(K^*, T)$ can be identified with the lattice of 1-parameter subgroups of T. Then the vector space $M^{\mathbf{Q}} := M \otimes \mathbf{Q}$ is dual to $N^{\mathbf{Q}} := N \otimes \mathbf{Q}$. Let (v, w) denote the relevant pairing for $v \in N, w \in M$.

Let $N \simeq \mathbf{Z}^k$ be a lattice contained in the vector space $N^{\mathbf{Q}} := N \otimes \mathbf{Q} \supset N$. By a *cone* in this paper we mean a convex set $\sigma = \mathbf{Q}_{\geq 0} \cdot v_1 + \ldots + \mathbf{Q}_{\geq 0} \cdot v_k \subset N^{\mathbf{Q}}$. A cone is *strictly convex* if it contains no line. For any cone $\sigma \subset N^{\mathbf{Q}}$ we denote by $\sigma^\vee$ the dual cone,

$$\sigma^\vee := \{m \in M^{\mathbf{Q}} \mid (v, m) \geq 0 \text{ for any } v \in \sigma\},$$

and by $(\sigma^\vee)^{\mathrm{integ}} := \sigma^\vee \cap M$ the monoid of the integral vectors in $\sigma^\vee$.

This associates with cone σ a toric affine variety $X_\sigma := \mathrm{Spec}(K[(\sigma^\vee)^{\mathrm{integ}}]) \supseteq T$.

de: fan

Definition 3.1.1. (see [Dan], [Oda88]). By a *fan* Σ in $N^{\mathbf{Q}}$ we mean a finite collection of finitely generated strictly convex cones σ in $N^{\mathbf{Q}} \supset N$ such that

- any face of a cone in Σ belongs to Σ,
- any two cones of Σ intersect in a common face.

By the *support* of the fan we mean the union of all its faces, $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$.

If σ is a face of σ' we shall write $\sigma \preceq \sigma'$.

For any set Σ of cones in N we denote by $\overline{\Sigma}$ the set

$$\overline{\Sigma} := \{\tau \mid \tau \preceq \tau' \text{ for some } \tau' \in \Sigma\}$$

To a fan Σ there is associated a *toric variety* $X_\Sigma \supset T$, obtained by gluing X_σ, where $\sigma \in \Sigma$.

It is a normal variety on which a torus T acts effectively with an open dense orbit (see [KKMSD73], [Dan], [Oda88], [Ful93]).

orbits

3.2. **The orbit stratification of toric varieties.** To each cone $\sigma \in \Sigma$ corresponds an open affine invariant subset X_σ and its unique closed orbit O_σ.

The orbits form a locally closed smooth stratification, and $\tau \preceq \sigma$ if and only if $O_\sigma \subset \overline{O_\tau}$.

de: star

Definition 3.2.1. Let Σ be a fan and $\tau \in \Sigma$. The *star* of the cone τ is defined as follows:

$$\mathrm{Star}(\tau, \Sigma) := \{\sigma \in \Sigma \mid \tau \preceq \sigma\},$$

The orbits in the closure $\overline{O_\sigma}$ of the orbit O_σ correspond to the cones of $\mathrm{Star}(\sigma, \Sigma)$.

Denote by $N_\sigma^{\mathbf{Q}}$ the subspace of $N^{\mathbf{Q}}$ spanned by the cone σ. It defines the lattice $N_\sigma := N \cap N_\sigma$. The dual space to $N_\sigma^{\mathbf{Q}}$ is isomorphic to $M_\sigma^{\mathbf{Q}} := M^{\mathbf{Q}}/\sigma^\perp$, where $\sigma^\perp := \{v \in M^{\mathbf{Q}} \mid (v, w) = 0 \mid w \in \sigma\}$.

Then the dual cone to

$$\overline{\sigma} := (\sigma, N_\sigma^{\mathbf{Q}})$$

is isomorphic to

$$\overline{\sigma}^\vee := \sigma^\vee/\sigma^\perp.$$

The subtorus T_σ corresponding to the sublattice $N_\sigma := N \cap N^{\mathbf{Q}}_\sigma$ is isomorphic to the stabilizer of the points in O_σ.

The quotient torus $\overline{T}_\sigma := T/T_\sigma$ corresponds to the quotient space $\overline{N}^{\mathbf{Q}}_\sigma := N^{\mathbf{Q}}/N^{\mathbf{Q}}_\sigma$.

It acts transitively on the big orbit $O_\sigma \subset \overline{O_\sigma}$ making $\overline{O_\sigma}$ into a toric variety.

Denote by $\pi : N^{\mathbf{Q}} \to \overline{N}^{\mathbf{Q}}_\sigma$ the projection map. The toric subvariety $\overline{O_\sigma} \supseteq O_\sigma$ corresponds to the quotient fan

$$\mathrm{Star}(\tau, \Sigma)/\tau := \{\pi(\sigma) \mid \sigma \in \mathrm{Star}(\tau, \Sigma)\}.$$

(see [KKMSD73], [Dan], [Oda88], [Ful93]).

Remark 3.2.2. The quotient complexes play important role in the algorithm. (Definition 4.4.1).

3.3. Birational morphisms of toric varieties.

Definition 3.3.1. (see [KKMSD73], [Oda88], [Dan], [Ful93]). A *birational toric morphism* of toric varieties $X_\Sigma \to X_{\Sigma'}$ is a morphism identical on $T \subset X_\Sigma, X_{\Sigma'}$.

Definition 3.3.2. (see [KKMSD73], [Oda88], [Dan], [Ful93]). A *subdivision* of a fan Σ is a fan Δ such that $|\Delta| = |\Sigma|$ and any cone $\sigma \in \Sigma$ is the union of cones $\delta \in \Delta$.

Theorem 3.3.3. *(see [KKMSD73], [Oda88], [Dan], [Ful93]) There exists a bijective correspondence between proper toric birational morphisms and the subdivisions of the fans.*

The theorem was originally stated over algebraically closed field but remains valid without this assumption with unchanged proof.

toroidal embeddings

3.4. Toroidal embeddings.

Toroidal embeddings were introduced in [KKMSD73] over algebraically closed field. The following definition over arbitrary field is essentially due to Mumford. It is closely related to the definition of Kato ([Kat89b]) who considered toroidal embeddings in a more general context of logarithmic geometry (and refer to them as logaritmically smooth varieties). It is also equivalent to Mumford's original definition over algebraically closed field.

Definition 3.4.1. A *strict toroidal embedding* (respectively toroidal embedding) is a variety X with an open subset U such that any point admits $x \in X$ an open neighborhood $V \subset X$ (respectively an étale neighborhood $f : V \to X$), and an étale morphism $\phi : (V, U_V) \to (X_\sigma, T)$, where $U_V = U \cap V$ (respectively $U_V = f^{-1}(U)$, and $\phi^{-1}(T) = U_Y$. Such a morphism is called an étale *chart*. (In the sequel and prequel we often represent a toroidal embedding (X, U) as (X, D) for the reduced divisor $D = X \setminus U$.)

Remark 3.4.2. Equivalently, a variety X over an algebraically closed field K with an open subset U is a *toroidal embedding* if for any $x \in X$ there is an isomorphism

$$\widehat{\phi} : \widehat{\mathcal{O}_{X,x}} \to \widehat{\mathcal{O}(X_\sigma)_y},$$

where X_σ is a toric variety containing a torus T and corresponding to the cone σ of the maximal dimension and $y \in O_\sigma$ is the closed orbit point, and $\widehat{\phi}$ takes the ideal of $X \setminus U$ to $X_\sigma \setminus T$.

A toroidal embedding is *strict* (or without self intersections) if , additionally, the irreducible components of the divisor $D = X \setminus U$ are normal (so they do not have intersections). ([KKMSD73])

The latter definition(s) are due to Mumford [KKMSD73]. Both conditions coincide over algebraically closed field as was shown, in particular in [Den] (see also Lemma 3.8.8). The whole theory of toric, and toroidal varieties was initially considered over algebraically closed fields, and mostly in the language of the completions of the local rings. As was observed, by Kato in [Kat89b] most of the results can be extended to the case of nonclosed fields in a more convenient language of logarithmic geometry which uses charts in the Zariski or étale topology and does not require assumption on algebraically closed base field.

conical

3.5. **Conical complexes.** The notion of the conical complex associated to strict toroidal embeddings is a natural extension of the fans associated with toric varieties.

The following definition is equivalent to the original definition from [KKMSD73]. We use this formalism, as in the later sections we are going to consider a variation of this notion in a more general setting of semicomplexes. (Definition 7.7.2)

complex1

Definition 3.5.1. By a *conical complex* Σ we mean a finite partially ordered set of finitely generated strictly convex cones σ of maximal dimension in $N^{\mathbb{Q}}_\sigma \supset N_\sigma$.

(1) For any $\tau \preceq \sigma$ there is linear injective map $i_{\tau,\sigma} : \tau \to \sigma$ such that $i_{\tau,\sigma}(\tau)$ is a face of σ, with $i_{\tau,\sigma}(N_\tau)$ saturated in $N^{\mathbb{Q}}_\sigma$. Moreover each face of σ can be presented in such a form.
(2) If $\tau \preceq \sigma \preceq \delta$ then $i_{\tau\delta} = i_{\sigma\delta} i_{\tau\sigma}$

The definition implies that the intersection of two cones is a union of common faces.

For any subset Σ_0 of the complex Σ by the *closure of* Σ_0 we shall mean the smallest subcomplex $\overline{\Sigma_0}$ of Σ containing Σ_0.

3.6. **Support of a complex.** By the *support* of a complex Σ we mean the topological space

$$|\Sigma| := \coprod_{\sigma\in\Sigma} \sigma / \sim$$

where $\sim$ is the equivalence relation generated by the inclusions $i_{\tau\sigma} : \tau \to \sigma$.

There is an inclusion $\phi_\sigma : \sigma \to |\Sigma|$, onto the closed subset $|\sigma| \subset |\Sigma|$ homeomorphic to σ.

Denote by $\mathrm{int}(\sigma)$ the relative interior of the cone σ. This means the interior of σ in $N^{\mathbb{Q}}_\sigma$. There is an inclusion $\mathrm{int}(\sigma) \to |\Sigma|$ onto locally subset $|\mathrm{int}(\sigma)|$ which allows to consider the support Σ as

$$|\Sigma| = \bigcup_{\sigma\in\Sigma} |\mathrm{int}(\sigma)|$$

In general, by the *support* of any subset Σ_0 of a complex Σ is defined as

$$|\Sigma_0| = \bigcup_{\sigma\in\Sigma_0} |\mathrm{int}(\sigma)|$$

3.7. **Mumford's definition of complexes.** Using the above we see that conical complexes define the topological which is covered by the closed cones. This allows to define conical complexes as topological spaces with a local cone structure.

Definition 3.7.1. ([KKMSD73] [Pay06]) A conical complex Σ is a topological space $|\Sigma|$ together with a finite collection Σ of closed subsets $|\sigma| \in \Sigma$ of $|\Sigma|$ such that

(1) For each $|\sigma|$ there is a finitely generated lattice M_σ of continuous functions on $|\sigma|$, and the dual lattice $N_\sigma = Hom(M, \mathbb{Z})$ in the vector space $N^{\mathbf{Q}}_\sigma = N_\sigma \otimes Q$.
(2) The natural map $\phi_\sigma : |\sigma| \to N^{\mathbf{Q}}_\sigma$ given by $x \mapsto (u \to u(x))$ maps σ homeomorphically onto a rational convex cone $\sigma := \phi_\sigma(|\sigma|)$.
(3) The inverse image under ϕ_σ of each face of σ is some $|\tau| \in \Sigma$, with $|\tau| \subset |\sigma|$ and $M_\tau = \{u_{||\tau|} : u \in M_\sigma\}$.
(4) The topological space $|\Sigma|$ is the disjoint union of the relative interiors of the $|\sigma| \in \Sigma$.

Identifying σ with $|\sigma|$ we obtain the natural maps $i_{\tau\sigma} := \phi_\sigma \phi_\tau^{-1} : \tau \to \sigma$ satisfying the conditions from Definition 3.5.1.

3.8. **Conical complexes associated with strict toroidal embeddings.**

3.8.1. *Stratifications on toroidal varieties.* Let (X, D) be a strict toroidal embedding. The intersections of the irreducible components D_i, where $i \in I$ of the divisor $D = X \smallsetminus U$ define a natural stratification S, with the closed strata defined by the irreducible components of the intersections $\bigcap_{i\in J} D_i$ for $J \subset I$, and strata defined by the irreducible components of locally closed sets

$$\bigcap_{i\in J} D_i \smallsetminus (\bigcup_{i\in I\smallsetminus J} D_i)$$

The closure of a stratum is a union of strata. This defines a natural order on the strata induced by generization:

$$s \leq s' \quad \text{iff} \quad \overline{s} \supseteq \overline{s'}.$$

3.8.2. *Stars and saturated subsets.*

Definition 3.8.3. Let (X, D) be a toroidal embedding. A subset U of X is called *saturated* if it is a union of strata.

Definition 3.8.4. By the *star* on (X, D) of a stratum $s \in S$ we mean

$$\mathrm{Star}(s, S) := \bigcup_{s \leq s'} s'$$

Immediately from the definition we get:

Lemma 3.8.5. $\mathrm{Star}(s, S)$ *is an open saturated subset. It is the smallest open subset containing a point* $x \in s$.

3.8.6. *Mumford's lemma.* The following useful results are essentially due to Mumford.

Mum **Lemma 3.8.7.** [KKMSD73], [Den] *Let $f : (X, D_X) \to (Y, D_Y)$ be an étale surjective smorphism of normal varieties mapping $x \in X$ to $y \in Y$, and such that*

(1) The components of D_Y are normal.
(2) $f^{-1}(D_Y) = D_X$.
(3) The components of D_X intersect at $x \in X$.

Then f defines a bijective correspondence between the components of D_X and the components of D_Y. The above correspondence extends to the bijective correspondence between the effective Cartier divisors supported on D_X and the effective Cartier divisors supported on D_Y

Proof. Such a correspondence exists locally for some neighborhoods U_x of $x \in X$, and U_y of $y \in Y$. It follows by adapting the arguments of the proof of [KKMSD73, Lemma 1, p.60], or more precisely the proof of [Den, Lemma 2.3]. Moreover the image of Cartier divisor at a point x is a Cartier divisor at a point y. The correspondence between Weil divisor follows from the fact that the normality of the components and thus their irreducibility is preserved. Since all the components intersect no two components of D_X can map to the same component of D_Y. This defines the correspondence between the components, and the Weil divisors supported on D_X and on D_Y.

The fact the the image of the effective Cartier divisor supported on D_X is Cartier on Y is a local question. It can be deduced from the property that $\mathcal{I}_D \subset \mathcal{O}_{Y,y}$ is principal if and only if $\mathcal{I}_D \cdot \widehat{\mathcal{O}_{Y,x}}$ is principal. The latter is equivalent to $\mathcal{I}_D \cdot \widehat{\mathcal{O}_{X,x}}$ being principal in $\widehat{\mathcal{O}_{X,x}} = \widehat{\mathcal{O}_{Y,x}} \otimes_{k(y)} k(x)$, which, in turn follows from the fact that $\mathcal{I}_D \mathcal{O}_{X,x}$ is principal. (Those facts easily follow from the observation that $\mathcal{I}_D \cdot \widehat{\mathcal{O}_{Y,x}}$ is principal iff the ideal of the initial forms $\mathrm{in}(\mathcal{I}_D) \subset \mathcal{O}/m_x \oplus \ldots \oplus m_x^n/m_x^{n+1}$ is principal, and from the faithful flatness of the completion of local ring.) Thus the bijection of the Weil divisors defines locally at every point the bijection between Cartier divisors at the point $x' \in X$ supported on D_X and the Cartier divisors at the corresponding point $y' \in Y$ supported on D_Y. Since the morphism is surjective the latter extends to the bijection between the Cartier divisors supported on D_X and on D_Y. ♣

The following result shows that the Mumford condition on normality of the components can be used to detect strict toroidal embeddings.

Mum2 **Corollary 3.8.8.** [KKMSD73, page 195 footnote]

Let (Y, D) be a toroidal embedding, and assume that the divisor D has normal components. Then (Y, D) is a strict toroidal embedding.

Proof. The question is local. Consider an étale neighborhood $(U, D_U) \to (Y, D_Y)$ of $y \in Y$ with the étale morphism $\phi : (U, D_U) \to (X_\sigma \times T, D_\sigma \times T)$ with closed orbit $O_\sigma \times T$, where σ is a cone of maximal dimension in N_σ. Replacing T with the affine space $\mathbb{A}^k$ and $X_\sigma \times T$ with $X_{\sigma_0} = X_\sigma \times \mathbb{A}^k$ we can assume that σ_0 is of maximal dimension $\dim(Y)$ and $\{x\} = \phi^{-1}(O_{\sigma_0})$ maps to y. This also enlarges the toroidal structure to (U, D^0_U) and locally on Y to (Y, D^0_Y). By the previous lemma the Cartier divisors on U which are pullbacks of the principal toric divisors on X_{σ_0}

descend to the Cartier divisors on Y in a neighborhood of y. The generators of the cone $(\sigma_0^\vee)^{\rm integ}$ generate the maximal ideal of m_y, (as they also define a maximal ideal m_x, and $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is faithully flat). They define the homomorphism $K(y)[[(\sigma_0^\vee)^{\rm integ}]] \to \widehat{O_{Y,y}}$ which extends to a sequence:

$$K[[(\sigma_0^\vee)^{\rm integ}]] \to K(y)[[(\sigma_0^\vee)^{\rm integ}]] \to \widehat{O_{Y,y}} \to \widehat{O_{X,x}} \simeq K(x)[[(\sigma_0^\vee)^{\rm integ}]],$$

where $K(y), K(x)$ are the residue fields, $K(y)[[(\sigma_0^\vee)^{\rm integ}]] \to K(x)[[(\sigma_0^\vee)^{\rm integ}]]$ is injective. Hence $K(y)[[\sigma_0^\vee]] \to \widehat{\mathcal{O}_Y, y}$ is injective and surjective. Hence

$$\phi_0 : Y \to X_{\sigma^0} = X_\sigma \times \mathbb{A}^k$$

is étale, transforming $D_\sigma \times \mathbb{A}^k$ to D_Y, where D_σ is the complement of the torus on X_σ. The morphism ϕ_0 can be easily modify into an étale morphism $\phi_1 : Y \to X_\sigma \times T$ by a generic translation along $\mathbb{A}^k$ and shrinking to T. ♣

3.8.9. *The monoids of Cartier divisors associated with strata.* .

Following [KKMSD73] we associate with a stratum the canonical monoids (commutative semigroups) and groups (lattices).

(1) $M_s^+ = {\rm Cart}(s,S)^+$ is the monoid of the Cartier divisors on $U_s := {\rm Star}(s,S)$ supported on $D \cap {\rm Star}(s,S)$
(2) M_s is the is the free abelian group of the Cartier divisors on ${\rm Star}(s,S)$ supported on $D \cap {\rm Star}(s,S)$. So M_s is a lattice which is the groupification of M_s^+
(3) $N_s := Hom(M_s, \mathbb{Z})$ is the dual lattice with the vector space $N_s^{\mathbf{Q}} := N_s \otimes_{\mathbb{Z}} \mathbb{Q}$.
(4) $\sigma_s = \{v \in N_s^{\mathbf{Q}} \mid F(v) \geq 0, \quad F \in M_s^+\}$ is the associated strictly convex cone of maximal dimension in $N_s^{\mathbf{Q}}$

toroidal-cartier **Lemma 3.8.10.** [KKMSD73] *Let $Y \subset {\rm Star}(s,S)$ be an open subset intersecting $s \in S$.*

Let $\phi : (Y,U) \to (X_\sigma, T)$ be an étale map mapping a certain $x \in s$ to a point t in the closed orbit $O_\sigma \subset X_\sigma$.

Then the group ${\rm Cart}(Y,U)$ (resp. ${\rm Cart}(Y,U)^+$) of Cartier divisors (resp. effective Cartier divisor) supported on $Y \setminus U$ is the pull-back of the group of toric Cartier divisors supported on $X_\sigma \setminus T$.

In particular,

$${\rm Cart}(Y,U) \simeq M_s \simeq M_{\overline{\sigma}} = M_\sigma / (\sigma^\perp)^{\rm integ},$$

$${\rm Cart}(Y,U)^+) \simeq M_s^+ \simeq (\overline{\sigma}^\vee)^{\rm integ} = (\sigma^\vee)^{\rm integ} / (\sigma^\perp)^{\rm integ},$$

where $\overline{\sigma} = (\sigma, N_\sigma^{\mathbf{Q}})$ is the associated cone of maximal dimension in $N_\sigma^{\mathbf{Q}} \subset N^{\mathbf{Q}}$.

The strata on Y are the pull-backs of the strata on X_σ. Moreover the relation preserves the order

$$s(\tau) \leq s(\tau') \quad \textit{iff} \quad O_\tau \leq O'_\tau \quad \textit{iff} \quad \tau' \preceq \tau.$$

The Lemma is a consequence of Lemma 3.8.7, and the properties of the orbits on toric varieties.

inclusion2 **Corollary 3.8.11.** *With the preceding notation, the cone $\overline{\sigma} = (\sigma, N_\sigma^Q)$ (defined by the chart) is dual to M_s^+. In particular the cone $\overline{\sigma}$ with the lattice N_σ is independent of chart and uniquely defined for the stratum $s \in S : \sigma_s = \overline{\sigma}$.* ♣

inclusion **Corollary 3.8.12.** [KKMSD73] *(see also* [Oda88],[Ful93]*) With the preceding notation, if* $s \leq s'$ *then* $\mathrm{Star}(s,S) \supset \mathrm{Star}(s',S)$*, is an open subset.*

For any $Y \subset \mathrm{Star}(s,S)$ *the restriction of an étale chart* $\phi : (Y,U) \to (X_\sigma, T)$ *to an open subset* $Y' = Y \cap \mathrm{Star}(s',S)$ *defines the étale morphism to* $X_{\sigma'}$*, with* $\sigma' \preceq \sigma$ *corresponding to* s'*.*

There is a natural surjection

$$\mathrm{Cart}^+(Y,U) \simeq M_s^+ \simeq (\overline{\sigma}^\vee)^{\mathrm{integ}} \quad \longrightarrow \quad \mathrm{Cart}^+(Y',U) \simeq M_{s'} \simeq (\overline{\sigma'}^\vee)^{\mathrm{integ}}$$

defined by the restriction of the Cartier divisors corresponds to the face inclusion $\overline{\sigma'} \hookrightarrow \overline{\sigma}$*.*

Proof. This translates into a a well known fact of the toric varieties and cones [KKMSD73],[Oda88], [Ful93]. If $\sigma' \preceq \sigma$, then the open immersion $X_{\sigma'} \hookrightarrow X_\sigma$ correspond to the localization $\mathcal{O}(X_{\sigma'}) = \mathcal{O}(X_\sigma)_m$ by a monomial $m \in \mathcal{O}(X_\sigma)$ corresponding to a vector $v \in (\sigma^\vee)^{\mathrm{integ}}$. Hence

$$((\sigma')^\vee)^{\mathrm{integ}} = (\sigma^\vee)^{\mathrm{integ}} + \mathbb{Z} \cdot v = (\sigma^\vee)^{\mathrm{integ}} + ((\sigma')^\perp)^{\mathrm{integ}}.$$

Consequently

$$(\overline{\sigma}^\vee)^{\mathrm{integ}} = (\sigma^\vee)^{\mathrm{integ}}/(\sigma^\perp)^{\mathrm{integ}} \twoheadrightarrow (\sigma^\vee)^{\mathrm{integ}}/(\overline{\sigma}^\perp)^{\mathrm{integ}} = (\overline{\sigma'}^\vee)^{\mathrm{integ}}.$$

♣

3.8.13. *Toroidal embeddings and logarithmic smoothness.*

KM **Theorem 3.8.14.** *(Kato-Mumford)* [Kat89b] *Let* (X,D) *be a strict toroidal variety, and* s *be the stratum through* $x \in X$*. Then*

(1) The stratum s *is locally the intersection of the components of the divisor* D
(2) $\widehat{\mathcal{O}}_{x,X} \simeq K(x)[[u_1,\ldots,u_k, M_s^+]]$*, where* $u_1,\ldots,u_k$ *generate* $\widehat{\mathcal{O}}_{x,X}/\mathcal{I}_s \simeq \widehat{\mathcal{O}}_{x,s}$*.*
(3) $\dim(s) + rank(M_s) = \dim(X)$*.*

Proof. Let $\phi : (X,U) \to (X_\sigma, T)$ be an étale map mapping a certain $x \in s$ to a point t in the closed orbit $O_\sigma \subset X_\sigma$.

The statements are valid for the toric varieties. Moreover, by corollary 3.8.12, we have $M_s^+ \simeq (\overline{\sigma}^\vee)^{\mathrm{integ}}$, so that

$$\widehat{\mathcal{O}}_{x,X} \simeq \widehat{\mathcal{O}}_{t,X_\sigma} \otimes_{K(t)} K(x) = K(x)[[u_1,\ldots,u_k,(\overline{\sigma}^\vee)^{\mathrm{integ}}]],$$

where $(\overline{\sigma}^\vee)^{\mathrm{integ}} = (\sigma^\vee)^{\mathrm{integ}}/(\sigma^\perp)^{\mathrm{integ}}$ is isomorphic to the semigroup M_s^+.

♣

3.8.15. *Conical complexes associated with strict toroidal embeddings.*

cc **Theorem 3.8.16.** *(*[KKMSD73]*) A strict toroidal embedding* (X,U) *determines uniquely an associated conical complex* Σ*. Moreover there is a bijective correspondence between the strata on a toroidal embedding* X *and faces of the complex.*

$$\tau \to s_\tau.$$

Moreover $\tau \preceq \sigma$ *iff* s_σ *is contained in the closure* $\overline{s_\tau}$ *of* s_τ*.*

Proof. The theorem was initially proven over algebraically closed field but it extends to nonclosed field using the results above: Lemma 3.8.10, Corollaries 3.8.11, 3.8.12. The conical complex $\Sigma := \{\sigma_s : s \in S\}$ is by obtained glueing of σ_s along the natural inclusion maps $\sigma_s \hookrightarrow \sigma_{s'}$ for $s \leq s'$ as in Corollary 3.8.12. The verification is streightforward. ♣

Corollary 3.8.17. *The piecewise linear functions $|\Sigma| \to \mathbb{Z}$ are in bijective correspondence with Cartier divisors on X supported on $D = X \setminus U$.*

The correspondence extends to the subsets.

saturated **Definition 3.8.18.** A subset of a toroidal embedding will be called *saturated* if it is the union of strata.

Note that this definition agrees with Definition 2.1.7

With any subset Σ_0 of Σ one can associate the constructible saturated subset

$$X(\Sigma_0) := \bigcup_{\tau \in \Sigma_0} s_\tau.$$

Then it follows immediately that

sat2 **Lemma 3.8.19.** *$X(\Sigma_0)$ is open (closed under generization) iff Σ_0 is a subcomplex.*

In particular, with a cone σ one can associate the open subset $X(\sigma) = \bigcup_{\tau \preceq \sigma} s_\tau$.

The notion of the support allows to interpret the topology of the subsets $|\Sigma_0|$ of $|\Sigma|$, and the corresponding subsets $X(\Sigma_0)$ of X

Lemma 3.8.20. *The following are equivalent for $\Sigma_0 \subset \Sigma$:*

(1) Σ_0 is a subcomplex
(2) $X(\Sigma_0) \subset X$ is open
(3) $|\Sigma_0|$ is a closed subset of $|\Sigma|$.

Proof. The property is local and can be verified for the the closed cover $|\sigma|$ of $|\Sigma|$. ♣

3.9. **Maps of conical complexes.**

map **Definition 3.9.1.** ([KKMSD73]) A map of conical complexes $f : \Sigma \to \Sigma'$, is a function which assigns to a cone $\sigma \in \Sigma$ a unique cone $\sigma' \in \Sigma'$, together with the linear map $f_{\sigma,\sigma'} : (\sigma, N^{\mathbf{Q}}_\sigma) \to (\sigma', N^{\mathbf{Q}}_{\sigma'})$, such that,

(1) $f_{\sigma,\sigma'}(N_\sigma) \subseteq (N_{\sigma'})$.
(2) $f_{\sigma,\sigma'}(\mathrm{int}(\sigma) \subset \mathrm{int}(\sigma')$.
(3) If $\tau \preceq \sigma$ then $\tau' \preceq \sigma'$ and $f_{\sigma,\sigma'} i_{\tau,\sigma} = i_{\tau',\sigma'} f_{\tau,\tau'}$.

The map f is thus induces a unique continuous map of topological spaces $|f| : |\Sigma| \to |\Sigma'|$.

Equivalently

Definition 3.9.2. ([KKMSD73], [Pay06]) A map of conical complexes $f : \Sigma \to \Sigma'$ is a continuous map of topological spaces $|\Sigma| \to |\Sigma'|$ such that, for each cone $\sigma \in \Sigma$ there is some $\sigma' \in \Sigma$ with $f(\sigma) \subseteq \sigma'$ and $f^* M_{\sigma'} \subseteq M_\sigma$.

locali **Definition 3.9.3.** A *subdivision* of a complex Σ is a map $\Delta \to \Sigma$ such that $|f|$ is a homeomorphism identifying $|\Delta| = |\Sigma|$, so that any cone $\sigma \in \Sigma$ is a union of cones $\delta \in \Delta$. A subdivision Δ of Σ which is regular is called *desingularization* of Σ.

A map $f : \Sigma \to \Sigma'$ will be called a *local isomorphism* (respectively *local linear isomorphism* if each $f_{\sigma,\sigma'}$ is an isomorphism (respectively a linear isomorphism injective on lattices).

If f is bijective and is a local isomorphism then f is called an *isomorphism*

If f is injective and is a local isomorphism then Σ is called a *subcomplex* of Σ'.

A map $f : \Sigma \to \Sigma'$ is called *a local projection* if for each $\sigma \in \Sigma$ there is a decomposition $\sigma = \sigma' \times \tau$, where τ is regular and $f_{\sigma,\sigma'} : \sigma = \sigma' \times \tau \to \sigma'$ is the projection on the first component.

Lemma 3.9.4. *Let $f : \Delta \to \Sigma$, be a subdivision, and $|f| : |\Delta| \to |\Sigma|$ be the induced homeomorphism of the topological spaces. Then*

$$\Delta^\sigma := \{\tau \in \Delta : |\tau| \subseteq |f|^{-1}(|\sigma|)\}$$

defines a fan which is the subdivision of the cone σ. ♣

3.10. **Toroidal morphisms of toroidal embeddings.** The following definitions are equivalent to definitions of log smooth morphisms in characteristic zero.

toro mor **Definition 3.10.1.** A morphism of strict toroidal embeddings $f : (X, U) \to (Y, V)$ is *strictly toroidal* if there exists the induced map of open neighborhoods $f' : (X', U') \to (Y', V')$, and a commutative diagram of

$$\begin{array}{ccc} (X', U') & \to & (X_{\sigma'}, T') \\ \downarrow f' & & \downarrow \\ (Y', V') & \to & (X_\sigma, T) \end{array},$$

with vertical morphisms étale and toric map $(X_{\sigma'}, T') \to (X_\sigma, T)$.

Definition 3.10.2. A morphism of toroidal embeddings $f : (X, U) \to (Y, V)$ is *toroidal* if there exists the induced map of open étale neighborhoods $f' : (X', U') \to (Y', V')$, and a commutative diagram as above.

3.11. **Canonical birational toroidal maps.** The following definition is equivalent to [KKMSD73, Definition 3 p.87, Definition 1 p.73] in view of Theorem 3.11.2.

Definition 3.11.1. A birational morphism of strict toroidal embeddings $f : (Y, U) \to (X, U)$ will be called *canonical toroidal* if for any $x \in s \subset X$ there exists an open neighborhood U_x of x, an étale morphism $U_x \to X_\sigma$ and a fan Δ^σ mapping to $\sigma = \sigma_s$ and the fiber square of morphisms of varieties

$$\begin{array}{ccccc} U_x \times_{X_{\sigma_s}} X_{\Delta^\sigma} & \simeq & (f^{-1}(U_x), f^{-1}(U_x) \cap U & \to & (X_{\Delta^\sigma}, T) \\ & & \downarrow f & & \downarrow \\ & & (U_x, U_x \cap U) & \to & (X_{\sigma_s}, T) \end{array}$$

Here $f^{-1}(U_x) := U_x \times_X Y$.

Note that the canonical toroidal morphisms are well defined and do not depend upon the choice of the charts. This fact can be described nicely using the following Hironaka condition:

For any geometric points x, y which are in the same stratum every isomorphism $\alpha : \widehat{X}^{\overline{K}}_x \to \widehat{X}^{\overline{K}}_y$ preserving stratification can be lifted to an isomorphism α' : $Y \times_X \widehat{X}^{\overline{K}}_x \to Y \times_X \widehat{X}^{\overline{K}}_y$ preserving stratification.

(Here $\overline{K}$ is the algebraic closure of K, $X^{\overline{K}} := X \times_{\operatorname{Spec}(K)} \operatorname{Spec}(\overline{K})$, and $\widehat{X}^{\overline{K}}_x := \operatorname{Spec}(\mathcal{O}_{X^{\overline{K}},x})$).

The Hironaka condition is extremely important when considering stratified toroidal varieties. As we can see the condition is satisfied for the canonical birational maps of strict toroidal embeddings. (Lemma 7.16.5).

sub **Theorem 3.11.2.** *(*[KKMSD73, Theorem 6 p.90]*) Let (X, U) be the strict toroidal embedding, and Σ be the associated semicomplex. Then there is a bijective correspondence between the subdivisions of Σ , and canonical proper birational toroidal maps.*

Proof. Again, the theorem was originally proven over algebraically closed field but it can be extended to arbitrary fields (with our definition). (It also can be further generalized to the case of the stratified toroidal varieties in Theorem 7.16.4).

If $Y \to X$ is any morphism of strict toroidal embeddings then the strata are mapped into strata.Moreover if $t \in T$ maps to $s \in S$ then its face $t' \preceq t$ maps to $s' \preceq s$. These and other properties of Definition 3.9.1 can be verified locally in the charts where they follow from the properties of the toric maps. Moreover we get the map of cones between the cones associated to the strata. In view of Lemma 3.8.10, and Corollaries 3.8.11, 3.8.11. The maps between the cones are independent of charts, as the correspond to the maps between the relevant groups of Cartier divisors.

Thus the restriction maps $(\operatorname{Star}(s, S_Y) \to (\operatorname{Star}(t, S_X)$, induce the maps $M^+_{s,X} \to M^+_{t,Y}$ and the dual maps of cones $\sigma_s \to \sigma_t$ commuting with face inclusions (by Lemma 3.8.12) and defining the maps of the conical complexes $f : \Sigma_Y \to \Sigma_X$. Moreover if $Y \to X$ is a canonical birational toroidal morphism then the correspondence between cones defined by the charts shows that the set $\Delta^\sigma := \{\tau \in \Delta_Y : |\tau| \subseteq |f|^{-1}(|\sigma|)\}$. Since map is proper by using valuative criterion of properness we see that $|\Delta^\sigma| = \sigma$, so f is, in fact a subdivision.

Conversely the subdivision of complexes defines locally for any chart $U \to X_\sigma$ the variety $\widetilde{U} := U \times_{X_\sigma} X_{\Delta^\sigma}$ over U. This map is independent of étale chart. The fact can be viewed using different arguments (See also Lemma 7.16.5 for a more general context). The subset $\widetilde{U}$ is the union of the canonical open subsets $\widetilde{U} = \bigcup_{\tau \in \Delta} \widetilde{U}_\tau$, where

$$\widetilde{U}_\tau := U \times_{X_\sigma} X_\tau = \operatorname{Spec} \mathcal{O}(U)(\tau^\vee)^{\mathrm{integ}} = \operatorname{Spec}(\sum_{D \in (\tau^\vee)^{\mathrm{integ}}} \mathcal{O}(U)(-D))$$

(see also [KKMSD73], page 74) Note that the Cartier divisors in $(\tau^\vee)^{\mathrm{integ}} \subset M_s$ are naturally contained in $M_s \subset \mathcal{K}(X)/\mathcal{O}^*_X$, where $K(X)$ is a constant sheaf of the rational functions on X, so $\sum_{D \in (\tau^\vee)^{\mathrm{integ}}} \mathcal{O}(U)(-D))$ is a subsheaf of $\mathcal{K}(X)$ over U.

The subsets $\widetilde{U}_\tau$ are independent of charts, and the canonically determined morphisms $\widetilde{U}_\tau \to U$ are birational. This allows to represent Y canonically by glueing the open subsets $\widetilde{U}_\tau$ over U along the subsets corresponding to their faces so that

$$Y = \bigcup_{\sigma\in\Sigma}\bigcup_{\tau\in\Delta^\sigma} \mathrm{Spec}(\sum_{D\in(\tau^\vee)^{\mathrm{integ}}} \mathcal{O}(U_{s_\sigma}(-D))$$

The natural projection $Y \to X$ is proper and separated as it is locally represented by the morphism $\widetilde{U}_\tau := U \times_{X_\sigma} X_\tau \to U$. ♣

As a corollary from the proof we get,

Corollary 3.11.3. *The strictly toroidal maps determine the maps of the conical complexes.*

Desi

4. Functorial desingularization of complexes

4.1. **Regular and singular subcomplexes.** In the sequel we shall call two cones *disjoint* of their intersection is the zero cone.

Definition 4.1.1. We say that any nonzero integral vector $v \in N_\sigma$ is *primitive* if it generates $\mathbb{Q}_{\geq 0} \cdot v \cap N_\sigma$.

Any strongly convex finitely generated cone can be written uniquely as

$$\sigma = \langle v_1, \ldots, v_k\rangle := \mathbb{Q}_{\geq 0}\cdot v_1 + \ldots + \mathbb{Q}_{\geq 0}\cdot v_k,$$

such that v_i are primitive vectors, and k is minimal. We shall call vectors v_i *vertices* of σ.

singul **Definition 4.1.2.** We say that a cone σ in $N^{\mathbb{Q}}$ is *regular* or *nonsingular* if it is generated by a part of a basis of the lattice $e_1, \ldots, e_k \in N$, written $\sigma = \langle e_1, \ldots, e_k\rangle$. If the cone is not regular it will be called *singular*. A complex Σ is *regular* or *nonsingular* if all cones $\sigma \in \Sigma$ are regular.

Lemma 4.1.3. *([KKMSD73]) Let (X, U) be the strict toroidal embedding, and Σ be the associated semicomplex. Then the cone $\sigma \in \Sigma$ is regular iff the open subset $X(\sigma)$ is nonsingular. In particular, Σ is regular if $X = X(\Sigma)$ is nonsingular.*

irreducible **Definition 4.1.4.** A cone σ is called *irreducible singular* or simply *irreducible* if it cannot be written as $\sigma = \tau \times \sigma_1$, with σ_1 being regular.

Any singular cone σ contains a unique irreducible singular face denoted by $\mathrm{sing}(\sigma)$, so we can write

$$\sigma = \mathrm{sing}(\sigma) \times \mathrm{reg}(\sigma),$$

where $\mathrm{reg}(\sigma)$ is the maximal regular face of σ disjoint with $\mathrm{sing}(\sigma)$. This follows from a simple observation that an irreducible face of $\tau \times \mathrm{reg}(\sigma)$ is contained in τ.

Denote by $\mathrm{sing}(\Sigma)$ the subset of all irreducible singular faces of Σ, and let $\mathrm{Sing}(\Sigma)$ denote its closure. The subset $\mathrm{sing}(\Sigma)$ describes the maximal components of the singular set on X.

On the other hand let $\mathrm{Reg}(\Sigma)$ the set of all the regular cones in Σ. Then $\mathrm{Reg}(\Sigma)$ corresponds to the open subset of nonsingular points on X. We see immediately from the definition that

local **Lemma 4.1.5.** *A local projection $f : \Sigma \to \Sigma'$ induces a local isomorphism*

$$\mathrm{Sing}(f) : \mathrm{Sing}(\Sigma) \to \mathrm{Sing}(\Sigma')$$

on the subcomplexes.

Proof. By definition 3.9.3, $\sigma \simeq (f(\sigma))\times\tau$, where τ is regular. So if $\sigma \in \mathrm{sing}(Sigma)$ then $f(\sigma) \simeq (\sigma)$ In particular $f(\sigma)$ is irreducible so it is in $\mathrm{Sing}(\Sigma')$. Thus f defines an isomorphism on the cones in $\mathrm{sing}(\Sigma)$ and their faces in $\mathrm{Sing}(\Sigma)$.

♣

4.1.6. *Toric divisors.*

Definition 4.1.7. Any T-stable divisor on a toric variety (X, T) will be called a *toric divisor*.

reg1 **Lemma 4.1.8.** *Let $D = \bigcup D_i$ be a toric divisor on X_σ with the components D_i such that the closed orbit O_σ is the the intersection of all D_i. Then (X_σ, D) is a strict toroidal embedding if and only if*

$$D := D_\sigma = X_\sigma \setminus T$$

(So D contains all the irreducible toric divisors on X_σ.

Proof. Let $x \in O_\sigma$ be a closed point. By Theorem 3.8.14, we get that the group of the Cartier divisors supported on D_σ on X_σ is given by

$$\mathrm{Cart}(X_\sigma, D_\sigma) = (\sigma^\vee)^{\mathrm{integ}}/(\sigma^\perp)^{\mathrm{integ}}.$$

Moreover

$$\mathrm{rank}(\mathrm{Cart}(X_\sigma, D_\sigma)) + \dim(O_\sigma) = \dim(X)$$

The latter implies that

$$\mathrm{rank}(\mathrm{Cart}(X_\sigma, D_\sigma)) = \mathrm{rank}(\mathrm{Cart}(X_\sigma, D)),$$

as D and D_σ define the same stratum O_σ. The group $\mathrm{Cart}(X_\sigma, D_\sigma)$ can be interpreted as the group of the integral functionals on $N_\sigma \subset N$, so

$$\mathrm{rank}(\mathrm{Cart}(D_\sigma)) = \mathrm{rank}(M_\sigma) = \dim(\sigma).$$

If $D \neq D_\sigma$, say there exists a component $E_i \in D_\sigma \setminus D$ corresponding to the one dimensional faces (rays) ρ_i of σ. Then $\mathrm{Cart}(X_\sigma, D)$ is a subgroup of $\mathrm{Cart}(X_\sigma, D_\sigma)$ which corresponds to a subgroup of the integral functionals vanishing on ρ_i. This implies that $\mathrm{rank}(\mathrm{Cart}(s_\sigma, D)) < \mathrm{rank}(\mathrm{Cart}(s_\sigma, D_\sigma))$, which is impossible. So $D = D_\sigma$ on X_σ. ♣

reg2 **Lemma 4.1.9.** *Let D be a toric divisor on X_σ. Then (X_σ, D) is a strict toroidal embedding if and only if $\sigma = \tau \times \sigma_1$, where σ_1 is regular, and $D = D_\tau \times X_{\sigma_1}$.*

Proof. Let O_τ be the generic orbit in the intersection of the divisor components D_i of D. Passing to X_τ we see that, by Lemma 4.1.8, that ρ_i generate τ, and $D_{X_\tau} = D_\tau$.

Now let $x \in O_\sigma$. If (X_σ, D) is a strict toroidal embedding then $s_\tau = \overline{O_\tau} = \bigcap D_i$ is the smooth toroidal stratum through x. The group $\mathrm{Cart}(X_\sigma, D) \simeq \mathrm{Cart}(X_\tau, D_\tau)$ is the subgroup of $\mathrm{Cart}(X_\sigma, D_\sigma)$ consisting of the toric Cartier divisors on X_σ supported on D.

Consequently any nonnegative integral functional F on τ extends to an integral functional $\overline{F}$ on σ, such that $\overline{F}_\rho = 0$ for all one dimensional rays $\rho = \rho_i$, of $\sigma \setminus \tau$. Consequently those rays form a face σ_1. We have the exact sequence of the monoids

$$0 \to (\tau^\vee)^{\mathrm{integ}} \to (\sigma^\vee)^{\mathrm{integ}} \to ((\sigma_1)^\vee)^{\mathrm{integ}} \to 0,$$

where $(\sigma^\vee)^{\mathrm{integ}} \to ((\sigma_1)^\vee)^{\mathrm{integ}}$ is defined by the restrictions, and the exact sequence of the corresponding lattice.

$$0 \to M_\tau \to M_\sigma \to M_{\sigma_1} \to 0.$$

Both exact sequences splits as we have the natural restriction map $(\sigma^\vee)^{\mathrm{integ}} \to ((\sigma_1)^\vee)^{\mathrm{integ}}$, and $M_\sigma \to M_{\sigma_1}$. So

$$M_\sigma \simeq M_\tau \times M_{\sigma_1},$$
$$(\sigma^\vee)^{\mathrm{integ}} \simeq (\tau^\vee)^{\mathrm{integ}} \times (\sigma_1^\vee)^{\mathrm{integ}}$$

Dualizing

$$\sigma \simeq \tau \times \sigma_1.$$

Moreover the orbit O_τ on $X_\sigma = X_{\tau\times\sigma_1}$ corresponds to $T_{\sigma_1} \times O_\tau$, and its closure is equal to $\overline{O_\tau} = O_\tau \times X_{\sigma_1}$ and since it is a smooth stratum we conclude that X_{σ_1} is smooth, and σ_1 is regular.

♣

4.2. **The determinants of subdivision.**

simplicial **Definition 4.2.1.** A cone σ is *simplicial* if it is generated over $\mathbb{Q}$ by linearly independent primitive vectors $v_1, \ldots, v_k$, written $\sigma = \langle v_1, \ldots, v_k\rangle$.

To control the singularities of the simplicial cones $\sigma = \langle v_1, \ldots, v_k\rangle$ one introduces the *multiplicity* or *determinant* $\det(\sigma)$ of the cone σ to be the absolute value of $\det(v_1, \ldots, v_k)$, where the determinant is computed with respect to any basis of the lattice N_σ. Set $\mathrm{Vert}(\sigma) := \{v_1, \ldots, v_k\}$ for the set of vertices of σ.

Let $N_{\mathrm{Vert}(\sigma)} := \bigoplus \mathbb{Z}v_i \subset N_\sigma$ be the sublattice generated by v_i.

Lemma 4.2.2. *Let $\sigma = \langle v_1, \ldots, v_k\rangle$ be a simplicial cone.*

The determinant or multiplicity $\det(\sigma)$ *of a cone σ is the index of the lattices $[N_\sigma : N_{\mathrm{Vert}(\sigma)}]$. In particular*

(1) σ is regular iff $\det(\sigma) = 1$.
(2) The order of the quotient group $\frac{N_\sigma}{N_{\mathrm{Vert}(\sigma)}}$ is equal to $n :=$ $\det(\sigma)$, *and the cosets in the quotient group $\frac{N_\sigma}{N_{\mathrm{Vert}(\sigma)}}$ have representatives which are integral vectors of N_σ of the form $\sum a_i v_i$, where $0 \le a_i < 1$, $a_i \in \frac{1}{n} \cdot \mathbb{Z}_{\ge 0}$.*

Proof. It is a well known fact. (1) $\det(\sigma) = 1$ means that the set $\mathrm{Vert}(\sigma)$ is a basis of N_σ. (2) By a triangular linear modification (which does not change determinant) one can transform $\mathrm{Vert}(\sigma)$ into a set $\{n_1e_1, \ldots, n_ke_k\}$ where $\{e_1, \ldots, e_k\}$ is a basis of N_σ so that $\det(\sigma) = \det(n_1e_1, \ldots, n_ke_k) = n_1 \cdot \ldots \cdot n_k = [N_\sigma : N_{\mathrm{Vert}(\sigma)}]$.

If $\sum a_i v_i \in N_\sigma$ then $\sum n a_i v_i \in N_{\mathrm{Vert}(\sigma)}$, so $na_i \in \mathbb{Z}$. ♣

minimall **Definition 4.2.3.** A *minimal point* or a *minimal vector* of a simplicial cone $\sigma = \langle v_1, \ldots, v_k\rangle$ is a nonzero integral vector of the form $v = a_1v_1 + \ldots + a_kv_k \in N_\sigma$ with $0 \le a_i < 1$.

Lemma 4.2.4. *All the minimal points in σ are necessarily in* $\mathrm{sing}(\sigma)$, *and conversely* $\mathrm{sing}(\sigma)$ *is the smallest face of σ containing all the minimal points.*

Proof. Follows from Definition.

♣

Abramovich **Example 4.2.5.** (D. Abramovich) Let $\sigma = \mathbb{Q}^3_{\geq 0}$, with $N^{\mathbb{Q}}_\sigma = \mathbb{Q}^3$, and the lattice

$$N_\sigma = \{(a_1, a_2, a_3) : a_i \in \mathbb{Z}, a_1 + a_2 + a_3 \in 2\mathbb{Z}\} \subset N^{\mathbb{Q}}_\sigma$$

Then σ is generated by $\{(1,1,0),(1,0,1),(0,1,1),(2,0,0),(0,2,0),(0,0,2)\}$, with the vertices $\{(2,0,0),(0,2,0),(0,0,2)\}$. Thus σ contains no minimal points in its relative interior.

Note that the semigroup (or monoid) of a cone τ containing no minimal vectors is necessarily generated by v_i, and thus τ is regular. Summarizing

min **Lemma 4.2.6.** *(1) A simplicial cone σ is regular if it contains no minimal points.*

(2) If $\det(\sigma) = n > 1$ then there exists a minimal point of the form σ, $v = a_1v_1 + \ldots + a_kv_k \in N_\sigma$ with $0 \leq a_i < 1$, where $a_i \in \frac{1}{n} \cdot N$.

4.2.7. *Smooth toroidal maps.*

toro mor2 **Definition 4.2.8.** A morphism of strictly toroidal varieties $f : (Y, D_Y) \to (X, D_X)$ will be called *smooth* (respectively *étale*) if it is a smooth (resp. étale) and $f^{-1}(D_X) = D_Y$

toro mor3 **Definition 4.2.9.** A morphism of strictly toroidal varieties $(Y, D_Y) \to (X, D_X)$ will be called *smooth toroidal* (respectively *étale toroidal*) if it is a smooth (resp. étale) and strictly toroidal. (see also Definition 3.10.1).

Remark 4.2.10. The smooth morphisms are locally étale equivalent to projections along tori, while the smooth toroidal morphisms are those corresponding to toric morphisms which are at the same time smooth.

The following Lemma illustrates some differences between two notions

Lemma 4.2.11. *(1) The smooth morphism of strictly toroidal varieties $f : (Y, D_Y) \to (X, D_X)$ is strictly toroidal morphism corresponding to local isomorphism of the associated conical complexes $\Sigma_X \to \Sigma_Y$.*

(2) The smooth toroidal morphisms determine the local projections (resp. local isomorphisms) of the associated conical complexes. Conversely if a strict toroidal map determines a local projection then it is a smooth toroidal morphism.

Proof. (1) Consider a local chart $\alpha : U \to X_\sigma$, and the induced smooth morphism $\beta := \alpha \circ f : V := f^{-1}(U) \to X_\sigma$ preserving strata. Then one can locally find a coordinate system $u_1, \ldots, u_s$ on the stratum $\beta^{-1}(O_\sigma)$ defining an étale chart $U \to X_\sigma \times \mathbb{A}^s$. We can assume that $u_1, \ldots, u_s$ are not zero so the chart is of the form $U \to X_\sigma \times T^s$.

(2) The problem reduces to the toric situation. We can assume that the relevant morphism of toric varieties is surjective:

Lemma 4.2.12. *The surjective toric map $X_\sigma \to X_\tau$ is smooth iff $\sigma \simeq \tau \times \delta$, where δ is regular.*

Proof. Since the morphism is smooth it the inverse images of toric divisor D_τ on X_τ is a toric divisor the toric divisor D on X_σ defining the toroidal structure. By Lemma 4.1.9, $D = D_{\tau_1} \times T_\delta$, where δ is regular, $\sigma = \tau_1 \times \delta$, and $X_\sigma = X_{\tau_1} \times X_\delta$, where $O(X_\delta) = K[x_1, \ldots, x_k]$.

Moreover the vertices of τ_1 map to the vertices of τ, and the vertices of δ map to zero. So the map $\sigma = \tau_1 \times \delta \to \tau$ sends δ to 0. It defines the map $(\tau_1, N_\sigma) \to (\tau, N_\tau)$ corresponding to the smooth morphism $\phi : X := X_{\tau_1, N_\sigma} \to Z := X_{\tau, N_\tau}$. Since the codimension of O_{τ_1}, and O_τ are the same it follows that $\dim(\tau_1) = \dim(\tau)$ so $\tau_1 \to \tau$ is a linear isomorphism. The fiber of a smooth morphism $\phi^{-1}(O_\tau)$ is a smooth T-invariant subvariety mapping to O_τ and thus it is equal $O_{\tau_1} \times X_\delta$, as it is the union of the orbits corresponding to the cones whose relative interiors are mapping to $\mathrm{int}(\tau)$, and those are exactly the orbits in $O_{\tau_1} \times X_\delta$. Considering the functions $x_1, \ldots, x_k$ (for which the differentials $dx_1, \ldots, dx_s$ form a basis of $\Omega_{Z/X}(O_\sigma)$) we construct the étale morphism

$$X_\sigma = X_{\tau_1, N_{\tau_1}} \times \mathbb{A}^s \to X_{\tau, N_\tau} \times \mathbb{A}^s$$

and the induced étale morphism of the subvarieties

$$\psi : X_{\tau_1, N_{\tau_1}} \to X_{\tau, N_\tau}.$$

Since ψ is unramified the ideal m_{O_τ} generates the ideal $m_{O_{\tau_1}}$. So $(\tau^\vee)^{\mathrm{integ}} \setminus \{0\}$ generates $(\tau_1^\vee)^{\mathrm{integ}}$, and M_τ generates M_{τ_1} . Since $\phi^* : M_\tau \hookrightarrow M_{\tau_1}$ is an injective map of the lattice of the same dimension, which is also surjective, it is an isomorphism. Thus $(\tau_1, N_{\tau_1}) \to (\tau, N_\tau)$ is an isomorphism. ♣

♣

4.3. **Star subdivisions.**

de: star

Definition 4.3.1. Let Σ be a conical complex and $\tau \in \Sigma$. The *star* of the cone τ, the *closed star*, and the *link* of τ are defined as follows:

$$\mathrm{Star}(\tau, \Sigma) := \{\sigma \in \Sigma \mid \tau \preceq \sigma\},$$

$$\overline{\mathrm{Star}}(\tau, \Sigma) := \overline{\mathrm{Star}(\tau, \Sigma)}$$

$$\mathrm{Link}(\tau, \Sigma) = \overline{\mathrm{Star}}(\tau, \Sigma) \setminus \mathrm{Star}(\tau, \Sigma)$$

Assuming that Σ is simplicial we define

$$\mathrm{Nerve}(\tau, \Sigma) = \{\sigma \in \overline{\mathrm{Star}}(\tau, \Sigma) \mid \tau \cap \sigma = \{0\}\}$$

If τ is a face of a simplicial cone σ then by $\mathrm{Nerve}(\tau, \sigma)$ we mean the maximal face of σ which is disjoint from τ.

Immediately from the definition we get

Nerve

Lemma 4.3.2. *Assume that Σ is simplicial. Then for any $\sigma \in \mathrm{Star}(\tau, \Sigma)$, we can write $\sigma = \tau + \delta$, where $\delta = \mathrm{Nerve}(\tau, \sigma) \in \mathrm{Nerve}(\tau, \Sigma)$* ♣

Definition 4.3.3. For any $\sigma \in \mathrm{Star}(\tau, \Sigma)$, by $\mathrm{Nerve}(\tau, \sigma)$ we mean the unique face $\delta \in \mathrm{Nerve}(\tau, \Sigma)$, such that $\sigma = \tau + \delta$.

de: star subdivision

Definition 4.3.4. Let Σ be a conical complex and v be a primitive vector in the relative interior of $\tau \in \Sigma$. Then the *star subdivision* $v \cdot \Sigma$ of Σ at v is defined to be

$$v \cdot \Sigma = (\Sigma \setminus \mathrm{Star}(\tau, \Sigma)) \cup \{\langle \sigma_v \mid \sigma \in \mathrm{Link}(\tau, \Sigma)\},$$

where $\sigma \preceq \sigma_v := \langle v \rangle + \sigma \subset \tau + \sigma$, with i_{σ, σ_v} is the restriction of the unique map$i_{\sigma, \tau + \sigma}$. Uniqueness follows

The vector v is the *center* of the star subdivision.

One can extend this definition for the purpose of functoriality

de: star subdivision **Definition 4.3.5.** Let Σ be a complex and $V = \{v_1, \dots, v_k\}$ be a set of the primitive vectors v_i in the relative interior of the cones $\tau_i \in \Sigma$ for $i = 1, \dots, k$ defining the disjoint stars $\mathrm{Star}(\tau_i, \Sigma)$.

The *star subdivision* $V \cdot \Sigma$ of Σ at V is defined to be

$$V \cdot \Sigma = v_1 \cdot \ldots \cdot v_k \cdot \Sigma = (\Sigma \setminus \bigcup_{i=1,\dots,k} \mathrm{Star}(\tau_i, \Sigma)) \cup \bigcup_{\sigma \in \mathrm{Link}(\tau_i, \Sigma)} \langle v_i \rangle + \sigma.$$

The set of vectors V is the *center* of the star subdivision.

Definition 4.3.6. A *multiple star subdivision* of Σ is a subdivision obtained as a sequence of star subdivisions at the consecutive centers $V_1, \dots, V_k$. A regular star subdivision is called a *star desingularization*

We shall assume the natural face relation on the faces in $v \cdot \Sigma$. That is If $\sigma_1 \preceq \sigma_2$, with σ_2 in $\mathrm{Link}(\tau, \Sigma)\}$ then $\sigma_1 \preceq \sigma_2(v) := \langle v \rangle + \sigma_2$, with the natural face inclusion $i_{\sigma_1, \sigma_2(v)}$

The star subdivisions at minimal points allow to resolve singularities of simplicial faces.

des **Lemma 4.3.7.** *([KKMSD73]) Let $v \in \mathrm{int}(\tau)$ be a minimal point of $\tau \in \Sigma$. Then for any cone $\sigma \in \mathrm{Star}(\tau, \Sigma)$ the resulting cones in $v \cdot \sigma$ in the the star subdivision $v \cdot \Sigma$ of the complex Σ have smaller determinants then $\det(\sigma)$.*

Proof. Let $\tau = \langle v_1, \dots, v_k \rangle$, and $\sigma = \langle v_1, \dots, v_s \rangle$, and write $v = a_1 v_1 + \ldots + a_k v_k$ with $0 \leq a_i < 1$. Then for the cone

$$\sigma_i = \langle v_1, \dots, \check{v}_i, \dots, v_s \rangle$$

we have

$$\det(\sigma_i) = |\det(v, v_1, \dots, \check{v}_i, \dots, v_s \rangle)| = a_i |\det(v_1, \dots, v_s)| = a_i \det(\sigma).$$

♣

Lemma 4.3.8. *Let Σ be a simplicial complex, and $\tau_1, \tau_2 \in \Sigma$, be two faces such that $\tau_1 \cap \tau_2 = \{0\}$. Then for any $\overline{v}_i \in \mathrm{int}(\sigma_i)$ the star subdivisions at $\overline{v}_1$ and $\overline{v}_2$ commute:*

$$\overline{v}_1 \cdot (\overline{v}_2 \cdot \Sigma) = \overline{v}_2 \cdot (\overline{v}_1 \cdot \Sigma) = \{\overline{v}_1, \overline{v}_2\} \cdot \Sigma.$$

Proof. It suffices to verify the property for a single cone $\sigma \in \Sigma$ which contains both τ_1, and τ_2. Write $\sigma = \langle v_1, \dots, v_k, w_1, \dots, w_r, u_1, \dots, u_s \rangle$, where $\tau_1 = \langle v_1, \dots, v_k \rangle$, $\tau_2 = \langle w_1, \dots, w_r \rangle$. Then the set of maximal cones of $v_1 \cdot (v_2 \cdot \sigma) = v_2 \cdot (v_1 \cdot \sigma)$ consists of the face

$$\langle \overline{v}_1, \overline{v}_2, v_1, \dots, \check{v}_i, \dots, v_k, w_1, \dots, \check{w}_j, \dots, w_r, u_1, \dots, u_s \rangle.$$

♣

subdivision **Lemma 4.3.9.** *Any multiple star subdivision or desingularization of $\mathrm{Sing}(\Sigma)$ with centers in $\mathrm{sing}(\Sigma)$ extends canonically to the subdivision and desingularization of Σ.*

Let Σ_0 be a subcomplex of Σ which contains $\mathrm{Sing}(\Sigma)$. Assume that any cone in $\Sigma \setminus \Sigma_0$ intersects Σ_0 along it face. Then any subdivision or desingularization of Σ_0 extends canonically to the subdivision or desingularization of Σ

Proof. Any subdivision of $\mathrm{sing}(\sigma)$ extends naturally to the subdivision of $\sigma = \mathrm{sing}(\sigma) \times \mathrm{reg}(\sigma)$. The extension commutes with faces so it defines subdivision of the complex.

♣

squotient

4.4. **Quotient complexes.**

quotient

Definition 4.4.1. If τ is a face of σ then we define the *quotient cone* σ/τ to be a cone $\sigma + N^{\mathbb{Q}}_{\tau}$ in a vector space $N^{\mathbb{Q}}_{\sigma}/N^{\mathbb{Q}}_{\tau}$.

In particular if $\tau \in \Sigma$ then there is a *quotient complex* $\Sigma(\tau)$ which is a collection of the quotient cones σ/τ, where $\sigma \in \mathrm{Star}(\tau, \Sigma)$. We shall write

$$\Sigma(\tau) = \mathrm{Star}(\tau, \Sigma)/\tau = \{\sigma/\tau \mid \sigma \in \mathrm{Star}(\tau, \Sigma)\}$$

It can be easily seen that the quotient complex $\Sigma(\tau)$ describes the toroidal variety associated with the closure of the stratum s_τ corresponding to τ. (See Section 3.2)

Nerve2

Lemma 4.4.2. *If Σ is simplicial and then there exists a canonical linear isomorphism between cones (without the lattice structures)*

$$\phi_\sigma : \mathrm{Nerve}(\tau, \sigma) \to \sigma/\tau,$$

(which is injective on lattices) for each $\sigma \in \mathrm{Star}(\tau, \Sigma)$.

Proof. Each face of $\sigma \in \mathrm{Star}(\tau, \Sigma)$ can be written in the form $\sigma = \langle v_1, \ldots, v_k, w_1, \ldots, w_r \rangle$, where $\tau = \langle v_1, \ldots, v_k \rangle$, and $\mathrm{Nerve}(\tau, \sigma) = \langle w_1, \ldots, w_r \rangle$ is in $\mathrm{Nerve}(\tau, \sigma)$. Also $\sigma/\tau = \langle w_1 + N^{\mathbb{Q}}_{\tau}, \ldots, w_r + N^{\mathbb{Q}}_{\tau} \rangle$. This defines a linear isomorphism $\mathrm{Nerve}(\tau, \sigma) \to \sigma/\tau$, $w_i \mapsto w_i + N^{\mathbb{Q}}_{\tau}$ (not preserving the lattices).

♣

product

Lemma 4.4.3. *If $\tau = \langle v_1, \ldots, v_k \rangle$, and $\sigma \langle v_1, \ldots, v_k, w_1, \ldots, w_k \rangle$, are simplicial cones. Assume that the images $w_1 + N^{\mathbb{Q}}_{\tau}, \ldots, w_1 + N^{\mathbb{Q}}_{\tau}$ are all primitive in the quotient space $N^{\mathbb{Q}}_{\sigma}/N^{\mathbb{Q}}_{\tau}$ with lattice $(N_\sigma + N^{\mathbb{Q}}_{\tau})/N^{\mathbb{Q}}_{\tau}$. Then*

$$\det(\sigma) = \det(\tau) \cdot \det(\sigma/\tau).$$

Proof. Consider the quotient map

$$\pi : N_\sigma \to N_{\sigma/\tau} = N_\sigma/N_\tau$$

with kernel N_τ.

Its restriction to $N_{\mathrm{Vert}(\sigma)} = \bigoplus_{v \in \mathrm{Vert}(\sigma)} \mathbb{Z} \cdot v$ defines the surjective homomorphism

$$\pi_{|N_{\mathrm{Vert}(\sigma)}} : N_{\mathrm{Vert}(\sigma)} \to \pi(N_{\mathrm{Vert}(\sigma)}) = N_{\mathrm{Vert}(\sigma/\tau)}$$

with kernel $N_{\mathrm{Vert}(\tau)}$.

This induces the surjective homomorphism $N_\sigma/N_{\mathrm{Vert}(\sigma)} \to N_{\sigma/\tau}/N_{\mathrm{Vert}(\sigma/\tau)}$ with the kernel $N_\tau/N_{\mathrm{Vert}(\tau)}$.

♣

4.5. **Canonical lifting.** Observe that for any primitive vector $[v]$ in the quotient cone σ/τ there are possibly many different lifts to the integral vectors in σ. They differ by certain integral vectors in N_τ.

The following observation is critical for the algorithm. It allows to lift canonically the desingularization of the quotient complexes to the subdivisions of the complexes.

lift **Lemma 4.5.1.** *Let $\tau = \langle v_1, \ldots, v_k\rangle$ be a regular face of a simplicial cone $\sigma = \langle v_1, \ldots, v_k, w_1, \ldots, w_s\rangle$, and $[w]$ be a primitive vector in σ/τ. Then there exists a unique primitive vector $w_{\mathrm{can}} \in \sigma$ called canonical lifting of $[w]$, such that*

$$w_{\mathrm{can}} + N_\tau^{\mathbb{Q}} = [w],$$

and

$$w_{\mathrm{can}} = \sum c_i v_i + \sum d_j w_j$$

has the minimal coefficients c_i. (The coefficients d_j are uniquely determined). Moreover

(1) *If $\sigma_j := \langle v_1, \ldots, v_k, w_j\rangle$ with $\det(\sigma_j) = n_j$ then the vector $[w_j] := \frac{1}{n_j} w_j + N_\tau^{\mathbb{Q}}$ is integral and primitive in the quotient lattice , and the quotient cone σ/τ can be written as $\sigma/\tau = \langle [w_1], \ldots, [w_s]\rangle$ and its canonical lifting is given by the vector*
$$w_{j\mathrm{can}} = \sum a_i v_i + \frac{1}{n} w_j \in \sigma,$$
with $0 \le a_i < 1$.

(2) *Let $[w] = \sum b_j [w_j] \in \sigma/\tau$ be a primitive vector in σ/τ. Then its canonical lifting $w_{\mathrm{can}} \in \sigma$ is of the form*
$$w_{\mathrm{can}} = \sum a_i v_i + \sum \frac{b_j}{n_j} w_j,$$
where $0 \le a_i < 1$,. If $[w]$ is a minimal vector then w_{can} is also a minimal vector.

Proof. (1) The group

$$N_{\sigma_j}/N_\tau = \frac{N_{\sigma_j} + N_\tau^{\mathbf{Q}}}{N_\tau^{\mathbf{Q}}} \subset \frac{N_{\sigma_j}^{\mathbf{Q}}}{N_\tau^{\mathbf{Q}}}$$

is isomorphic to $\mathbb{Z}$, and its cyclic subgroup $N_{\mathrm{Vert}(\sigma_j)}/N_\tau$ is of index n_j and generate by the coset $w_j + N_\tau$. Thus $\frac{1}{n_j} w_j + N_\tau^{\mathbf{Q}}$ defines an integral primitive vector in the one dimensional lattice $N_{\sigma_j}/N_\tau = \frac{N_{\sigma_j} + N_\tau^{\mathbf{Q}}}{N_\tau^{\mathbf{Q}}}$. So we can write $\sigma/\tau = \langle [w_1], \ldots, [w_s]\rangle \subset N_\sigma^{\mathbf{Q}}/N_\tau^{\mathbf{Q}}$.

(2) The vector $[w] = \sum b_j [w_j]$ is integral in $N_\sigma/N_\tau =$, there is an integral vector in w_{can} in

$$N_{\sigma_j}/N_\tau = \frac{N_{\sigma_j} + N_\tau^{\mathbf{Q}}}{N_\tau^{\mathbf{Q}}} \subset \frac{N_{\sigma_j}^{\mathbf{Q}}}{N_\tau^{\mathbf{Q}}},$$

which can be written in the form

$$w_{\mathrm{can}} = a_1 v_1 + \ldots + a_k v_k + \sum \frac{b_j}{n_j} w_j$$

with rational coefficients a_i. Since a_i are define in the presentation up to integers and w_{can} is the vector with minimal coordinates we see that $0 \le a_i < 1$ that it is unique. ♣

Definition 4.5.2. We shall call the vector $w_{\rm can} \in |{\rm Star}(\tau,\Sigma)|$, *the canonical lifting* of $[w] = w + N^{\mathbb{Q}}_\tau \in |{\rm Star}(\tau,\Sigma)/\tau|$.

4.6. **Regular lifting.**

Lift **Corollary 4.6.1.** *Let $\tau = \langle v_1,\ldots,v_k\rangle$ and $\sigma = \langle v_1,\ldots,v_k,w_1,\ldots,w_s\rangle$ and assume σ/τ and τ are regular. Set $\sigma_j := \langle v_1,\ldots,v_k,w_j\rangle$. Then*

1. *Each σ_j/τ is regular then $[w_j] := \frac{1}{n_j}w_j + N^{\mathbb{Q}}_\tau$ is primitive, where $n_j := \det(\sigma_j)$, and $\sigma/\tau = \sigma/\tau = \langle [w_1],\ldots,[w_s]\rangle$*
2. *The canonical lifting of $[w_j]$ is of the form*
$$(w_j)_{\rm can} = a^j_1 v_1 + \ldots + a^j_k v_k + \frac{1}{n}w_j,$$
for some $0 \le a^j_i < 1$.
We set $(w_j)_{\rm reg} := (w_j)_{\rm can}$
3. *The cone $\langle v_1,\ldots,v_k,(w_1)_{\rm reg},\ldots,(w_s)_{\rm reg}\rangle$ is regular.*
4. *The vector $[w_\sigma] := \sum [w_j] \in \sigma/\tau$ is primitive and admits a lifting of $[w]$ of the form*
$$(w_\sigma)_{\rm reg} := \sum (w_j)_{\rm reg} = \sum_{i,j} a^j_i v_i + \sum \frac{1}{n_j} w_j,$$
which is a primitive vector.
5. *If $n_j > 1$ for all j then $(w_\sigma)_{\rm reg}$ is the sum of the minimal vectors in σ.*

Proof. (1), and (2) follow from Lemma 4.5.1

(3)
$$\det(\langle v_1,\ldots,v_k,(w_1)_{\rm reg},\ldots,(w_s)_{\rm reg}\rangle) = \det(\sigma/\tau)\cdot\det(\tau) = 1,$$
by Lemma 4.4.3.

(4) The vector $(w_\sigma)_{\rm reg} := \sum (w_j)_{\rm reg}$ is primitive since $\langle (w_1)_{\rm reg},\ldots,(w_s)_{\rm reg}\rangle$ is regular.

(5) Each vector $(w_j)_{\rm reg} = a^j_1 v_1 + \ldots + a^j_k v_k + \frac{1}{n}w_j$ is primitive.

♣

Definition 4.6.2. We shall call the vector $w_{\rm reg} \in \sigma$, *the regular lifting* of $[w] \in \sigma/\tau$.

Remark 4.6.3. Note that the canonical and regular lifting coincide for the rays (of the quotient cones) but are different for the midpoints. The regular liftings are used for the quotient cones which are regular.

They lift regular quotient cones to the regular cones. However, in the case of singular cones they do not give minimal vectors.

Example 4.6.4.
1. If $\sigma = \langle v_1,\ldots,v_k,w\rangle$ is regular then $[w] := w + N^{\mathbb{Q}}_\tau$ is primitive and its canonical lifting is $w_{\rm can} = w_{\rm reg} = w$.
2. If $\sigma = \langle v_1,\ldots,v_k,w_1,\ldots,w_s\rangle$ is regular then σ/τ is regular and generated by $[w_i] := w_i + N^{\mathbb{Q}}_\tau$ and $[w] = \sum [w_i] \in \sigma/\tau$ is primitive with the canonical lifting of $[w]$ equal to $w_{\rm can} = w_{\rm reg} = \sum (w_i)_{\rm can}$.

lift1A **Corollary 4.6.5.** *The star subdivision of Σ at $w_{\rm can}$ (resp $w_{\rm reg}$) defines the star subdivision of $\Sigma(\tau)$ at*
$w_{\rm can} + N^{\mathbb{Q}}_\tau = w + N^{\mathbb{Q}}_\tau$ (resp $w_{\rm reg} + N^{\mathbb{Q}}_\tau = w + N^{\mathbb{Q}}_\tau$) such that
$$({\rm Star}(\tau, w_{\rm can}\cdot\Sigma))/\tau = (w_{\rm can} + N^{\mathbb{Q}}_\tau)\cdot(\Sigma/\tau)$$

Proof. Follows from the definition of star subdivision. ♣

4.7. **Barycenters and irreducible barycentric subdivisions.** In our desingularization algorithm one considers the canonical centers of the star subdivisions which are in the relative interiors of irreducible cones. There are several ways of doing this. One could associate with any cone $\sigma = \langle v_1, \ldots, v_k \rangle$ the canonical center $v_1 + \ldots + v_k$. However for the applications in Section 7 we need other choice of the centers in the desingularization of semicomplexes.

mi **Definition 4.7.1.** By a *minimal internal vector* of σ we mean an integral vector $v \in \mathrm{int}(\sigma)$ which and cannot be represented as the sum of nonzero integral vectors in σ, such that at least one of them is in the relative interior $\mathrm{int}(\sigma)$.

bar **Lemma 4.7.2.** *Let σ be an irreducible cone. Then the sum of all its minimal internal vectors*

$$v_\sigma := w_1 + \ldots + w_r$$

is in the relative interior of σ so can be chosen as the canonical barycenter.

barycenter **Definition 4.7.3.** Let Σ be a conical complex. By the *canonical irreducible barycentric subdivision* of Σ we mean the sequence of star subdivision at the sets of all barycenters of all the irreducible faces of the given dimension starting from the top dimension.

bar **Lemma 4.7.4.** *If Δ is a canonical irreducible barycentric subdivision of Σ then Δ is simplicial.*

Proof. If δ is a face of Δ then all its new rays (vertices) are linearly independent of the other rays. So δ has a unique maximal face τ which is in Σ, and this face is regular, as it contains no irreducible face and thus $\mathrm{sing}(\tau) = \{0\}$. Consequently $\mathrm{Vert}(\delta) \setminus \mathrm{Vert}(\tau)$ are linearly independent from $\mathrm{Vert}(\tau)$, and $\mathrm{Vert}(\tau)$ are linearly independent.

♣

4.8. **Marked complexes.** The procedure described in Lemma 5.8.1 allows to resolve singularities of simplicial complexes by applying the star subdivisions at the minimal points. Unfortunately the choice of such minimal point is highly noncanonical. In order to eliminate choices we introduce here the concept of marking.

Definition 4.8.1. A *marking* on a complex Σ is a partially ordered subset V of the set of all vertices $\mathrm{Vert}(\Sigma)$ of Σ such that the following conditions are satisfied.

(1) For any cone σ in Σ the set $V(\sigma) := V \cap \mathrm{Vert}(\sigma)$ is linearly independent of the remaining vertices in $\mathrm{Vert}(\sigma) \setminus V$. It means $\sum_{v_i \in \mathrm{Vert}(\sigma)} c_i v_i = 0$ implies that $v_i = 0$ for each $v_i \in V(\sigma)$.
(2) The order on V is total on each subset $V(\sigma)$

A complex with a marking will be called *marked*. We say that a face τ of a complex Σ, is *completely marked* if $V(\sigma) = \mathrm{Vert}(\sigma)$. A subcomplex Σ_0 is *completely marked* if all its faces are completely marked. A face is *unmarked* if $V(\sigma) = \emptyset$. The set of all unmarked faces of Σ forms the *maximal unmarked subcomplex* $U(\Sigma)$.

The emptyset $V = \emptyset$ defines the trivial marking on a complex.

mark2 **Remark 4.8.2.** Given any completely marked simplicial subcomplex Σ_0 of a marked complex, one can define the canonical order $\leq_V$ on the set of the vectors in $|\Sigma_0|$. We order the set of vertices of any face $\sigma = \langle v_1, \ldots, c_k \rangle$ of Σ_0 according to the order on V, which is total on each face of Σ. For any $v \in \sigma$ we define the lexicographic order on the the coefficients c_i in the presentation $v = \sum c_i v_i$. (Note that the face σ is simplicial, and vectors v_i are linearly independent.)

The marking naturally occurs for star subdivisions of complexes.

mark **Lemma 4.8.3.** *Let Σ be a cone with marking $V \subset \mathrm{Vert}(\Sigma)$. Let Σ' be obtained by a sequence of star subdivisions of a complex Σ at the consecutive centers $v_1, \ldots, v_k$. Then there exists a natural marking $V' := V \cup \{v_1, \ldots, v_k\} = (V \setminus \{v_1, \ldots, v_k\}) \cup \{v_1, \ldots, v_k\}$, which extends the order on $V \setminus \{v_1, \ldots, v_k\}$ and such that*

(1) All v_i are greater than vertices in $V \setminus \{v_1, \ldots, v_k\}$
(2) $v_i < v_j$ if $i < j$, and v_i, v_j are in a face of Σ.

Proof. All the new vertices of Σ' are centers of the star subdivisions. Note that the center of a star subdivision is linearly independent from other vertices in the newly constructed faces, and the properties is preserved under the consecutive star subdivisions.

Thus for any face σ' of Σ' the new vertices (the centers of the star subdivisions) are linearly independent and marked.

♣

The following is the auxiliary result used in the proof of Theorem 4.9.1

can mar **Lemma 4.8.4.** *Let Σ be a completely marked complex. There exists a canonical multiple star desingularization $V_1 \cdot \ldots \cdot V_k \cdot \Sigma$ of Σ such that*

(1) The centers lie in $|\mathrm{sing}(\Sigma)|$, and no regular faces are affected.
(2) The centers of the consecutive star subdivisions are minimal points in the interior of singular irreducible faces.
(3) The algorithm is functorial with respect to local projections and local isomorphisms of complexes, preserving the order, in the sense that the centers transform functorially with the trivial subdivisions removed.

Proof. By Remark 4.8.2, the complete marking V given for Σ defines the order on the set $|\Sigma|$.

The property is satisfied also for any subdivision of Δ. Define the following invariant for the vectors $v \in |\Delta| = |\Sigma|$:

$$\mathrm{inv}(v) = (\dim(\tau), \det(\tau), -d_1, \ldots, -d_k, \tau)$$

where $v \in \mathrm{int}(\tau), \tau \in D(\Delta), \tau = \langle w_1, \ldots, w_k \rangle$, $w_1 < \ldots < w_k$ and

$$v = d_1 w_1 + \ldots + d_k w_k, \quad 0 \leq d_i < 1,$$

Note that there is a partial order on the set of cones in Σ, defined by the comparison of the vertices ordered lexicographically. Any two cones of the same dimension which are contained in a certain face can be compared.

We apply the star subdivision at the sets of the minimal points of the consecutive subdivisions $D(\Delta)$ of Δ for which the invariant inv is maximal. If there are several points for which the invariant is minimal then any cone in $D(\Delta)$ contains at most one such a point. This is because we always take centers in the interior of the irreducible faces of maximal dimension of Σ, and in each such a face there is exactly one such a minimal point. So the corresponding stars in $D(\Delta)$ (which are contained in the stars in Σ) have disjoint the relative interia and the star subdivisions commute. We run the algorithm until the faces of the resulting subdivision have no minimal points and thus are regular. By Lemma 4.3.7, during each subdivision the maximal determinant of the subdivided faces of maximal dimension drops. So the procedure terminates. ♣

4.9. **Canonical desingularization of conical complexes.**

can des

Theorem 4.9.1. *There exists a canonical desingularization Σ' of a conical complex Σ, that is a sequence of star subdivisions $(\Sigma_i)_{i=0}^n$ of Σ such that $\Sigma_0 = \Sigma$ and $\Sigma_n = \Sigma'$ is regular. Moreover*

1. *All the centers are in $|\mathrm{sing}(\Sigma)|$, where $\mathrm{sing}(\Sigma)$ is the subset of all the irreducible singular faces of Σ [15]. The centers are in the relative interiors of irreducible faces of the intermediate subdivisions.*
2. *The centers of the consecutive star subdivisions are either sets of the sums of minimal points [16] in the interior of singular simplicial irreducible faces of the induced subdivisions, or the sets of the canonical barycenters [17]that is is the sums of the minimal internal vectors of irreducible faces.*
3. *The subdivision does not affect the set $\mathrm{Reg}(\Sigma)$ of all the regular cones.*
4. *The algorithm is functorial with respect to local projections and local isomorphisms [18] of complexes, in the sense that the centers transform functorially with the trivial subdivisions removed. In the case of surjective local isomorphisms the centers of the star subdivisions transform functorially.*

Proof. **Step 1** Recall that $\mathrm{Sing}(\Sigma)$ denotes the closure of $\mathrm{sing}(\Sigma)$. While $\mathrm{sing}(\Sigma)$ consists of irreducible singular cones its closure $\mathrm{Sing}(\Sigma)$ contains some regular cones which shall not be subdivided.

By Lemma 4.3.9, any (regular) star subdivision of $\mathrm{Sing}(\Sigma)$ extends uniquely to a (regular) star subdivision of Σ. Let $\mathrm{Reg}(\mathrm{Sing}(\Sigma))$ denote the subcomplex of $\mathrm{Sing}(\Sigma)$ consisting of all regular cones in $\mathrm{Sing}(\Sigma)$.

Consider the canonical irreducible barycentric subdivision $B(\Sigma)$ of Σ (as in Lemma 5.9.2). By Lemma 4.8.3, we create some marking on the set of new vertices , defined by the dimensions of the subdivided cones. The subdivision will not affect any regular cones, and all the cones from $\mathrm{sing}(\Sigma)$ will be subdivided. This induces a simplicial subdivision of Σ, with the induced marking V and such that its maximal unmarked subcomplex $U(B(\mathrm{Sing}(\Sigma))$ of $B(\mathrm{Sing}(\Sigma))$ consists of $\mathrm{Reg}(\mathrm{Sing}(\Sigma))$- the regular part of $\mathrm{Sing}(\Sigma)$. Indeed, the untouched faces of $\mathrm{Sing}(\Sigma)$ are exactly the regular cones.

Step 2 For simplicity denote by $\mathrm{Sing}(\Sigma)$ the subdivision $B(\mathrm{Sing}(\Sigma))$ after Step 1 as well as all the further subdivision of $\mathrm{Sing}(\Sigma)$. We shall induct on the dimension $k := \dim(U(\mathrm{Sing}(\Sigma)))$.

[15]Definition 4.1.4
[16]Definition 4.2.3
[17]Lemma 4.7.4
[18]Definition 3.9.3

Let $U(\mathrm{Sing}(\Sigma))(k)$ be the subset of all the faces in $U(\mathrm{Sing}(\Sigma))$ of the maximal dimension k. Observe that, by maximality the corresponding stars

$$\mathrm{Star}(\tau, \mathrm{Sing}(\Sigma))$$

are necessarily disjoint for distinct $\tau \in U(\mathrm{Sing}(\Sigma))(k)$.

Write any such a cone τ as $\tau = \langle v_1, \ldots, v_k\rangle$. Let $\sigma \in \mathrm{Star}(\tau, \mathrm{Sing}(\Sigma))$. Then $\tau = \langle v_1, \ldots, v_k\rangle$ is regular. Note that all vectors $w_1, \ldots, w_r$ are marked. Indeed otherwise, say w_1 is unmarked. Then the face $\langle v_1, \ldots, v_k, w_1\rangle$ is untouched, and as such it is regular which contradicts to the maximality of τ. Thus, in particular, $\sigma = \langle v_1, \ldots, v_k, w_1, \ldots, w_r\rangle$ is simplicial.

The star $\mathrm{Star}(\tau, \mathrm{Sing}(\Sigma))$ determines the quotient complex

$$\Sigma_{\mathrm{sing}}(\tau) := \mathrm{Star}(\tau, \mathrm{Sing}(\Sigma))/\tau = \{\sigma/\tau \mid \sigma \in \mathrm{Star}(\tau, \mathrm{Sing}(\Sigma))\},$$

(see Definition 4.4.1) which is completely marked by the previous remark.

By Lemma 4.8.4, there exists a canonical desingularization

$$D_\tau(\Sigma_{\mathrm{sing}}(\tau)) := \overline{u}_s \cdot \ldots \cdot \overline{u}_1 \cdot \Sigma_{\mathrm{sing}}(\tau)$$

of $\Sigma_{\mathrm{sing}}(\tau)$ of the completely marked quotient complex $\Sigma_{\mathrm{sing}}(\tau)$ at $\overline{u}_1 := u_1 + N^{\mathbb{Q}}_\tau, \ldots, \overline{u}_s := u_s + N^{\mathbb{Q}}_\tau$.

Using Lemma 4.5.1, we shall lift the centers $\overline{u}_i$ of the star subdivisions consecutively in the canonical way. The lifting $(u_i)_{\mathrm{can}}$ of $\overline{u}_i$ is done within the cone of $\mathrm{Star}(\tau, (u_{i-1})_{\mathrm{can}} \cdot \ldots \cdot (u_1)_{\mathrm{can}} \cdot \mathrm{Sing}(\Sigma))$ corresponding to the cone of $\overline{u}_{i-1} \cdot \ldots \cdot \overline{u}_1 \cdot \Sigma_{\mathrm{sing}}(\tau)$ containg $\overline{u}_i$ in its relative interior.

The procedure is done for all such τ of dimension k. By Lemma 4.6.5, by lifting the canonical star subdivisions at $\overline{u}_1, \ldots, \overline{u}_s$ we obtain the canonical multiple star subdivision

$$D_\tau(\mathrm{Sing}(\Sigma)) := (u_s)_{\mathrm{can}} \cdot \ldots \cdot (u_1)_{\mathrm{can}} \cdot \mathrm{Sing}(\Sigma)$$

of $\mathrm{Sing}(\Sigma)$, for which

$$\mathrm{Star}(\tau, D_\tau(\mathrm{Sing}(\Sigma)))/\tau = D_\tau(\Sigma_{\mathrm{sing}}(\tau))).$$

Note that by Lemma 4.5.1, we used the canonical lifts of the centers are necessary minimal vectors of the cones in the intermediate subdivisions.

All the cones σ'/τ in $D_\tau(\Sigma_{\mathrm{sing}}(\tau)))$ are regular. However the relevant cones $\sigma' \in \mathrm{Star}(\tau, D_\tau(\mathrm{Sing}(\Sigma)))$ need not to be regular yet. Moreover the procedure of desingularization the cones in $\mathrm{Star}(\tau, D_\tau(\mathrm{Sing}(\Sigma)))$ shall be done in a canonical way.

The vertices in the quotient complexes $D_\tau(\Sigma_{\mathrm{sing}}(\tau)))$ correspond to $k{+}1$-dimensional faces $\rho = \tau + \langle v\rangle \in \mathrm{Star}(\tau, D_\tau(\mathrm{Sing}(\Sigma)))$, where $\langle v\rangle = \mathrm{Nerve}(\sigma, \tau)$ is the corresponding one dimensional face.

For any singular irreducible face $\sigma' \in \mathrm{Star}(\tau, D_\tau(\mathrm{Sing}(\Sigma)))$ write

$$\sigma' = \langle v_1, \ldots, v_k, w_1, \ldots, w_r\rangle$$

with $\tau = \langle v_1, \ldots, v_k\rangle$. In order to apply Lemma 4.4.3 , and deduce regularity of σ', we need to ensure that the induced vectors $\overline{w}_j = w_j + N^{\mathbb{Q}}_\tau$ in the quotient space $(N(\sigma') + N^{\mathbb{Q}}_\tau)/N^{\mathbb{Q}}_\tau$ are primitive. This is the case for all vertices $(u_i)_{\mathrm{can}}$ which are the canonical liftings.

In other words we need to ensure that all $k{+}1$- dimensional faces $\rho = \langle v_1, \ldots, v_k, w\rangle$ in $\mathrm{Star}(\tau, D_\tau(\mathrm{Sing}(\Sigma)))$ are regular.

Let $\rho_i = \tau + \langle w_i \rangle$ be the set of all $k+1$- dimensional singular faces in $\mathrm{Star}(\tau, D_\tau(\mathrm{Sing}(\Sigma)))$.

Let $(w_1)_{\mathrm{can}} = (w_1)_{\mathrm{reg}}, \ldots, \ldots (w_s)_{\mathrm{can}} = (w_s)_{\mathrm{reg}}$ denote their canonical (so also regular) liftings of all the corresponding vectors $\overline{w_i} = w_i + N_\tau^{\mathbb{Q}} \in \rho_i/\tau$. By Lemma 4.6.1, the correspondence $\overline{w}_i \mapsto (w_i)_{\mathrm{reg}}$ extends to the correspondence between the regular cones $\tau' = \langle \overline{w}_{i_1}, \ldots, \overline{w}_{i_s} \rangle \in D^0_\tau(\Sigma_{\mathrm{sing}}(\tau)))$ and the regular cones $\tau_{\mathrm{reg}} := \langle (w_{i_1})_{\mathrm{reg}}, \ldots, (w_{i_s})_{\mathrm{reg}} \rangle \subset \langle (w_{i_1})_{\mathrm{reg}}, \ldots, (w_{i_s})_{\mathrm{reg}}, v_1 \ldots, v_k \rangle$. Consequently we define the midpoints $\overline{w}_\tau := \overline{w}_{i_1} + \ldots + \overline{w}_{i_s}$, and their regular liftings $(w_\tau)_{\mathrm{reg}} := (w_{i_1})_{\mathrm{reg}} + \ldots + (w_{i_s})_{\mathrm{reg}}$.

Then consider the star subdivision $D'_\tau(\Sigma_{\mathrm{sing}}(\tau)))$ of $D_\tau(\Sigma_{\mathrm{sing}}(\tau)))$ at the centers $\overline{w}_{\tau'}$ for all τ' with vertices in Vert^0, starting from the cones τ' of top dimensions down to in one dimensional faces. This order will ensure the canonicity of the subdivisions, as no cone will contain more than one center in the process.

Let $D'_\tau(\mathrm{Sing}(\Sigma))$ be the corresponding star subdivision of $D_\tau(\mathrm{Sing}(\Sigma))$ at w^{can}_σ. Then as before

$$\mathrm{Star}(\tau, D'_\tau(\mathrm{Sing}(\Sigma)))/\tau = D'_\tau(\Sigma_{\mathrm{sing}}(\tau))).$$

Then $D'_\tau(\Sigma_{\mathrm{sing}}(\tau)))$ is regular. Moreover all the cones σ'/τ in $D_\tau(\Sigma_{\mathrm{sing}}(\tau)))$ have primitive vertices:

$$\mathrm{Vert}(\sigma'/\tau) = \{w_j \mid j \in J\},$$

such that

$$\mathrm{Vert}(\sigma') = \{\overline{w}^{\mathrm{can}}_j \mid j \in J\} \cup \mathrm{Vert}(\tau)$$

Thus, by Lemma 4.4.3,

$$\det(\sigma'/\tau)\det(\tau) = \det(\sigma') = 1,$$

and all the subdivision $\mathrm{Star}(\tau, D'_\tau(\mathrm{Sing}(\Sigma))$ are regular. In the process we use regular lifts which are the sums of minimal vectors.

Then, for the constructed subdivision $D(\Sigma)$ of Σ, the unmarked subcomplex $U(\mathrm{Sing}(D(\Sigma))$ is contained in $U(\mathrm{Sing}(\Sigma))$ since the subdivisions do not create new unmarked subdivision. On the other hand the unmarked complex $U(\mathrm{Sing}(D(\Sigma))$ is disjoint from the faces in $U(\mathrm{Sing}(\Sigma))(k)$ of dimension k, since the latter are no longer in the closure of the singular locus $\mathrm{Sing}(D(\Sigma))$, as all the cones in their stars are regular.

Consequently for the subdivision $D(\Sigma)$ of Σ, constructed above, the unmarked complex $U(\mathrm{Sing}(D(\Sigma))$ is of dimension $\leq k-1$.

Replacing Σ with $D(\Sigma)$, defines a new complex for which $\dim(U(\mathrm{Sing}(\Sigma)) \leq k-1$.

We repeat the procedure down to dimension 0. We obtain a (canonical) subdivision $D(\Sigma)$ of Σ such that

$$\dim(U(\mathrm{Sing}(D(\Sigma))) = \{0\}$$

. This means that $\mathrm{Sing}(D(\Sigma))$ is completely marked.

Step 3 By Lemma 4.8.4 the produced completely marked subdivision $\mathrm{Sing}(D(\Sigma))$ can be canonically resolved. The desingularization of $\mathrm{Sing}(D(\Sigma))$ determines the desingularization of $D(\Sigma)$.

Commutativity with local projections follows immediately from Lemmas 4.1.5, and 4.8.4.

♣

relative complexes reli

5. Desingularization of relative complexes

5.1. Relative complexes.

Definition 5.1.1. By a *relative conical complex* (Σ, Ω) we mean a pair of a conical complex Σ and its subcomplex $\Omega \subseteq \Sigma$.

We say that a cone $\sigma \in \Sigma$ is *balanced* if it contains a unique maximal face ω which is in Ω. In such a case we shall call (σ, ω) a *pair* in (Σ, Ω), and write $(\sigma, \omega) \in (\Sigma, \Omega)$. The relative complex (Σ, Ω) is balanced if any $\sigma \in \Sigma$ is balanced.

Let $(\sigma, \omega) \in (\Sigma, \Omega)$, where $\sigma = \langle v_1, \ldots, v_k, w_1, \ldots, w_s\rangle \in \Sigma$ be a cone with the maximal face $\omega = \langle w_1, \ldots, w_s\rangle \in \Omega$. We say that a pair (σ, ω) is *regular* if $\mathrm{sing}(\sigma) \preceq \omega$. We say that (σ, ω) is *simplicial* if $v_1, \ldots, v_k$ are linearly independent from $w_1, \ldots, w_s$. Equivalently we say σ is *relatively regular* (respectively *relatively simplicial*) if σ is a a balanced cone with a maximal face $\omega \in \Omega$, and (σ, ω) is regular (resp. simplicial). If σ is not relatively regular (in particular, if it is not balanced or simplicial) it will be called *relatively singular*.

The relative complex (Σ, Ω) is balanced (respectively regular or simplicial) if any cone $\sigma \in \Sigma$ is is balanced (resp. relatively regular or relatively simplicial).

reli2

5.2. Relative irreducibility.

As in the previous case of $\Omega = \{0\}$ we can introduce the relative versions of irreducibility. Let (Σ, Ω) be a relative complex. We say that σ is *relatively irreducible* if it contains no proper face τ which contains both $\mathrm{sing}(\sigma)$ and $\sigma \cap |\omega|$. Any relatively irreducible face which is not in Ω is singular.

Any cone σ contains a unique maximal relatively irreducible face denoted by $\mathrm{sing}_\Omega(\sigma)$, which is the minimal face containing $\mathrm{sing}(\sigma)$ and $\sigma \cap |\omega|$.

By definition $\mathrm{sing}(\sigma) \subseteq \mathrm{sing}_\Omega(\sigma)$, and thus as before we can write

$$\sigma = \mathrm{sing}_\Omega(\sigma) \times \mathrm{reg}_\Omega(\sigma),$$

where $\mathrm{reg}_\Omega(\sigma)$ is the maximal regular face of σ disjoint with $\mathrm{sing}_\Omega(\sigma)$. We shall need the following

relative regular

Lemma 5.2.1. $\mathrm{sing}_\Omega(\sigma) \in \Omega$ *iff* σ *is relatively regular.*

Proof. If $\omega := \mathrm{sing}_\Omega(\sigma) \in \Omega$ then $\omega = \sigma \cap |\omega|$ is a unique maximal face of σ in Ω. Moreover $\mathrm{sing}(\sigma) \subset \omega$.

Conversely, suppose that σ is regular. Then its balanced and $\mathrm{sing}(\sigma) \subset \omega$, where ω is a unique maximal face of σ in Ω. Then $\mathrm{sing}_\Omega(\sigma) = \omega \in \Omega$.

♣

Denote by $\mathrm{sing}(\Sigma, \Omega)$ the subset of all relatively irreducible faces of Σ, and let $\mathrm{Sing}(\Sigma, \Omega)$ denote its closure (the smallest subcomplex of Σ containing $\mathrm{sing}(\Sigma, \Omega)$).

On the other the set $\mathrm{Reg}(\Sigma, \Omega)$ the set of all the relatively regular cones in Σ.

Lemma 5.2.2. *(1)* $\mathrm{sing}(\Sigma, \Omega)$ *contains* Ω.
(2) $\mathrm{Reg}(\Sigma, \Omega)$ *is a subcomplex of* Σ *containing* Ω.
(3) $\mathrm{sing}(\Sigma, \Omega) \cap \mathrm{Reg}(\Sigma, \Omega) = \Omega$.

Proof. (1) follows from definition.

(2) if σ is relatively regular with maximal cone $\omega \preceq \sigma$ which is in Ω. and τ is its face then $\omega \cap \tau \in \Omega$ is maximal in τ. Moreover $\mathrm{sing}(\tau) \subseteq \mathrm{sing}(\sigma) \subseteq \omega$, hence $\mathrm{sing}(\tau) \subseteq \omega \cap \tau$. So $\mathrm{Reg}(\Sigma, \Omega)$ is a subcomplex of Σ.

(3) If σ is relatively regular and irreducible then $\sigma = \mathrm{sing}_\Omega(\sigma) = \omega$. ♣

Remark 5.2.3. The results are interpreted geometrically in Lemma 7.5.8 in the toric situation. In the toroidal case they have exactly the same meaning. The set $\mathrm{sing}(\Sigma, \Omega)$ describes the strata in S in the singular locus defined by the singularity type and the divisor $D = \overline{D_\Omega}$ (as in Definition 5.3.3. The meaning of the complex $\mathrm{Reg}(\Sigma, \Omega)$ is explained also in Lemma 7.5.8 (in the toric situation) as the saturation of the set defined by Ω.

5.3. Regular relative complexes and SNC divisors.

toroidd **Definition 5.3.1.** Let (X, D_X) be a toroidal embedding. By a *toroidal divisor* on X we mean a reduced divisor $D \subset D_X$. By the *closed strata* of $D = \bigcup_{i\in J} D_i$ with irreducible Weil components D_i we mean the irreducible components of the intersections $\bigcap_{i\in I} D_i$ of D_i. Denote by S_D the induced stratification with strata the components of

$$\bigcap_{i\in I} D_i \smallsetminus (\bigcup_{i\in(J\smallsetminus I)} D_i),$$

and by $\overline{S}_D$ the set of the induced closed strata.

Note that the strata of D are the union of some strata on the toroidal embeddings (X, D_X) (defined by D_X). This means that S_D is coarser then the canonical stratification on (X, D_X) and the strata in S_D are not necessarily smooth.

rr **Lemma 5.3.2.** *(X, D_X) be a strict toroidal embedding with associated complex Σ. There exists a bijective correspondence between the saturated open subsets $V = X(\Omega)$ of (X, D_X) and the subcomplexes $\Omega \subset \Sigma$.*

Proof. The Lemma is rephrasing of Lemma 3.8.19.

♣

omega **Definition 5.3.3.** Let (X, D_X) be a toroidal embedding with $U = X \smallsetminus D_X$ and with associated complex Σ. Then for any complex $\Omega \subset \Sigma$ we define $D_\Omega := X(\Omega) \smallsetminus U$.

Definition 5.3.4. A subcomplex $\Omega \subset \Sigma$ will be called *saturated* if any cone of σ with vertices (one dimensional faces) in Ω is in Ω.

Recall that by Theorem 3.8.16, there is a bijective correspondence between the strata in the canonical stratification S on a strict toroidal embedding, (X, D) with the associated complex Σ and faces $\sigma \in \Sigma$: $\sigma \mapsto s_\sigma$

cr **Lemma 5.3.5.** *Let (X, D_X) be a strict toroidal embedding, with associated complex Σ. There is a bijective correspondence between the toroidal divisors $D \subset D_X$ and the saturated subcomplexes Ω of Σ. Let D be a toroidal divisor on X. Then there is a unique saturated subcomplex $\Omega \subset \Sigma$ such that*

(1) The cones $\sigma \in \Omega$ are defined by the closed strata $\overline{s_\sigma}$ which are in $\overline{S}_D$, and for which (X, D) is a toroidal at s_σ.
(2) The open saturated subset $X(\Omega)$ intersects all strata of D such that $D = \overline{D_\Omega}$.
(3) The toroidal locus of (X, D) is defined by the (saturated) relative complex $\mathrm{Reg}(\Sigma, \Omega)$. *(Definition 2.1.9)*

Proof. Note that Ω is defined by the set of vertices in Σ, corresponding to the components of D, and the cones $\sigma \in \Sigma$ which are generated by these vertices. If $\sigma \in \Sigma$ defines the closed stratum $\overline{s_\sigma}$ in S_D, and for which (X, D) is a toroidal at s_σ then, by Lemma 4.1.8, D locally corresponds to D_σ in a neighborhood of s_σ, so σ and all the rays and all the faces of σ are in Ω.

In particular, Ω is a complex saturated in Σ. It defines the open saturated subset $V = X(\Omega)$ of X. Moreover by the above $V = X(\Omega)$ intersects all the divisorial strata of D, so all the strata on V are toroidal, and $S_{D|V} = S_{|V}$, and $(V, D \cap V)$ is a toroidal embedding, with the stratification defined by $D_{|V} = D_\Omega$, corresponding to the subcomplex Ω, and D is the closure of D_Ω. Part (3) follows immediately from Lemma 5.3.6 below. ♣

regular **Lemma 5.3.6.** *Let (X, D_X) be a strict toroidal embedding with the associated complex Σ. Let (Σ, Ω) be a regular relative complex. Then*

(1) $D = \overline{D_\Omega}$
(2) $E := \overline{D_X \setminus D}$ is a relative SNC divisor on (X, D).
(3) (X, D) is the saturation of $(X(\Omega), D_\Omega)$.

Proof. With the preceding notation, we reduce the theorem to the toric variety $X_\sigma = X_\omega \times X_\tau = X_\omega \times \mathbb{A}^k$. By Lemma 4.1.9, the divisor $D \cap X(\Omega) = D_\omega$ corresponds the maximal toric divisor on $D_\omega \times T$ with its closure $D_\omega \times \mathbb{A}^k$ defining the toroidal structure $(X_\sigma, D_\omega \times \mathbb{A}^k) = (X_\omega \times \mathbb{A}^k, D_\omega \times \mathbb{A}^k)$ on X_σ. The divisor $E = X \setminus V$ corresponds to $X_\omega \times D_\tau$ has SNC crossings with $D_\omega \times \mathbb{A}^k$. ♣

5.4. Subdivisions of relative complexes.

Definition 5.4.1. A map $f : (\Sigma, \Omega) \to (\Sigma', \Omega')$ is a map of complexes $\Sigma \to \Sigma'$ which induces a map of the subcomplexes $\Omega \to \Omega$.

A *subdivision* of a relative complex (Σ, Ω) is a subdivision $\Delta \to \Sigma$ which is identical on $\Omega \subset \Delta$. A subdivision Δ of Σ which is regular is called *desingularization* of Σ.

A map $f : (\Sigma, \Omega) \to (\Sigma', \Omega')$ is a *local isomorphism* (respectively *local linear isomorphism*) if each $f_{\sigma,\sigma'}$ is an isomorphism (respectively a linear isomorphism injective on lattices) mapping $\Omega \cap \sigma$ to $\Omega' \cap \sigma'$.

A map $f : (\Sigma, \Omega) \to (\Sigma', \Omega')$ is an isomorphism if f is a bijection of the sets and a local isomorphism.

A map $f : (\Sigma, \Omega) \to (\Sigma', \Omega')$ is called *a local projection* if for each $\sigma \in \Sigma$ there is a decomposition $\sigma \simeq \sigma' \times \tau$, where τ is regular, which takes $\Omega \cap \sigma$ to $(\Omega' \cap \sigma') \times \tau$, such that $f_{\sigma,\sigma'} : \sigma \simeq \sigma' \times \tau \to \sigma'$ is the projection on the first component.

We see immediately that

local2 **Lemma 5.4.2.** *A local projection $f : (\Sigma, \Omega) \to (\Sigma', \Omega')$ induces a local isomorphism*

$$\operatorname{Sing}(f) : \operatorname{Sing}(\Sigma, \Omega) \to \operatorname{Sing}(\Sigma', \Omega')$$

on the subcomplexes.

5.5. Minimal vectors in relative complexes.

minrel **Definition 5.5.1.** Let $\sigma = \langle v_1, \dots, v_k, w_1, \dots, w_s \rangle \in \Sigma$ be a relatively simplicial cone with a unique maximal face $\omega = \langle w_1, \dots, w_s \rangle \in \Omega$.

A *minimal vector* of (σ,ω) is a vector which can be written down in the form $\sum c_iv_i+\sum d_jw_j$, where

(1) $0\leq c_i<1$
(2) at least one $c_i\neq 0$

Lemma 5.5.2. *Assume that the pair (σ,ω) is simplicial. Then (σ,ω) is regular if there is no minimal vector of (σ,ω).*

Proof. If (σ,ω) is singular simplicial then the images of vectors $v_1,\ldots,v_k$ in N_σ/N_τ are linearly independent but do not generate the whole lattice N_σ/N_τ, just its sublattice of finite rank. Then there are the coefficients $0\leq c_i<1$ not all equal zero for which $\sum c_iv_i+N_\tau\in N_\sigma$. Thus there exists a minimal vector

$$\sum c_iv_i+\sum d_jw_j\in N_\sigma\cap\sigma,\quad d_j\geq 0$$

of (σ,ω). Conversely, if (σ,ω) is regular then $\sigma=\tau\times\omega$, where $\tau=\langle v_1,\ldots,v_k\rangle$ is regular, and $\omega=\langle w_1,\ldots,w_s\rangle$, so (σ,ω) contains no minimal vectors.

♣

5.6. **Determinants of relative subdivision.** We introduce the measure for the singularity of the relative simplicial cones:

If (σ,ω) is simplicial with $\omega=\langle w_1,\ldots,w_s\rangle$, and $\sigma=\langle v_1,\ldots,v_k,w_1,\ldots,w_s\rangle$ then we put

$$\det(\sigma,\omega):=N_\sigma/(N_{\mathrm{Vert}(\sigma)}+N_\omega)=\det(\overline{v}_1,\ldots,\overline{v}_k),$$

where $\overline{v}_1,\ldots,\overline{v}_k$ are the images of $v_1,\ldots,v_k$ are in N_σ/N_ω. The following lemma is a consequence of the definition:

regular2

Lemma 5.6.1. *If (σ,ω) is simplicial then* $\det(\sigma,\omega)=1$ *iff (σ,ω) is regular.*

Proof. If $\det(\sigma,\omega)=1$ then $v_1,\ldots,v_k$ generate that lattice N_σ/N_ω. So $\sigma\simeq\langle v_1,\ldots,v_k\rangle\times\omega$ is relatively regular.

♣

product2

Lemma 5.6.2. *If $\omega=\langle v_1,\ldots,v_k\rangle$, and $\sigma\langle v_1,\ldots,v_k,w_1,\ldots,w_k\rangle$, are both simplicial then*

$$\det(\sigma)=\det(\omega)\cdot\det(\sigma,\omega).$$

Proof.

$$\det(\sigma,\omega)=N_\sigma/(N_{\mathrm{Vert}(\sigma)}+N_\omega)=\frac{N_\sigma/N_{Ver(\sigma)}}{N_\omega/N_{Ver(\omega)}}$$

♣

5.7. **Star subdivisions.** Let (Σ,Ω) be a relative simplicial complex.

de: star subdivision2

Definition 5.7.1. Let (Σ,Ω) be a relative conical complex and v be a primitive vector in the relative interior of $\tau\in\Sigma\smallsetminus\Omega$. Then the *star subdivision* $v\cdot(\Sigma,\Omega)$ of (Σ,Ω) at v is defined to be

$$v\cdot(\Sigma,\Omega):=(v\cdot\Sigma,\Omega)$$

Analogously if $V = \{v_1, \ldots, v_k\}$ is a set of the primitive vectors v_i in the relative interior of the cones $\tau_i \in \Sigma$ for $i = 1, \ldots, k$ defining the disjoint stars $\mathrm{Star}(\tau_i, \Sigma)$ then the *star subdivision* $V \cdot (\Sigma, \Omega)$ of (Σ, Ω) at V is defined to be

$$V \cdot (\Sigma, \Omega) = (V \cdot \Sigma, \Omega)$$

A *multiple star subdivision* of (Σ, Ω) is a subdivision obtained as a sequence of star subdivisions at the consecutive centers $V_1, \ldots, V_k$. A *multiple star subdivision* of (Σ, Ω) which is a regular relative complex is called *desingularization.*

subdivision2 **Lemma 5.7.2.** *Any multiple star subdivision or star desingualrization of* $\mathrm{Sing}(\Sigma, \Omega)$ *at centers contained in* $\mathrm{sing}(\Sigma, \Omega)$, *extends canonically to the multiple star subdivision or the desingularization of* Σ.

Proof. (1) If $\sigma \in \Sigma$ then $\sigma = \mathrm{sing}_\Omega(\sigma) \times \mathrm{reg}_\Omega(\sigma)$, where $\mathrm{sing}_\Omega(\sigma) \in \mathrm{Sing}(\Sigma, \Omega)$.

Any subdivision of $\mathrm{sing}_\Omega(\sigma)$ extends naturally to the subdivision of σ . The extension commutes with faces so it defines the subdivision of the complex.

♣

5.8. **The invariant of relative cones.** We introduce a somewhat richer invariant measuring the progress under star subdivision at the minimal vectors. Set $\mu(\sigma, \omega) := (\dim(\omega), \det(\sigma, \omega))$ in $\mathbb{N}^2$ ordered lexicographically.

The following extends Lemma 4.3.7([KKMSD73])

des2 **Lemma 5.8.1.** *Assume* (Σ, Ω) *is simplicial and let* $(\tau, \omega) \in (\Sigma, \Omega)$ *Let* $\omega \in \mathrm{int}(\tau)$ *be a minimal point of* (τ, ω). *Then for any cone* $\sigma \in \mathrm{Star}(\tau, \Sigma)$ *the resulting cones in* $v \cdot \sigma$ *in the the star subdivision* $v \cdot \Sigma$ *of the complex* Σ *have smaller invariant* μ *determinants than* $\mu(\sigma, \tau)$.

Proof. Let $\omega = \langle w_1, \ldots, w_k \rangle$, $\tau = \langle v_1, \ldots, v_l, w_1, \ldots, w_k \rangle$, $\sigma = \langle v_1, \ldots, v_r, w_1, \ldots, w_k \rangle$, and write $v = a_1 v_1 + \ldots + a_k v_k + c_1 w_1 + \ldots + c_t w_t$ with $0 \le a_i < 1$.

There are two types of faces of $v \cdot \sigma$: $\sigma_i = \langle v, v_1, \ldots, \check{v}_i, \ldots, v_k, w_1, \ldots, w_k \rangle$, for $i \le l$, and $\delta_j = \langle v_1, \ldots, v_r \rangle + \omega_j$, where ω_j is a facet (codimension one face) of ω. The cones δ_j contain the maximal faces $\omega_j \in \Omega$, so $\mu(\delta_j, \omega_j) < \mu(\sigma, \tau)$.

On the other hand for each cone σ_i we have

$$\det(\sigma_i, \tau) = |\det(\overline{v}, \overline{v}_1, \ldots, \check{\overline{v}}_i, \ldots, \overline{v}_k)| = a_i |\det(\overline{v}_1, \ldots, \overline{v}_k)| = a_i \det(\sigma, \tau) < \det(\sigma, \tau),$$

where $\overline{v_i}$ are the images of v_i in N_σ / N_ω. ♣

5.9. **Barycentric subdivision.**

Lemma 5.9.1. *Let* σ *be a relatively irreducible cone. Let* w_σ *be the sum of all vertices of* $\Omega \cap \sigma$, *and* z_σ *be the sum of all the minimal internal vectors in* $\mathrm{sing}(\sigma)$:

$$z_\sigma := w_1 + \ldots + w_r$$

(Definition 4.7.1). Then the sum

$$v_\sigma := w_\sigma + z_\sigma$$

is in the relative interior of σ *(So can be chosen as the canonical barycenter).*

Proof. Otherwise v_σ is in a relative interior of certain proper face τ of σ containing $\mathrm{sing}(\sigma)$ and ω and σ is not irreducible.

♣

barycenter **Definition 5.9.2.** Let (Σ, Ω) be a relative conical complex. By the *canonical irreducible barycentric subdivision* of (Σ, Ω) we mean the sequence of the star subdivisions at the sets of all barycenters v_σ of all the irreducible faces in $\Sigma \setminus \Omega$ of the same dimension starting from the top dimension to the dimension zero.

bar2 **Lemma 5.9.3.** *If (Δ, Ω) is a canonical irreducible barycentric subdivision of (Σ, Ω) then (Δ, Ω) is simplicial.*

Proof. If δ is a face of Δ and then all its new rays (vertices) are linearly independent of the other rays. So τ has a unique maximal face ω which is in Σ. Then its irreducible face $\mathrm{sing}_\Omega(\tau)$ is in Ω, which implies, by Lemma 5.2.1 that τ is relative regular. Moreover the rays defined by the vertices $\mathrm{Vert}(\delta) \setminus \mathrm{Vert}(\tau)$ are not in Ω, so ω is a unique maximal face of δ which is in Ω, so δ is balanced. Since $\mathrm{Vert}(\delta) \setminus \mathrm{Vert}(\tau)$ are linearly independent from $\mathrm{Vert}(\tau)$, and $\mathrm{Vert}(\tau) \setminus \mathrm{Vert}(\omega)$ are linearly independent from $\mathrm{Vert}(\omega)$ we conclude that $\mathrm{Vert}(\delta) \setminus \mathrm{Vert}(\omega)$ are linearly independent from $\mathrm{Vert}(\omega)$.

♣

5.10. **Quotient complexes.** One can extend the notion of the nerve defined for simplicial cones and its faces to the case relative simplicial cones and some of their faces.

nerve0 **Definition 5.10.1.** Let (Σ, Ω) be a simplicial relative complex. Let τ be a (simplicial) face of Σ such that $\tau \cap |\Omega| = \{0\}$. Then for any face $\sigma \in \mathrm{Star}(\tau, \Sigma)$, by $Nerve(\tau, \sigma)$ we mean the maximal face of σ which is disjoint from τ.

Remark 5.10.2. Note that the vertices of τ are linearly independent of the remaining vertices of σ, and those generate $Nerve(\tau, \sigma)$ as in the simplicial (absolute) case before.

nerve3 **Lemma 5.10.3.** *Let (Σ, Ω) be a simplicial relative complex. Let τ be a (simplicial) face of Σ such that $\tau \cap |\Omega| = \{0\}$, and σ be a cone in $\mathrm{Star}(\tau, \Sigma)$. Then $\sigma = \tau + \mathrm{Nerve}(\tau, \sigma)$.*

Proof. By the assumption a face $\sigma \in \mathrm{Star}(\tau, \Sigma)$ can be written in the form $\sigma = \langle v_1, \ldots, v_k, w_1, \ldots, w_s, z_1, \ldots, z_r\rangle$, where $\tau = \langle v_1, \ldots, v_k\rangle$, $\omega = \rangle z_1, \ldots, z_r\rangle$ is the maximal face of σ which is in Ω. Since $v_1, \ldots, v_k$, are linearly independent from $w_1, \ldots, w_s, z_1, \ldots, z_r$, the cone $\mathrm{Nerve}(\tau, \sigma) = \langle w_1, \ldots, w_s, z_1, \ldots, z_r\rangle$ is a face of σ, and $\sigma = \tau + \mathrm{Nerve}(\tau, \sigma)$. ♣

Nerve34 **Definition 5.10.4.** Let (Σ, Ω) be a simplicial complex. Let σ be a cone in Σ. Let τ be a (simplicial) face of σ in Σ such that $\tau \cap |\Omega| = \{0\}$ then we define the *quotient complex*

$$(\Sigma, \Omega)(\tau) = (\Sigma(\tau), \Omega(\tau)^0),$$

where

(1) $\Sigma(\tau) = \mathrm{Star}(\tau, (\Sigma))/\tau$,
(2) $\mathrm{Star}(\tau, \Omega)^0 := \{(\sigma \in \mathrm{Star}(\tau, (\Sigma)) \mid \mathrm{Nerve}(\tau, \sigma) \in \Omega\}$
(3) $\Omega(\tau)^0 := \mathrm{Star}(\tau, \Omega)^0/\tau$

Nerve4 **Lemma 5.10.5.** *If (Σ, Ω) is simplicial then there exists a canonical isomorphism between relative cones (without the lattice structures)*

$$(\mathrm{Nerve}(\tau, \sigma), \mathrm{Nerve}(\tau, \sigma')), \to (\sigma/\tau), (\sigma'/\tau)),$$

(which is injective on lattices), where

- $(\sigma, \sigma') \in \mathrm{Star}(\tau, (\Sigma), \mathrm{Star}(\tau, \Omega)^0)$, *(So* σ' *a unique maximal face of* σ *with* $\mathrm{Nerve}(\tau, \sigma')) \in \Omega$*),*
- $(\sigma/\tau), (\sigma'/\tau)) \in (\Sigma(\tau), \Omega(\tau)^0)$

Proof. The proof is identical to the proof of Lemma 4.4.2. Note that since the vertices of τ are independent of ω we have that $\sigma' = \tau + \omega$ is a face of $\mathrm{Star}(\tau, (\Sigma))$, so $\mathrm{Star}(\tau, \Omega)^0$, whenever $\omega \in \mathrm{Nerve}(\tau, \Omega)^0$.

♣

product3 **Lemma 5.10.6.** *Let* $\sigma = \langle v_1, \ldots, v_k, w_1, \ldots, w_k, z_1, \ldots, z_s \rangle$, $\tau = \langle v_1, \ldots, v_k \rangle$, *and* $\omega = \langle z_1, \ldots, z_s \rangle$. *Assume that* (σ, ω) *is simplicial and* $w_i + N_\tau$ *are primitive in* N_σ/N_τ. *Then*

$$\det(\sigma, \omega) = \det(\omega + \tau, \omega)) \cdot \det(\sigma/\tau, (\omega + \tau)/\tau).$$

Proof. Recall that $\det(\sigma, \omega) = |N_\sigma/(N_{\mathrm{Vert}(\sigma)} + N_\omega)|$. By the assumption,

$$\det(\sigma/\tau, (\omega + \tau)/\tau) = |N_{\sigma/\tau}/(N_{\mathrm{Vert}(\sigma/\tau)} + N_{(\omega+\tau)/\tau))}|,$$

where $N_{\sigma/\tau} = N_\sigma/N_\tau$, $N_{\mathrm{Vert}(\sigma)}$ is the subgroup of N_σ/N_τ generated by images $w_i + N_\tau$ and the primitive generators in the rays spanned by $z_j + N_\tau$. Finally $N_{(\omega+\tau)/\tau))}$ is the subgroup of N_σ/N_τ spanned by the image of N_ω. This implies that

$$\det(\sigma/\tau, (\omega + \tau)/\tau) = |\frac{N_\sigma}{N_{\mathrm{Vert}(\sigma)} + N_\tau + N_\omega}|$$

Finally $|\det(\omega + \tau, \omega)) = N_\tau/(N_{\mathrm{Vert}(\tau)} + N_\omega)|$.

The kernel of the surjective quotient map

$$\pi : N_\sigma/(N_{\mathrm{Vert}(\sigma)} + N_\omega) \to \frac{N_\sigma}{N_{\mathrm{Vert}(\sigma)} + N_\tau + N_\omega}$$

is given by

$$\frac{(N_{\mathrm{Vert}(\sigma)} + N_\omega) + N_\tau}{N_{\mathrm{Vert}(\sigma)} + N_\omega} = \frac{N_\tau}{(N_{\mathrm{Vert}(\sigma)} + N_\omega) \cap N_\tau} = \frac{N_\tau}{N_{\mathrm{Vert}(\tau)} + N_\omega}$$

♣

5.11. **Canonical lifting.** The following lemma extends Lemma 4.5.1 to the relative case.

lift2 **Lemma 5.11.1.** *Let* $\tau = \langle v_1, \ldots, v_k \rangle$ *be a regular face of a simplicial relative cone* $(\sigma, \omega) \in (\Sigma, \Omega)$ *such that* $\tau \cap |\Omega| = \{0\}$.

Write $\sigma = \langle v_1, \ldots, v_k, w_1, \ldots, w_s, z_1, \ldots, z_r \rangle$, *where* $\tau = \langle v_1, \ldots, v_k \rangle$, *and* $\omega = \langle z_1, \ldots, z_r \rangle$ *is a maximal face of* σ *which is in* Ω.

Let $[w]$ *be a primitive vector in* $\sigma/\tau \setminus (\omega + \tau)/\tau$. *Then there exists a unique primitive vector* $w_{\mathrm{can}} \in \sigma \setminus \omega$ *called canonical lifting of* $[w]$, *such that* $w_{\mathrm{can}} + N^{\mathbb{Q}}_\tau = [w]$, *and*

$$w_{\mathrm{can}} = \sum a_i v_i + \sum b_j w_j + \sum c_k z_k$$

has the minimal coefficients a_i. *(The part* $\sum b_j w_j + \sum c_k z_k$ *is uniquely determined.)*

Moreover let $[w_j]=(1/n_j)w_j+N^{\mathbb{Q}}_\tau$, *and* $[z_k]=(1/n_k)z_k+N^{\mathbb{Q}}_\tau$ *be the primitive vectors in the rays* $\mathbb{Q}_{\geq 0}w_i+N^{\mathbb{Q}}_\tau/N^{\mathbb{Q}}_\tau$, *and, respectively* $\mathbb{Q}_{\geq 0}z_j+N^{\mathbb{Q}}_\tau/N^{\mathbb{Q}}_\tau$, *where* $n_j=\det(\sigma_j)$, $n_k=\det(\sigma_j)$, *and* $\sigma_j:=\langle v_1,\ldots,v_k,w_j\rangle$, $\sigma_k:=\langle v_1,\ldots,v_k,z_k\rangle$

Let

$$[w]=\sum b_j[w_j]+\sum c_k[z_k]\in\sigma/\tau$$

be a primitive vector in σ.

Then its canonical lifting $w_{\rm can}$ *is of the form*

$$w_{\rm can}=\sum a_iv_i+\sum\frac{b_j}{n_j}w_j+\sum\frac{c_k}{n_k}z_k$$

in σ , *where* $0\leq a_i<1$,.

If $[w]$ *is a minimal vector of* $(\sigma/\tau,\omega+\tau/\tau)$ *then* $w_{\rm can}$ *is also a minimal vector for* (σ,ω).

Proof. Identical as the proof od Lemma 4.5.1. Note that since

$$\mathrm{Nerve}(\tau,\sigma)=\langle w_1,\ldots,w_s,z_1,\ldots,z_r\rangle\to\sigma/\tau$$

is a linear isomorphism the part $\sum b_jw_j+\sum c_kz_k$ in the lifting $w_{\rm can}$ is uniquely determined.

♣

5.12. **Regular lifting of relative cones.**

Lift2 **Corollary 5.12.1.** *With the notation of Lemma 4.6.1, let*

$$\sigma=\langle v_1,\ldots,v_k,w_1,\ldots,w_s,z_1,\ldots,z_r\rangle,$$

where $\tau=\langle v_1,\ldots,v_k\rangle$, $\omega=\langle z_1,\ldots,z_r\rangle$ *and assume that* (σ,ω) *is simplicial and both the pair* $(\sigma/\tau,(\omega+\tau)/\tau)$ *and the cone* τ *are regular. Then* $(\omega+\tau,\omega)$ *is regular. Moreover consider the regular liftings*

$$(w_j)_{\rm reg}:=(w_j)_{\rm can}=a^j_1v_1+\ldots+a^j_kv_k+\frac{1}{n_j}w_j,$$

of w_j *as in Lemma 4.6.1. (Here as before* $n_j=\det(\sigma_j)$, $\sigma_j:=\langle v_1,\ldots,v_k,w_j\rangle$*). Then*

1. *The pair* $(\langle v_1,\ldots,v_k,(w_1)_{\rm reg},\ldots,(w_s)_{\rm reg},z_1,\ldots,z_r\rangle,\omega)$ *is regular.*
2. *Each pair* $(\langle v_1,\ldots,v_k,(w_1)_{\rm reg},\ldots,(\check{w}_j)_{\rm reg},\ldots(w_s)_{\rm reg},(w_\sigma)_{\rm reg},z_1,\ldots,z_r\rangle,\omega)$ *is regular, where* $(w_\sigma)_{\rm reg}=(w_1)_{\rm reg}+\ldots+(w_s)_{\rm reg}$.
3. *If* $n_j>1$ *for all* j *then* $(w_j)_{\rm reg}$ *is the sum of the minimal vectors in* (σ,ω).

Proof. (1) Note that since τ is regular the pair $(\tau+\omega,\omega)$ is also regular. Then, by Lemma 5.10.6:

$$\det(\langle v_1,\ldots,v_k,(w_1)_{\rm reg},\ldots,(w_s)_{\rm reg},z_1,\ldots,z_r\rangle\rangle)=$$

$$\det(\sigma/\tau,\omega+\tau)/\tau)\cdot\det(\omega+\tau,\omega)=1.$$

(2) Follows from (1) ♣

5.13. Marked relative complexes.

Definition 5.13.1. A *marking* on a relative complex (Σ, Ω) is a partially ordered subset V of the set of all vertices $\mathrm{Vert}(\Sigma)$ of Σ such that the following conditions are satisfied.

(1) $\mathrm{Vert}(\Omega) \subseteq V$
(2) Set $V(\sigma) := V \cap \mathrm{Vert}(\sigma)$ for any $\sigma \in \Sigma$. The set $V(\sigma) \setminus \Omega$ is linearly independent of the remaining vertices in $\mathrm{Vert}(\sigma) \setminus V(\sigma)$.
(3) For any $\sigma \in \Sigma$ the order on V is total on each subset $V(\sigma)$.

A subcomplex $\Sigma_0 \supseteq \Omega$ is *completely marked* if $V = \mathrm{Vert}(\Sigma_0)$. A face σ is *unmarked* if as before $V(\sigma) = \emptyset$. The set of all unmarked faces of (Σ, Ω) forms the *maximal unmarked subcomplex* $U(\Sigma, \Omega)$. Note that $|U(\Sigma, \Omega)| \cap |\Omega| = \{0\}$.

mark3 **Lemma 5.13.2.** *Given any completely marked (relative) subcomplex (Σ_0, Ω) of a marked relative complex (Σ, Ω), one can define the canonical order $\leq_V$ on the set of all the vectors in $|\Sigma_0| \subset |\Sigma|$.*

Proof. We order the set of vertices of any face $\sigma = \langle v_1, \ldots, v_k, w_1, \ldots, w_s\rangle$ of Σ_0, with maximal face $\omega = \langle w_1, \ldots, w_s\rangle$ according to the order on V, which is total on each face of Σ. For any $v \in \sigma$ we define the lexicographic order on the the coefficients c_i, and d_j in the presentation $v = \sum c_i v_i + \sum d_j w_j$. Although the presentation is not unique we consider the smallest presentation.

♣

can mar2 **Lemma 5.13.3.** *Let (Σ, Ω) be a completely marked simplicial relative complex. There exists a canonical multiple star desingularization $V_1 \cdot \ldots \cdot V_k \cdot \Sigma$ of Σ such that*

(1) The centers lie in $|\mathrm{sing}_\Omega(\Sigma)|$, and no faces in $\mathrm{Reg}_\Omega(\Sigma)$ are affected.
(2) The centers of the consecutive star subdivisions are minimal vectors (definition 5.5.1) in the interior of nonempty relatively irreducible faces of Σ which are not in Ω.
(3) The algorithm is functorial with respect to local projections and local isomorphisms of complexes, preserving the order, in the sense that the centers transform functorially with the trivial subdivisions removed.

Proof. The proof is nearly identical to the proof of Lemma 5.13.3. By Lemma 5.13.2, the complete marking V given for Σ defines the order on the set $|\Sigma|$.

The property is satisfied also for any subdivision of Δ. Define the following invariant for the vectors $v \in |\Delta| = |\Sigma|$:

$$\mathrm{inv}(v) = (\dim(\tau), \dim(\omega), \det(\tau, \omega)), -d_1, \ldots, -d_k, \tau)$$

where $v \in \mathrm{int}(\tau), \tau \in D(\Delta), \tau = \langle w_1, \ldots, w_k\rangle$, $w_1 < \ldots < w_k$, $(\tau, \omega) \in (\Delta, \Omega)$ and

$$v = d_1 w_1 + \ldots + d_k w_k, \quad 0 \leq d_i < 1,$$

We apply the star subdivision at the sets of the minimal points of the consecutive subdivisions $(D(\Delta), \Omega)$ of (Δ, Ω) for which the invariant inv is maximal. We run the algorithm until the relative faces of the resulting subdivision have no minimal points and thus are regular. By Lemma 5.8.1, after each star subdivision the invariant $\mu(\sigma) = \dim(\omega), \det(\tau, \omega)$ of the faces of maximal dimension drops. So the procedure terminates. ♣

5.14. **Canonical desingularization of relative conical complexes.**

can des2

Theorem 5.14.1. *Let (Σ,Ω) be a relative conical complex, with a completely marked subcomplex Ω.*

There exists a canonical desingularization (Δ,Ω) of a relative conical complex $(\Sigma\Omega)$, that is a sequence of star subdivisions $(\Sigma_i,\Omega))_{i=0}^n$ of (Σ,Ω) such that $(\Sigma_0,\Omega)=(\Sigma,\Omega)$ and $(\Sigma_n,\Omega)=(\Delta,\Omega)$ is regular, that is $\operatorname{sing}(\Delta,\Omega)=\Omega$. [19]

Moreover

1. *All the centers are in $|\operatorname{sing}(\Sigma,\Omega)|$. The centers are in the relative interiors of relatively irreducible faces σ*[20] *of the intermediate subdivisions.*
2. *The centers of the consecutive star subdivisions are either the barycenters of the relatively irreducible faces or the sets of sums of minimal vectors of the relatively simplicial irreducible cones σ* [21].
3. *The subdivision does not affect the set $\operatorname{Reg}(\Sigma,\Omega)$.*
4. *The algorithm is functorial with respect to local projections and local isomorphisms of complexes preserving the order on Ω, in the sense that the centers transform functorially with the trivial subdivisions removed.*

Proof. The proof is a slight modification of the proof of Theorem 4.9.1.

Step 1 By Lemma 5.7.2, any regular star subdivision of $\operatorname{Sing}(\Sigma,\Omega)$ at centers $\operatorname{sing}(\Sigma,\Omega)$ extends uniquely to a regular subdivision of (Σ,Ω). It suffices to desingularize $\operatorname{Sing}(\Sigma,\Omega)$.

Consider the canonical relative barycentric subdivision $B(\Sigma)$ of Σ as in Lemma 5.9.3. This induces a simplicial subdivision of (Σ,Ω), with the induced marking V and such that its maximal unmarked subcomplex $U(B(\operatorname{Sing}(\Sigma,\Omega)))$ is regular (as in the absolute case).

Step 2 For simplicity denote by Σ its subdivision $B(\Sigma)$ after Step 1. We shall induct on the dimension $k:=\dim(U(\operatorname{Sing}(\Sigma,\Omega)))$.

Let $U(\operatorname{Sing}(\Sigma,\Omega))(k)$ be the subset of all the faces in $U(\operatorname{Sing}(\Sigma,\Omega))$ of the maximal dimension k. Observe that, by maximality the corresponding stars

$$\operatorname{Star}(\tau,\operatorname{Sing}(\Sigma))$$

are necessarily disjoint for distinct $\tau\in U(\operatorname{Sing}(\Sigma,\Omega))(k)$.

Write $\tau=\langle v_1,\ldots,v_k\rangle$, and let

$$\sigma=\langle v_1,\ldots,v_k,w_1,\ldots,w_r,z_1,\ldots,z_s\rangle\in\operatorname{Star}(\tau,\operatorname{Sing}(\Sigma))$$

with maximal $\omega=\langle z_1,\ldots,z_s\rangle\in\Omega$. Then $\tau=\langle v_1,\ldots,v_k\rangle$ is regular, and the pair (σ,ω) is simplicial.

Note that by maximality all vectors $w_1,\ldots,w_r$ are marked so σ/τ is marked for each $\sigma\in\operatorname{Star}(\tau,\operatorname{Sing}(\Sigma))$.

With the notation of Definition 5.10.4, the star $\operatorname{Star}(\tau,\operatorname{Sing}(\Sigma))$ determines the quotient relative complex $(\Sigma_{\operatorname{sing}}(\tau),\Omega^0(\tau)$, where

- $\Sigma_{\operatorname{sing}}(\tau):=\operatorname{Star}(\tau,\operatorname{Sing}(\Sigma))/\tau$
- $\operatorname{Star}(\tau,\Omega)^0:=\{(\sigma\in\operatorname{Star}(\tau,(\Sigma))\mid\operatorname{Nerve}(\tau,\sigma)\in\Omega\}$
- $\Omega(\tau)^0:=\operatorname{Star}(\tau,\Omega)^0/\tau$

[19] Section 5.2
[20] Section 5.1
[21] Definition 5.5.1

Then $\Sigma_{\rm sing}(\tau)$ is completely marked, since by maximality of the unmarked face τ, all the remaining vertices in the cones in . Moreover all the cones in $\mathrm{Star}(\tau, \mathrm{Sing}(\Sigma))$ are marked.

Moreover all the relative cones in $\mathrm{Star}(\tau, \Omega)^0$ are regular. If such a face were singular then it would have been subdivided in Step 1, as a cone of the original complex Σ.

By Lemma 5.13.3, there exists a canonical desingularization

$$D_\tau(\Sigma_{\rm sing}(\tau), \Omega^0(\tau)) := \overline{u}_1 \cdot \ldots \cdot \overline{u}_s \cdot (\Sigma_{\rm sing}(\tau), \Omega^0(\tau))$$

at $\overline{u}_1 := u_1 + N^{\mathbb{Q}}_\tau, , \ldots, \overline{u}_s :=:= u_s + N^{\mathbb{Q}}_\tau$, of the completely marked quotient complex

$$(\Sigma_{\rm sing}(\tau), \Omega^0(\tau)).$$

Using Lemma 5.11.1, we shall lift the centers of the star subdivisions (for all τ) in the canonical way. By lifting the star subdivisions $D_\tau(\Sigma_{\rm sing}(\tau), \Omega^0(\tau))$ we obtain the canonical multiple star subdivision

$$D_\tau(\mathrm{Sing}(\Sigma), \Omega) := (u_1)_{\rm can} \cdot \ldots \cdot (u_s)_{\rm can} \cdot (\mathrm{Sing}(\Sigma), \Omega)$$

of $(\mathrm{Sing}(\Sigma), \Omega)$, for which

$$\mathrm{Star}(\tau, D_\tau(\mathrm{Sing}(\Sigma)))/\tau = D_\tau(\Sigma_{\rm sing}(\tau)), \Omega^0(\tau)).$$

All the cones σ'/τ in $D_\tau(\Sigma_{\rm sing}(\tau)), \Omega^0(\tau))$ are relatively regular.

The vertices in $D_\tau(\Sigma_{\rm sing}(\tau)))$ correspond to $k+1$-dimensional faces $\rho = \tau + \langle v \rangle \in \mathrm{Star}(\tau, D_\tau(\mathrm{Sing}(\Sigma, \Omega)))$, where $\langle v \rangle = \mathrm{Nerve}(\sigma, \tau)$ is the corresponding one dimensional face.

For any singular irreducible face $\sigma' \in \mathrm{Star}(\tau, D_\tau(\mathrm{Sing}(\Sigma, \Omega)))$ write

$$\sigma' = \langle v_1, \ldots, v_k, w_1, \ldots, w_r, z_1, \ldots, z_s \rangle$$

with $\tau = \langle v_1, \ldots, v_k \rangle$, and $\omega = \langle z_1, \ldots, z_s \rangle$. In order to apply Lemma 5.10.6 , and deduce regularity of (σ', ω), from regularity of $(\sigma'/\tau, (\omega+\tau)/\tau)$ and $(\omega+\tau), \omega)$ we need to ensure that the induced vectors $\overline{w}_j = w_j + N^{\mathbb{Q}}_\tau$ in the quotient space $N_R(\sigma')/N^{\mathbb{Q}}_\tau$ are primitive.

Let $\rho_i = \tau + \langle w_i \rangle + \omega_i$ be the all $k+1$- dimensional in $\mathrm{Star}(\tau, D_\tau(\mathrm{Sing}(\Sigma, \Omega)))$, for which (ρ_i, ω_i) are singular, where ω_i is the maximal face of ρ_i which is in Ω.

Let $(w_1)_{\rm can} = (w_1)_{\rm reg}, \ldots, \ldots (w_s)_{\rm can} = (w_s)_{\rm reg}$ denote their canonical (so also regular) liftings of all the corresponding vectors $\overline{w_i} = w_i + N^{\mathbb{Q}}_\tau \in \rho_i/\tau$.

By Lemma 5.12.1, the correspondence $\overline{w}_i \mapsto (w_i)_{\rm reg}$ extends to the correspondence between the regular cones $\tau' = \langle \overline{w}_{i_1}, \ldots, \overline{w}_{i_s} \rangle \in D^0_\tau(\Sigma_{\rm sing}(\tau)))$ and the regular cones $\tau_{\rm reg} := \langle (w_{i_1})_{\rm reg}, \ldots, (w_{i_s})_{\rm reg} \rangle \subset \langle (w_{i_1})_{\rm reg}, \ldots, (w_{i_s})_{\rm reg}, v_1 \ldots, v_k \rangle$. Consequently we define the midpoints $\overline{w}_\tau := \overline{w}_{i_1} + \ldots + \overline{w}_{i_s}$, and their regular liftings $(w_\tau)_{\rm reg} := (w_{i_1})_{\rm reg} + \ldots + (w_{i_s})_{\rm reg}$.

Then consider the star subdivision $D'_\tau(\Sigma_{\rm sing}(\tau), \Omega^0(\tau))$ of $D_\tau(\Sigma_{\rm sing}(\tau), \Omega^0(\tau)))$ at the centers $\overline{w}_{\tau'}$ for all τ' with vertices in Vert^0, starting from the cones τ' of top dimensions down to in one dimensional faces. Let $D'_\tau(\mathrm{Sing}(\Sigma, \Omega))$ be the corresponding star subdivision of $D_\tau(\mathrm{Sing}(\Sigma, \Omega))$ at $w^{\rm can}_\sigma$. Then as before

$$\mathrm{Star}(\tau, D'_\tau(\mathrm{Sing}(\Sigma, \Omega)))/\tau = D'_\tau(\Sigma_{\rm sing}(\tau))).$$

Then $D'_\tau(\Sigma_{\rm sing}(\tau)))$ is regular. Moreover all the cones σ'/τ in $D_\tau(\Sigma_{\rm sing}(\tau)))$ have primitive vertices:

$$\mathrm{Vert}(\sigma'/\tau) = \{w_j \mid j \in J\},$$

such that

$$\mathrm{Vert}(\sigma') = \{\overline{w}_j^{\mathrm{can}} \mid j \in J\} \cup \mathrm{Vert}(\tau)$$

Thus, by Lemma 5.10.6,

$$\det(\sigma'/\tau, (\omega+\tau)/\tau)\det(\tau+\omega,\omega) = \det(\sigma',\omega) = 1,$$

and all the relative cones in $\mathrm{Star}(\tau, D'_\tau(\mathrm{Sing}(\Sigma,\Omega))$ are regular.

Then, for the constructed subdivision $D(\Sigma,\Omega)$ of (Σ,Ω), the unmarked subcomplex $U(\mathrm{Sing}(D(\Sigma,\Omega))$ is contained in $U(\mathrm{Sing}(\Sigma,\Omega))$ since the subdivisions do not create new unmarked subdivision. On the other hand the unmarked complex $U(\mathrm{Sing}(D(\Sigma,\Omega))$ is disjoint from the faces in $U(\mathrm{Sing}(\Sigma,\Omega))(k)$, since the latter are no longer in the closure of the singular locus $\mathrm{Sing}(D(\Sigma,\Omega))$.

Consequently for the subdivision $D(\Sigma,\Omega)$ of Σ,Ω, constructed above, the unmarked complex $U(\mathrm{Sing}(D(\Sigma,\Omega))$ is of dimension $\leq k-1$.

Replacing (Σ,Ω) with $D(\Sigma,\Omega)$, defines a new complex for which

$$\dim(U(\mathrm{Sing}(\Sigma,\Omega)) \leq k-1.$$

We repeat the procedure down to dimension 0. We obtain a (canonical) subdivision $D(\Sigma,\Omega)$ of (Σ,Ω) such that

$$\dim(U(\mathrm{Sing}(D(\Sigma,\Omega))) = \{0\}$$

. This means that $\mathrm{Sing}(D(\Sigma))$ is completely marked.

Step 3 By Lemma 4.8.4 the produced completely marked subdivision $\mathrm{Sing}(D(\Sigma))$ can be canonically resolved. The desingularization of $\mathrm{Sing}(D(\Sigma))$ determines the desingularization of $D(\Sigma)$.

Commutativity with local projections follows immediately from Lemmas 5.4.2, and 5.13.3.

♣

5.15. **Finite group actions and obstructions to an equivariant relative desingularization.**

obstruction **Example 5.15.1.** Consider the monoid

$$\sigma \cap N_\sigma = \{(a_1,a_2,a_3) : \sum a_i \in 2\mathbb{Z}\} \quad \cap \quad \mathbb{Q}^3_{\geq 0},$$

generated by $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, and $(2,0,0)$, $(0,2,0)$, $(0,0,2)$ as in the Abramovich Example 4.2.5. Let $G = \mathbb{Z}_2$ acts on N_σ by permuting first two coordinates. Let

$$\omega = \langle (2,0,0),(0,2,0)\rangle.$$

be the face defining the submonoid $\omega \cap N_\sigma$ generated by $(2,0,0),(1,1,0),(0,2,0)$.

Suppose that there is a G-equivariant desingularization (Δ,ω) of (σ,ω). Then there is a unique 3-dimensional relatively regular simplicial cone $\sigma_0 \in \Delta$ containing the face ω. But such a cone is G-stable. So it can be written as

$$\sigma_0 = \langle (2,0,0),(0,2,0),w\rangle,$$

, where w is G-invariant and has a form

$$w = (a,a,2b),$$

where $a,b \in \mathbb{Z}_{>0}$. But then the pair (σ_0,ω) is not regular.

This shows the necessity to order the vertices of Ω and to forbid permutations violating such an order.

6. Functorial desingularization of strict toroidal and toroidal embeddings

desing-toroid

6.1. Canonical stratification defined by an NC divisor. If divisor E has SNC on a toroidal variety (X, D) then one can consider the stratification with the closed strata defined by the intersections of the divisorial components. This stratification is preserved by étale morphisms.

If $\phi : (U, E_U) \to (X, E_X)$ is an étale neighborhood with $V = \phi(U)$ for which the inverse image $\phi^{-1}(E_X) = E_U$ of the NC divisor E_X is an SNC divisor E_U. We define the strata of $E_X \cap V$ to be the images of strata of E_U. This is well defined, and is independent of choice of étale neighborhoods. If $x \in s$ is the image $x = \phi(y) = \phi(y')$. Then the points y, y' define a point y'' in the component U'' of the étale morphism of fiber product $U \times_X U \to U$, and thus the strata s_y and $s_{y'}$ through y, and y' are the images of the stratum $s_{y''}$ on U''. So the images of s_y and $s_{y'}$ and $s_{y''}$ coincide in a neighborhood of y.

Moreover the closure of the stratum is the union of strata. If s is a stratum through x, and assume that $x \in \overline{s_1}$. Then for a point y over x, there is a stratum s_U containing y and dominating s. Moreover since ϕ is open and thus closed under generization there is a stratum s_{U1} dominating s_1, and such that $\overline{s_{U1}}$ contains y. So the stratum s_U intersects $\overline{s_{U1}}$, and thus $s_U \subset \overline{s_{U1}}$. Consequently the image of $\phi(s_U)$ is contained in the image $\phi(\overline{s_{U1}}) \subset \overline{s_1}$. So $s \subset \overline{\phi(s_U)} \subset \overline{s_1}$.

6.2. Transforming a relative NC divisor to a relative SNC divisor.

Definition 6.2.1. We say that a locally closed subscheme on a toroidal embedding (X, D) is *relatively smooth* if its ideal is locally generated by a set of free parameters on (X, D).

Lemma 6.2.2. *Let E be a relative NC divisor on a toroidal embedding (X, D). Then all the locally closed strata of E are relatively smooth.*

Proof. Let s be a stratum of E, and $x \in s$. We can assume that it is minimal and thus closed and irreducible in a Zariski neighborhood V. Consider an étale neighborhood $\pi : U \to V \subset X$ of x, such that $\pi^{-1}(E)$ is a relatively SNC divisor on U. Let $\overline{x} \in U$ be a point over x. Then by shrinking U, if necessary, we can assume that the inverse image $s_U := \pi^{-1}(s)$ is irreducible, contains s, and thus is a minimal stratum on U. Consider the ideal $I_s \subset \mathcal{O}_{x,X}$. By definition of a relative NC divisor, s_U is relatively smooth on $(U, \pi^{-1}(D))$ and the induced ideal $I_{s_U,\overline{x}} = I_s \cdot \mathcal{O}_{\overline{x},U} \subset \mathcal{O}_{\overline{x},U}$ is generated by free parameters $(u_1, \ldots, u_k)$ at $\overline{x}$. Since $\mathcal{O}_{x,s} \to \mathcal{O}_{\overline{x},\overline{s}}$ is a étale there is a natural isomorphism

$$\widehat{\mathcal{O}_{\overline{x},U}} = \widehat{\mathcal{O}_{x,X}} \otimes_{K(x)} K(\overline{x})$$

Consider a set of generators of $I_{s_U,\overline{x}} \subset \mathcal{O}_{x,X}$, and choose a subset $v_1, \ldots, v_k$ of k functions with linearly independents linear parts at x. Then $(v_1, \ldots, v_k)$ generates $\widehat{I_{s_U,\overline{x}}} = (u_1, \ldots, u_k) = (v_1, \ldots, v_k)$ since it generates

$$(I_{s_U,\overline{x}} + m^2_{\overline{x},s_U})/m^2_{\overline{x},s_U} = ((I_{x,s} + m^2_{x,s})/m^2_{x,s}) \otimes_{K(x)} K(\overline{x})$$

so by Nakayama, it generates $I_{\overline{s}}$ and I_s. ♣

Lemma 6.2.3. *Let C be a closed relatively smooth center having SNC (respectively NC) with a relatively SNC (respectively NC) divisor E on a toroidal embedding (X, D). Consider the blow-up $X' \to X$ of C, and denote by E_C is the exceptional divisor, and by E' the strict transforms of E.*

Then the induced divisor $E' \cup E$ has SNC (respectively NC) on (X', D').

Proof. Let (X, D) be a toroidal variety. Assume that E has SNC on (X, D). Then there is a system of free parameters $u_1, \ldots, u_r$ on a neighborhood U of X such that the center C is described by the ideal $I_C = (u_1, \ldots, u_k)$, and E by the equation $u_{j_1} \cdot \ldots \cdot u_{j_l} = 0$. Then in a neighborhood and $x \in U$ we have

$$\widehat{\mathcal{O}_{x,U}} = K(x)[[u_1, \ldots, u_r, P_\sigma]].$$

The blow-up will transform C into the exceptional divisor E_C defined locally without loss of generality by $x := u'_1 := u_1$, with $u'_i = u_i/u_1$, for $i = 1, \ldots, k$, $u'_j = u_j$ for $j = k+1, \ldots, r$. The local rings and their completions will be transformed accordingly

$$\widehat{\mathcal{O}_{\overline{y},U'}} = K(\overline{y})[[u'_1, \ldots, u'_r, P_\sigma]],$$

where $u'_1, \ldots, u'_r$ are free parameters on X.

The divisor $E' \cup E$ is up to mutiplicities described as $\sigma^*(u_{j_1} \cdot \ldots \cdot u_{j_l}) = u'_{j_1} \cdot \ldots \cdot u'_{j_l} \cdot (u'_1)^d$ so has SNC on X.

The proof for NC divisor E is the same except we need to consider the effect of the blow-up in an étale neighborhood. ♣

normal

Proposition 6.2.4. *Let E be a relative NC divisor on a toroidal embedding (X, D). Then there exists a canonical sequence of blow-ups of strata of E transforming E into a relative SNC divisor on the resulting toroidal embedding (X', D'), where D' is the closure of $D \smallsetminus E$ on X'.*

Proof. Consider the blow-ups of all the strata of E of the minimal dimension r. By minimality the strata are disjoint closed, and relatively smooth. The blow-up will create a new NC divisor of the form $E^1 \cup E_1$, where E_1 is the exceptional divisor, and E^1 is a strict transform. The divisor E_1 is relatively SNC, and both E^1 and $E^1 \cup E_1$ are a relatively NC divisor, with E^1 having no strata in dimension $\leq r$. (All such strata were blowned up). Then we blow-up the strata of E^1 of the minimal dimension $r+1$. They have SNC with E_1. We create $E^2 \cup E_2$, where E_2 is the union of the exceptional divisors and the strict transforms of E_1, and E^2 are the strict transforms of E^1. As before E_2 is a relatively SNC, and E^2 and $E^2 \cup E_2$ are a relatively NC divisors, with E^2 having no strata in dimension $\leq r+1$.

♣

6.3. **Blow-ups of toroidal valuations.**

6.3.1. *Toric valuations.* We interpret the star subdivisions in the desingularization theorem 4.9.1 as the blow-ups at some functorial centers associated wit the sets of valuations.

Each vector $v \in N$ defines a linear integral function on M which determines a discrete valuation $\mathrm{val}(v)$ on X_Σ.

For any regular function $f = \sum_{w \in M} a_w x^w \in K[T]$ set

$$\mathrm{val}(v)(f) := \min\{(v, w) \mid a_w \neq 0\}.$$

Thus N can be perceived as the lattice of all T-invariant integral valuations of the function field of X_Σ.

The vector $v \in |\Sigma|$, and the valuation $\mathrm{val}(v)$ define the coherent ideal sheaves on X_Σ:
$$\mathcal{I}_{\mathrm{val}(v),a} = \{\{f \in \mathcal{O}_{X_\Sigma} \mid \mathrm{val}(v)(f) \geq a\}$$
for all natural $a \in \mathbb{N}$.

6.3.2. *Toroidal valuations.*

Lemma 6.3.3. *Let Σ be a fan, and X_Σ be the corresponding toric variety.*

Any vector v in $|\Sigma|$ defines a piecewise linear function $F_{v,a} : |\Sigma| \to Q$, such that for $w \in \sigma \subset |\Sigma|$,
$$F_{v,a}(w) := \min\{L(w) : L \in \sigma^\vee, \quad L(v) = a\}$$
.

Moreover $F_{v,a}$ is a convex on each face and piecewise linear function.

It is integral if a is sufficiently divisible. For such a, $I_{F_{v,a}} = I_{\mathrm{val}(v),a}$.

Proof. It is a well known fact. To construct $F_{v,a}$, we consider the star subdivision $\langle v\rangle \cdot \Sigma$ of Σ, and define $F_{v,a}$ on all vertices of 1-dimensional rays of $\langle v\rangle \cdot \Sigma$:

$F(u) = 0$ for $u \in \mathrm{Vert}(\Sigma) \setminus \{v\}$, and $F(v) = a$. This canonically extends to all faces of $\langle v\rangle \cdot \Sigma$ by linearity.

The function $F_{v,a}$ defines an ideal sheaf $I_{\mathrm{val}(v),a}$ for sufficiently divisible a. ♣

Lemma 6.3.4. *Let (X, D) be a strict toroidal embedding .*

Any vector v in $|\Sigma|$ defines a piecewise linear function $F_{v,a} : |\Sigma_X| \to Q$, as above which is is a convex on each face and piecewise linear function. It is integral if a is sufficiently divisible. For such a, there is a unique coherent ideal $I_{F_{v,a}}$ and a locally toric valuation $\mathrm{val}(v), X)$ *on X, such that $I_{F_{v,a}} = I_{\mathrm{val}(v),X,a}$.*

Proof. To define the valuation on X. Consider the canonical birational transformation $Y \to X$ associated with the star subdivision $\langle v\rangle \cdot \Sigma$. Then the valuation $\mathrm{val}(v), Y)$ can be easily described as the one corresponding to the exceptional divisor D of $Y \to X$. Since $Y \to X$ is birational $\mathrm{val}(v), Y)$ determines a unique valuation on $\mathrm{val}(v), X)$ on X, with $K(X) = K(Y)$.

♣

6.3.5. *Blow-ups at toroidal valuations.*

de: blow

Definition 6.3.6. Let X be a toric variety (respectively strict toroidal embedding), with associated fan (respectively complex Σ). Let $v \in |\Sigma$ be an integral vector.

By the *blow-up* $\mathrm{bl}_{\mathrm{val}(v)}(X)$ of X at a toric valuation $\mathrm{val}(v), a)$ we mean the normalization of
$$\mathrm{Proj}(\mathcal{O} \oplus \mathcal{I}_{\mathrm{val}(v),1} \oplus \mathcal{I}_{\mathrm{val}(v),2} \oplus \ldots).$$

Denote by $\mathrm{bl}_{\mathcal{J}}(X) \to X$ the blow-up of any coherent sheaf of ideals $\mathcal{J}$.

le: blow-up valuation

Lemma 6.3.7. *(5.2.8) Let X_Σ be a toric variety (resp. strict toroidal embedding) associated to a fan (resp. with a conical complex) Σ and $v \in |\Sigma|$ be an integral vector. Then* $\mathrm{bl}_{\mathrm{val}(v)}(X_\Sigma)$ *is the toric variety (toroidal embedding) associated to the* *subdivision* $\langle v\rangle\cdot\Sigma$ *of* Σ. *Moreover for any sufficiently divisible integer* d, $\mathrm{bl}_{\mathrm{val}(v)}(X_\Sigma)$ *is the normalization of the blow-up of* $\mathcal{I}_{\mathrm{val}(v),d}$. *The valuation* ν *is induced by an irreducible Weil (*$\mathbf{Q}$*-Cartier) divisor on the variety* $\mathrm{bl}_{\mathrm{val}(v)}(X)$.

Proof. By [KKMSD73], Ch. II, Th. 10, the blow-up of $\mathcal{I}_{\mathrm{val}(v),d}$ corresponds to the minimal subdivision Σ' of Σ such that f is linear on each cone in Σ'.

♣

divisors

Corollary 6.3.8. *Let* ν *be a toroidal valuation on a locally toric variety* X, *and* $\pi:\mathrm{bl}_\nu(X)=\mathrm{bl}_{\mathcal{I}_{\nu,d}}(X)\to X$ *be the associated blow-up with the exceptional Weil,* $\mathbb{Q}$*-Cartier divisor* D.

Then for the ideals $\mathcal{I}_{D,n}=\mathcal{I}_{\mathrm{val}_D,n}=\{f\in\mathcal{O}_X\mid\nu_D(f)\geq a\}$ *we have*

$$\pi_*(\mathcal{I}_{D,n})=\mathcal{I}_{\nu,d}$$

Thus the valuation ν *is induced by an irreducible exceptional Weil (*$\mathbf{Q}$*-Cartier) divisor on the variety* $\mathrm{bl}_{\mathrm{val}(v)}(X)$.

Lemma 6.3.9. *Let* (X,D) *be a strict toroidal embedding, and* Σ *be the associated conical complex. Then any integral vector* $v\in|\Sigma|$ *determines a locally monomial valuation* $\nu:=\mathrm{val}(v)$, *and its normalized blow-up* $\mathrm{bl}_{\mathrm{val}(v)})(X)\to X$ *corresponds to the star subdivision* $\langle v\rangle\cdot\Sigma$. *Moreover for any sufficiently divisible integer* d, $\mathrm{bl}_{\mathrm{val}(v)}(X)$ *is the normalization of the blow-up of* $\mathcal{I}_{\nu,d}$.

Proof. Locally we use Lemma 6.3.7. By quasicompactness of X one can find a common sufficiently divisible d which defines the center of the blow-up $\mathcal{I}_{\nu,d}$

♣

6.3.10. *Stable ideals and functoriality.* The centers of the blow-ups of the valuations ν are naturally represented by the set of ideals $\mathcal{I}_{\nu,a}$. It is possible to associate with-the center one of these ideals for a sufficiently divisible a, but this correspondence won't be functorial. In order to maintain the functoriality property important for glueing properties we consider the centers of multiple ideals defined by the locally toric valuations. This leads to the following definition

multiple

Definition 6.3.11. By the *multiple center* of the blow-up we mean the set of ideals $\{\mathcal{I}_n\}_{n\in\mathbb{N}}$ on a variety X such we have the equality of the blow-ups:

$$\mathrm{Proj}(\mathcal{O}\oplus\mathcal{I}_1\oplus\mathcal{I}_2\oplus\ldots)=\mathrm{bl}_{\mathcal{I}_d}(X).$$

for sufficiently divisible d.

6.4. Canonical desingularization of strict toroidal embeddings.

th: resolution

Theorem 6.4.1. *For any strict toroidal embedding* X *over a field* K *there exists a canonical resolution of singularities i.e. a birational projective* $f:Y\to X$ *such that*

(1) f *is an isomorphism over the open set of nonsingular points.*
(2) *The inverse image* $f^{-1}(\mathrm{Sing}(X)$ *of the singular locus* $(\mathrm{Sing}(X)$ *is a simple normal crossing divisor on* Y.
(3) f *is a composition of the normalized blow-ups of the locally monomial multiple centers* $\{\mathcal{J}_{in}\}_{n\in\mathbb{N}}$ *defined by the sets of valuations.* [22]

[22] Definition 6.3.11

(4) *f commutes with smooth and smooth toroidal morphisms* [23] *and field extensions, in the sense that the centers are transformed functorially, and the trivial blow-ups are omitted.*

Proof. Consider the associated semicomplex Σ. Its desingularization Δ defines the desingularization of (X, D). The theorem is a consequence of the combinatorial desingularization in Theorem 4.9.1. ♣

6.5. **Canonical desingularization of toroidal embeddings.**

des toroidal

Theorem 6.5.1. *Let (X, U) be a toroidal embedding over a field of any characteristic.*

There exists a canonical desingularization (Y, U) of (X, U), that is a projective birational toroidal morphism $\phi_X : (Y, U) \to (X, U)$ such that

(1) *$\overline{X}$ is a smooth variety, and $D_Y := Y \setminus U$ is an SNC -divisor (respectively NC- divisor).*
(2) *If $\mathrm{Sing}(X)$ is the set of the singular points on X then $\phi_X^{-1}(\mathrm{Sing}(X))$ is an SNC divisor.*
(3) *ϕ is an isomorphism for all points where X is smooth ,and D_X is and an SNC- divisor (respectively NC divisor).*
(4) *f is a composition of the normalized blow-ups of the locally monomial multiple centers $\{\mathcal{J}_{in}\}_{n\in\mathbb{N}}$ defined by the sets of valuations.*
(5) *ϕ commutes with smooth and smooth toroidal morphisms.*
(6) *ϕ commutes with the field extensions*

Proof. Consider an étale cover X_0 of X consisting of Zariski locally toric varieties, and let $\pi_i : X_1 := X_0 \times_X X_0$, $i = 1, 2$ be two natural projections. By the functoriality the centers lift to $\pi_1^*(\mathcal{I}_{jn}) = \pi_2^*(\mathcal{I}_{jn})$, and thus by the flat descent the ideals $\mathcal{I}_{jn}$ on X_0 generate the ideals $\mathcal{I}_{jnX}$ on X. The blow-ups at $\mathcal{I}_{jnX}$ determine the functorial birational modification Y of X which is smooth in étale topology, and hence it is smooth. The inverse image of the $f^{-1}(\mathrm{Sing}(X)$ of the singular locus $(\mathrm{Sing}(X)$ is SNC in étale topology and thus it is NC in Zariski topology. By Proposition 6.2.4, the exceptional locus D_Y can be further modify to an SNC divisor D'_Y (if needed) by a sequence of blow-ups at smooth strata.

♣

6.6. **Canonical partial desingularization of toroidal and strict toroidal embeddings.**

des toroidal2

Theorem 6.6.1. *Let (X, D) be a toroidal embedding over a field of any characteristic with $U = X \setminus D$. Let $V \subset X$ be an open saturated toroidal subset in X. Denote by $D_V = V \cap D$ the induced divisor on V, and assume thet D_V has locally ordered components.* [24]

There exists a canonical resolution of singularities of (X, D) except of V i.e. a projective birational toroidal morphism $f : Y \to X$ such that

(1) *f is an isomorphism over the open set V .*

[23] Definitions 4.2.8, 4.2.9
[24] Definitions 2.1.20

(2) *f is toroidal with respect to $(Y,U) \to (X,U)$*
(3) *The variety (Y, D_Y) is a strict toroidal embedding, where $D_Y := \overline{D_V}$ is the closure of the divisor D_V in Y.*
(4) *The variety (Y, D_Y) is the saturation of (V, D_V).*[25]
(5) *The complement $E = Y \smallsetminus V$ and thus the exceptional divisor $f^{-1}(X \smallsetminus V) = Y \smallsetminus V \subset V$ E of f have simple normal crossings with D_Y.*[26] *(So $(Y, D_Y \cup E)$ is also a strict toroidal embedding.)*
(6) *In particular, if V is smooth and D_V is an SNC divisor on V then Y is smooth and $D_Y \cup E$ is an SNC divisor.*
(7) *f commutes with field extensions and smooth and smooth toroidal morphisms preserving the order of the components D_V, in the sense that the centers are transformed functorially, and the trivial blow-ups are omitted.*

Proof. Assume first that (X, D) is a strict toroidal embedding with associated complex Σ. We can also assume that (V, D_V) is saturated in (X, D_X), where D_X is the closure of D_X taking the appropriate saturation (enlarging V). Hence V contains U moreover by using charts we see that V is saturated in (X, D). Thus it has a form $V = X(\Omega)$, where (Σ, Ω) is a relative semicomplex. Consider its canonical relative desingularization (Δ, Ω). This defines the transformation $Y \subset V$, such that $(Y, \overline{D_V})$ is a toroidal embedding $E = Y \smallsetminus V$ has SNC with $\overline{D_V}$.

Now if (X, D) is a toroidal embedding then we repeat the reasoning from the proof of Theorem 6.6.1.

♣

stratified

7. Canonical desingularization of locally toric varieties

7.1. Toroidal and locally binomial varieties.

locally

Definition 7.1.1. A variety X is called *a locally toric* (respectively *a étale locally toric*) if for any $x \in X$ there is an étale morphism called a *chart* $\varphi_x : U_x \to X_\sigma$ from an open neighborhood (respectively an étale neighborhood) from to a toric variety X_σ. Similarly X will be called *locally binomial* (respectively *étale locally binomial*) if it locally (respectively étale locally) admits an étale morphism to a variety $Z \subset \mathbb{A}^n = \operatorname{Spec}(K[x_1 \ldots, x_n])$ with the (reduced) ideal $\mathcal{I}_Z = (F_1, \ldots, F_s)$ defined by the binomials

$$F_i = x^{\alpha_i} - x^{\beta_i},$$

with $\alpha_i, \beta_i \in \mathbb{Z}^k$.

Consider the homomorphism of tori $\phi : T^k \to T^s$, given by

$$\phi(x) = (x^{\alpha_1 - \beta_1}, \ldots, x^{\alpha_1 - \beta_1}).$$

Its kernel defines a variety $\ker(\phi) := T^k \cap Z$. By the assumption $T_Z = \ker(\phi)$ is a subvariety $\ker(\phi)$, and, in fact subtorus $T_Z = \ker(\phi)$, which makes Z into a not normal toric variety $Z = K[G_1, \ldots, G_s]$, where $G_i = y^{\gamma_i}$ is a Laurent monomial in $K[T_Z] = K[y_1, y_1^{-1}, \ldots, y_r, y_r^{-1}]$. The normalization transforms Z into a normal toric variety.

Similarly the normalization of étale locally binomial variety transforms it into a étale locally toric variety.

[25] Definition 2.1.7
[26] Definition 2.1.14

sse: stratified toric

7.2. **Stratified toric varieties.** Locally toric varieties posses a natural stratification making them into a more general class of *stratified toroidal* varieties.

7.2.1. *Stratified toric varieties.*

Definition 7.2.2. [Wło03, Definition 3.1.3] A *stratified toric variety* is a toric variety (X, T) over a field K with a T-stable equisingular stratification, such that for any two geometric points $x, x' \to s$ their completion of local rings $\widehat{\mathcal{O}}_{X,x}$ and $\widehat{\mathcal{O}_{X,x'}}$ are isomorphic.

The important difference between toric varieties and stratified toric varieties is that the latter come with a stratification which may be coarser than the one given by orbits. As a consequence, the combinatorial object associated to a stratified toric variety, called a *semifan*, consists of those faces of the fan of the toric variety corresponding to strata. The faces which do not correspond to strata are ignored (see Definition 7.2.7).

7.2.3. *Demushkin theorem and equisingularity.* Recall that $\mathrm{sing}(\sigma)$ is the unique maximal irreducible singular face of σ. (Definition 4.1.4)

The equisingularity condition can be well understood by applying the following criterion:

th: Dem

Theorem 7.2.4. *([Dem82], [Wło03, Theorem 2.5.1]) Let σ and τ be two cones of maximal dimension in isomorphic lattices $N_\sigma \simeq N_\tau$. Set $\widehat{X}^K_\sigma = \mathrm{Spec}(K[[(\sigma^\vee)^{\mathrm{integ}}]]$, $\widehat{X}^K_\tau = \mathrm{Spec}(K[[(\sigma^\vee)^{\mathrm{integ}}]]$ for any field K. Let $\overline{K}$ denote the algebraic closure of K.*

Then the following conditions are equivalent:

(1) $\sigma \simeq \tau$.
(2) $\widehat{X}^K_\sigma \simeq X^K_\tau$.
(3) $\widehat{X}^{\overline{K}}_\sigma \simeq \widehat{X}^{\overline{K}}_\tau$.
(4) $\mathrm{sing}(\sigma) \simeq \mathrm{sing}(\tau)$.

Remark 7.2.5. Theorem 7.2.4 was originally proven over algebraically closed field. But we have obvious implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

7.2.6. *Semifans.*

de: embedded

Definition 7.2.7. [Wło03, Definition 3.1.5] An *embedded semifan* is a subset $\Omega \subset \Sigma$ of a fan Σ in a lattice N such that for every $\sigma \in \Sigma$ there is a unique maximal $\omega(\sigma) \in \Omega$ satisfying

(1) $\omega(\sigma) \preceq \sigma$ and any other $\omega \in \Omega$ with $\omega \preceq \sigma$ is a face of $\omega(\sigma)$,
(2) $\sigma = \omega(\sigma) \oplus \mathrm{r}(\sigma)$ for some regular cone $\mathrm{r}(\sigma) \in \Sigma$.

An *embedded fan* is an embedded semifan $\Omega \subset \Sigma$, where Ω is a subfan of Σ.

A *semifan* in a lattice N is a set Ω of cones in $N^{\mathbb{Q}}$ such that the set Σ of all faces of the cones of Ω is a fan in $N^{\mathbb{Q}}$ and $\Omega \subset \Sigma$ is an embedded semifan.

By the *support* of the semifan Ω we mean the union of all its faces, $|\Omega| = \bigcup_{\sigma\in\Omega} \sigma$.

Example 7.2.8. [Wło03, Example 3.1.2] Consider any fan Σ, and the subset

$$\mathrm{sing}(\Sigma) := \{\mathrm{sing}(\sigma) \mid \sigma \in \Sigma\}.$$

Then $\mathrm{sing}(\Sigma) \subset \Sigma$ is an embedded semifan.

The idea of the embedded fan is to associate the sets of cones with the orbits in strata. In fact we have

semifans correspondence

Proposition 7.2.9 (Proposition 3.1.7)**.** [Wło03] *Let Σ be a fan with a lattice N, and let X denote the associated toric variety. There is a canonical bijective correspondence between the toric equisingular stratifications of X and the embedded -semifans $\Omega \subset \Sigma$:*

1. *If S is a toric stratification of X, then the corresponding embedded semifan $\Omega \subset \Sigma$ consists of all those cones $\omega \in \Sigma$ that describe the big orbit of some stratum $s \in S$.*
2. *If $\Omega \subset \Sigma$ is an embedded semifan, then the strata of the associated toric stratification S_Ω of X arise from the cones of Ω via*

$$\omega \mapsto \operatorname{strat}(\omega) := \bigcup_{\omega(\sigma)=\omega} O_\sigma.$$

Proof. (1) Since strata of S are T-invariant and disjoint, each orbit O_τ belongs to a unique stratum s. Let $\omega \in \Omega$ describe the big open orbit of s. Then O_τ is contained in the closure of O_ω. Hence ω is a face of τ. Moreover by definition 7.2.7,there exists an isomorphism of the completion of the local rings $\overline{\mathcal{O}_{X_\tau,x}}$ and $\overline{\mathcal{O}_{X_\omega,y}}$ of two points $x \in O_\tau$ and $y \in O_\omega$. By Theorem 7.2.4 and since $\operatorname{sing}(\tau) \supset \operatorname{sing}(\omega)$, we infer that $\operatorname{sing}(\tau) = \operatorname{sing}(\omega)$. Hence $\omega = \operatorname{sing}(\omega) \oplus \mathrm{r}(\omega)$ and $\tau = \operatorname{sing}(\tau) \oplus \mathrm{r}(\tau) = \omega \oplus \mathrm{r}'(\tau)$, where $\mathrm{r}(\tau) = \mathrm{r}(\omega) \oplus \mathrm{r}'(\tau)$.

(2) By definition the strata $\operatorname{strat}(\omega)$ in S_Ω correspond to the collection of orbits:

$$\{\sigma \in \Sigma \mid \omega(\sigma) = \omega)\} = \operatorname{Star}(\omega, \Sigma) \smallsetminus \bigcup_{\omega \prec \omega' \in \Omega} \operatorname{Star}(\omega', \Sigma),$$

So the closure

$$\overline{\operatorname{strat}(\omega)} = \operatorname{Star}(\omega, \Sigma)$$

Hence all the defined subsets $\operatorname{strat}(\omega)$ are locally closed. The closure of each $\operatorname{strat}(\omega)$ corresponds to $\operatorname{Star}(\omega, \Sigma)$ and hence it is a union of the sets $\operatorname{strat}(\omega')$, where $\omega \prec \omega'$. Each $\operatorname{strat}(\omega)$ is a toric variety with a fan

$$\Sigma' := \{\tau/\omega \quad | \quad \omega(\tau) = \omega\}$$

in $(N^{\mathbf{Q}})' := N^{\mathbf{Q}}/\operatorname{span}(\omega)$.

Since $\tau = \omega \oplus \mathrm{r}(\tau)$ the cone τ/ω is isomorphic to the regular cone $\mathrm{r}(\tau)$ in $(N^{\mathbf{Q}})'$. Thus the stratum $\operatorname{strat}(\omega)$ is smooth. Moreover since $\operatorname{sing}(\tau) = \operatorname{sing}(\omega)$, so they have isomorphic local rings (and their completions). ♣

7.3. Isomorphisms of embedded semifans over partially ordered sets.

Definition 7.3.1. By an automorphism of $\Omega \subset \Sigma$ we mean an automorphism α of the vector space $N^{\mathbf{Q}}$ which defines an automorphism of the fan Σ such that $\alpha(\omega) = \omega$ for any $\omega \in \Omega$. The group of the automorphisms will be denoted by $\operatorname{Aut}(\Sigma, \Omega)$.

Remark 7.3.2. The automorphisms in $\operatorname{Aut}(\Sigma, \Omega)$ of the embedded semifan $\Omega \subset \Sigma$ induce the automorphisms of the toric variety X_Σ preserving the strata in S_Ω (and the torus $T \subset X_\Sigma$).

The cones in Ω are in the natural bijective correspondence with the strata of S_Ω, and this bijection respects the natural partial order. The other cones of Σ have no geometric meaning in general and are dependent of a particular torus action. When comparing two embedded semifans associated with the same stratification S it is convenient to consider the natural bijection $\Omega \to S_\Omega$ of the partially ordered sets.

This leads to the following slightly more general definition.

Definition 7.3.3. Let S be a partially ordered set. By *an embedded semifan over* S we mean a semifan $\Omega \subset \Sigma$ with an injective map $j_{\Omega,S} : \Omega \to S$ of the partially ordered sets respecting the order.

Let $\Omega_i \subset \Sigma_i$ in $N_i^{\mathbf{Q}} \supset N_i$, for $i = 1, 2$, be two embedded semifans over S

By the *isomorphism* $(\Sigma_1, \Omega_1) \to (\Sigma_2, \Omega_2)$ *of embedded semifans over* S we mean a vector space isomorphism $i_{N_1,N_2} : N_1^{\mathbf{Q}} \to N_2^{\mathbf{Q}}$ preserving the lattice structures which defines an isomorphism of the fans $j_{\Sigma_1,\Sigma_2} : \Sigma_1 \to \Sigma_2$ and induces a bijection $j_{\Omega_1,\Omega_2} : \Omega_1 \to \Omega_2$ of the sets of isomorphic cones, commuting with the maps to S:

$$j_{\Omega_2,S} \circ j_{\Omega_1,\Omega_2} = j_{\Omega_1,S}$$

Remark 7.3.4. Any embedded semifan $\Omega \subset \Sigma$ is defined over the stratification S_Ω with the natural map $j_{\Omega,S} : \Omega \to S_\Omega$ which identifies the cones of Ω with strata of S_Ω. This interpretation is useful for glueing properties when comparing different charts.

Any embedded semifan $\Omega \subset \Sigma$ can be considered naturally as an embedded semifan over $S := \Omega$ with the identical map $\Omega \to \Omega = S$. Then isomorphisms of $\Omega \subset \Sigma$ into itself over S are simply the automorphisms in $\mathrm{Aut}(\Sigma, \Omega)$.

Analogously one can define the stratified toric varieties over S and their isomorphisms.

Definition 7.3.5. Let S be a partially ordered set. By *a stratified toric variety over* S we mean a stratified toric variety (X, T) with an injective map $j_{T,S} : T \to S$ of the partially ordered sets respecting the order.

By the *isomorphism* $(X_1, T_1) \to (X, T_2)$ *of stratified toric varieties over* S we mean an isomorphism of toric varieties preserving stratifications which defines a bijection $j_{T_1,T_2} : T_1 \to T_2$ of the sets of strata, commuting with the maps to S:

$$j_{T_2,S} \circ j_{T_1,T_2} = j_{T_1,S}$$

7.4. **Stratification on toric varieties by the singularity type.** We define the singularity type $\mathrm{sing}(x)$ for any closed point $x \in X_\Sigma$ with the residue fielf $K(x)$ to be the isomorphism class of the cone of maximal dimension $(\mathrm{sing}(\sigma), N^{\mathbf{Q}}_{\mathrm{sing}(\sigma)})$, where σ is defined by $\widehat{X}_x = \widehat{X_\sigma^{\overline{K}}}$. By the Demushkin Theorem 7.2.4 it is independent of the toric structure on $X = X_\Sigma$.

strat1 **Lemma 7.4.1.** [Wło03, Lemma 4.1.7] *The canonical stratification* $\mathrm{Sing}(X_\Sigma)$ *defined by the singularity type* sing *on a toric variety* X_Σ *corresponds to the embedded semifan* $\mathrm{sing}(\Sigma) \subset \Sigma$.

That is, for a stratum $s \in \mathrm{Sing}(X_\Sigma)$ *through* $x \in X_\Sigma$ *we have*

$$s \cap U = \{x \in U \mid \mathrm{sing}(x) = \mathrm{sing}(y)\}.$$

Proof. Consider the stratification corresponding to the embedded semifan $\mathrm{Sing}(\Sigma) \subset \Sigma$.

Let $x \in s = \mathrm{strat}(\sigma)$ where σ is an irreducible face in $\mathrm{Sing}(\Sigma)$. Then $x \in O_\tau$, with τ satisfying $\mathrm{sing}(\tau) = \sigma$ in the open set X_τ. We can write $\tau = \mathrm{sing}(\tau) \oplus \mathrm{reg}(\tau)$, so the singularity type sing (defined up to isomorphism of cones) is the same for all points of the stratum $\mathrm{strat}(\sigma)$ and equal to $\mathrm{sing}(\sigma)$. It differs from the points in $X_\tau \setminus s$ which have the singularity types of the irreducible cones of smaller dimension. They are isomorphic to proper irreducible faces of $\mathrm{sing}(\sigma)$. Thus any stratum $\mathrm{strat}(\sigma)$, where $\sigma \in \mathrm{sing}(\Sigma)$ can be characterized as the set of points with locally constant singularity type equal σ.

♣

7.5. **Toric varieties with toric divisors.** One can adapt the terminology of relative complexes to the fans.

Definition 7.5.1. By a relative fan we mean a pair (Σ, Ω) where $\Omega \subseteq \Sigma$ are fans. A relative fan is *regular* if any cone in σ contains a unique maximal face $\omega \in \Omega$ such that $\sigma = \omega \oplus \tau$ for a regular cone τ. A relative fan will be called *saturated* if any face of Σ with vertices (one dimensional rays) in Ω, is in fact in Ω. For any $\sigma \in \Sigma$ by $\mathrm{sing}_\Omega(\sigma)$ we mean the smallest face containing $\mathrm{sing}(\sigma)$, and all faces of σ which are in Ω. Then $\sigma = \mathrm{sing}_\Omega(\sigma) \oplus \tau$, where τ is a regular cone. Moreover $\mathrm{sing}_\Omega(\sigma)$ is a unique minimal face of σ which give such decomposition.

embedd **Example 7.5.2.** Consider a relative fan (Σ, Ω). Let

$$\mathrm{sing}(\Sigma, \Omega) := \{\mathrm{sing}_\Omega(\sigma) \mid \sigma \in \Sigma\}.$$

Then $\mathrm{sing}(\Sigma, \Omega) \subset \Sigma$ is an embedded semifan.

For any toric variety $X_\Delta \supset T$ associated with the fan Δ denote by $D_\Delta := X_\Delta \setminus T$ the toric divisor of the complement of the big torus T.

cr2 **Lemma 7.5.3.** *There is a bijective correspondence between the toric divisors D on a toric variety X_Σ and the saturated subfans Ω of Σ.*

Let D be a toric divisor D on X_σ. Let $\overline{S}_D$ be the set of closed strata defined by D. Then there is a unique saturated subfan $\Omega \subset \Sigma$ such that

(1) The cones $\sigma \in \Omega$ are defined by the closed strata $\overline{O_\sigma}$ which are in $\overline{S}_D$, and for which (X, D) is a toroidal at O_σ.
(2) The open saturated subset X_Ω intersects all strata of D such that $D = \overline{D_\Omega}$.
(3) The toroidal locus of (X, D) is equal to

$$V = (X, D)^{\mathrm{tor}} = X_{\mathrm{Reg}(\Sigma,\Omega)} = (X_\Omega, D_\Omega)^{\mathrm{sat}}.$$

Proof. Identical as the proof of Lemma 5.3.5.

♣

7.5.4. *Embedded fans.* There is a relation between to notions of embedded fans and relative fans as both represent properties of stratifications.

Lemma 7.5.5. *A relative fan (Σ, Ω) is an embedded fan iff it is a regular relative fan. Moreover if (Σ, Ω) is an embedded fan then $\Omega \subset \Sigma$ is saturated subfan, and* $\mathrm{Reg}((\Sigma, \Omega) = \Omega$.

embedded fan **Corollary 7.5.6.** *There is a bijective correspondence between toroidal embeddings (X, D) defined by the toric variety $X = X_\Sigma$ with the toric divisor D and the embedded fans $\Omega \subset \Sigma$. If (X, D) is a toric variety with toroidal structure then Ω is subset of the generic orbits in the strata of (X, D), and the open subset X_Ω is the smallest toric open subset intersecting all the components of D.*

Conversely, if $\Omega \subset \Sigma$ is an embedded fan then

1. *$(X_\Sigma, \overline{D_\Omega})$ is a strict toroidal embedding, where $\overline{D_\Omega}$ is the closure of D_Ω.*
2. *Any divisor E contained in $X_\Sigma \setminus X_\Omega$ has SNC on $(X_\Sigma, \overline{D_\Omega})$.*
3. *$(X_\Sigma, \overline{D_\Omega})$ is the saturation of (X_Ω, D_Ω).*

Proof. Identical to the proof of the analogous statement for complexes in Lemma 5.3.6.

♣

7.5.7. *Canonical stratification on toric varieties with toric divisors.* Let (X, D) be a pair of a toric variety and a toric divisor. Consider the associated saturated relative fan (Σ, Ω).

Let $x \in X$, $x \in O_\sigma$, with $\Omega_\sigma := \sigma \cap \Omega$. We define the singularity type at x to be

$$\mathrm{sing}_D(x) = (\mathrm{sing}_x, n_D(x)),$$

where $n_D(x)$ is the number of the components of D through $x \in X$.

strat2 **Lemma 7.5.8.** *Let $D = \overline{D_\Omega}$* [27] *be a toric divisor on X_Σ corresponding to a saturated relative fan $\Omega \subseteq \Sigma$. Then the stratification $\mathrm{Sing}_\Omega(X_\Sigma)$ on X_Σ corresponding to the embedded semifan*

$$\mathrm{sing}(\Sigma, \Omega) \subset \Sigma.$$

is defined by the singularity type $\mathrm{sing}_D(x)$.

1. *The strata of $\mathrm{Sing}_\Omega(X_\Sigma)$ on X_Σ correspond to the cones in $\mathrm{sing}(\Sigma, \Omega)$. In particular $\mathrm{Sing}_\Omega(X_\Sigma)$ contains the strata which are extensions of the orbits in X_Ω. That is $\mathrm{sing}(\Sigma, \Omega)$ contains Ω.*
2. *The saturation of (X_Ω, D_Ω) in $(X_\Sigma, \overline{D_\Omega})$ is an open subset $X_{\mathrm{Reg}(\Sigma,\Omega)}$ corresponding to the subfan $\mathrm{Reg}(\Sigma, \Omega)$ is a of Σ containing Ω.*
3. *The strata in the saturation of (X_Ω, D_Ω) correspond to*
$$\mathrm{sing}(\Sigma, \Omega) \cap \mathrm{Reg}(\Sigma, \Omega) = \Omega.$$
4. *The saturation of (X_Ω, D_Ω) defines the toroidal locus of (X_Σ, D).*

Proof. Consider the embedded semifan $\mathrm{sing}(\Sigma, \Omega) \subset \Sigma$. (Example 7.5.2) It defines a stratified toric variety. If $x \in \mathrm{strat}(\sigma)$, where σ is relatively irreducible cone of Σ, with its subfan $\Omega_\sigma = \Omega \cap \sigma$ then $\mathrm{sing}_D(x)$ is the pair (σ, n_D) where n_D is the number of vertices in the subfan Ω_σ.

Moreover this invariant is locally constant on $\mathrm{strat}(\sigma)$. If $x \in \mathrm{strat}(\sigma)$. Then $x \in O_\tau$ with $\mathrm{sing}_\Omega(\tau) = \sigma$, so all faces of Ω and all the singular faces are in σ: $\Omega \cap \tau = \Omega_\sigma$, and $\mathrm{sing}(\tau) = \mathrm{sin}(\sigma)$. If $y \in X_\tau$, and $y \notin \mathrm{strat}(\sigma)$ then $y \in O_{\tau'}$, where $\tau' \leq \tau$ with $\mathrm{sing}_\Omega(\tau') \preceq \sigma$. But $\mathrm{sing}_\Omega(\tau')$ is the smallest face $\mathrm{sing}(\tau')$ and $\Omega(\tau')$. So either $\mathrm{sing}(\tau') < \sigma$ or $\Omega_{\tau'} \subsetneq \Omega_\sigma$. But $\Omega_{\tau'}$ is saturated in $\overline{\tau}$ by Corollary 7.5.6. So if $\Omega_{\tau'} \subsetneq \Omega_\sigma$ then the number of the vertices in $\Omega_{\tau'}$ which $n_D(y)$ is smaller

[27] see Corollary 7.5.6

than the number of vertices of the vertices in $\Omega_{\tau'}$ equal to $n_D(x)$. In any case $(\dim(\sigma_y), n_D(y)) < (\dim(\sigma_x), n_D(x))$ (with lexicographic order).

(1) follows from Definition. (2) Follows from Lemma 7.5.3. (3) Follows from (1) and (2). (4) All the strata in the toroidal locus of (X_Σ, D) intersect X_Ω. ♣

7.6. **Stratified toroidal varieties.**

Definition 7.6.1. [Wło03, Definition 4.1.6] By a *stratified toroidal variety* we mean a stratified variety (X, S) such that for any $x \in s$, $s \in S$ there is an étale map called a *chart* $\varphi : (U, S \cap U) \to (X_\sigma, S_\sigma)$ from an open neighborhood to a stratified toric variety (X_σ, S_σ) such that all strata in U_x are preimages of strata in S_σ.

over **Remark 7.6.2.** The stratified toric varieties (X_σ, S_σ) corespond to embedded semifans $\overline{\sigma} \supseteq \Omega_\sigma$. Moreover the definition implies existence of the natural injective map from $\Omega_\sigma \to S$, associating with faces the corresponding strata. So the induced semifans can be considered as semifans over the stratification S. This observation is important and helps to understand how gluing works.

Toroidal embeddings with the natural stratification are particular example of a stratified toroidal variety.

7.6.3. *Canonical stratification on locally toric varieties.* Demushkin theorem allows to consider the canonical stratification on a locally toric variety.

For any closed point $x \in X$, with residue field $K(x)$ set the invariant $\mathrm{sing}(x)$ to be the isomorphism class of the cone $\mathrm{sing}(\sigma)$ of maximal dimension, where $\widehat{X}_x \simeq \widehat{X}_\sigma^{K(x)}$. The invariant $\mathrm{sing}(x)$ is well defined by the Demushkin theorem 7.2.4.

can0 **Lemma 7.6.4.** [Wło03, Lemma 4.2.1] *Let X be locally toric variety. There is a locally closed stratification $\mathrm{Sing}(X)$ with smooth irreducible strata $s \in \mathrm{Sing}(X)$, such that for any $y \in X$, and the stratum s though y there is an open neighborhood U such that*

$$s \cap U = \{x \in U \mid \mathrm{sing}(x) = \mathrm{sing}(y)\}.$$

Moreover the stratification $\mathrm{Sing}(X)$ is locally defined via charts by the inverse images of strata on $(X_\sigma, \mathrm{Sing}(X_\sigma)$ associated with the embedded semifan $(\overline{\sigma}, \mathrm{sing}(\overline{\sigma}))$.

Proof. Consider the étale chart $U \to X_\sigma$. By Lemma 7.4.1 the stratification $\mathrm{Sing}(X_\sigma)$ on X_σ is defined by the singularity type $\mathrm{sing}(x)$. So the stratification $\mathrm{Sing}(X)$ is induced locally by the stratification $\mathrm{Sing}(X_\sigma)$. The strata are the inverse images of strata in $\mathrm{Sing}(X_\sigma)$. ♣

7.6.5. *Canonical stratification on locally toric varieties with divisors.*

divisor **Definition 7.6.6.** Let X be locally toric variety (respectively étale locally toric variety). A divisor D on X will be called a *locally toric divisor* if there exists locally (respectively étale locally) a map $U \to X_\sigma$, where $D \cap U$ is the inverse image of a toric divisor on X_σ.

The above extends to the relative case. Let X be a locally toric variety, and D be a locally toric divisor on X. For any closed point $x \in X$ set as before

$$\mathrm{sing}^1_D(x) := (\mathrm{sing}(\sigma), n_D(x)).$$

can **Theorem 7.6.7.** *Let X be locally toric variety with a locally toric divisor D. There is a locally closed toroidal stratification $\mathrm{Sing}_D(X)$ defined by the invariant $\mathrm{sing}^1_D(x)$ with smooth irreducible strata $s \in \mathrm{Sing}_D(X)$, and such that for any $y \in X$, and the stratum s though y there is an open neighborhood U such that*

$$s \cap U = \{x \in U \mid \mathrm{sing}^1_D(x) = \mathrm{sing}^1_D(y)\}.$$

Moreover the stratification $\mathrm{Sing}_D(X)$ is locally defined via local étale charts $\phi : (X, D) \to (X_\sigma, D_{\Omega_\sigma})$, where Ω_σ is a subfan of $\overline{\sigma}$, by the inverse images of strata on $(X_\sigma, \mathrm{Sing}_D(X_\sigma)$ associated with the embedded semifan $(\overline{\sigma}, \mathrm{sing}(\overline{\sigma}, \Omega_\sigma))$.

The proof is identical as before. Uses the first part of Theorem 7.5.8.

7.7. **Conical semicomplexes.**

7.7.1. *Semicones.* Next we generalize the notion of cone. The stratified toroidal varieties are locally described by the charts to affine stratified toric varieties (X_σ, S_σ). The induces semifan consists of the cone σ and some of their faces -those corresponding to the strata in S_σ. We will call such semifans semicones. In analogy to usual cones we denote the semicones by small Greek letters σ, τ, etc.:

We also shall consider *semicones over partially ordered* set S. i.e. semifan over S consisting of the faces of σ_s.

Again by the previous Remark 7.6.2 the semicones arising from the charts are naturally defined over the stratification S.

semii **Definition 7.7.2.** [Wło03, Definition 4.3] Let N be a lattice in the vector space $N^{\mathbb{Q}}$. A *semicone* in $N^{\mathbb{Q}}$ is a semifan σ in $N^{\mathbb{Q}}$ such that the support $|\sigma|$ of σ occurs as an element of σ.

This implies that the semifan σ consists of the cone $|\sigma|$ and some of their faces. This defines an embedded semifan $(\overline{\sigma}, \sigma)$, where $\overline{\sigma}$ is the the fan consisting of the faces of the cone $|\sigma|$. Moreover one can write the semicone σ as the collection of the faces $\sigma = \{|\tau| \mid \tau \leq \sigma\}$.

The *dimension* of a semicone is the dimension of its support. Moreover, for an injection $\imath\colon N^{\mathbb{Q}} \to (N')^{\mathbb{Q}}$ of vector spaces, the *image* $\imath(\sigma)$ of a semicone σ in $N^{\mathbb{Q}}$ is the semicone consisting of the images of all the elements of σ.

Note that every cone becomes a semicone by replacing it with the set of all its faces. Moreover, every semifan is a union of maximal semicones. Generalizing this observation, we build up in the next sections the *semicomplexes* associated with stratified toroidal varieties from semicones.

7.7.3. *Generalized Demushkin's theorem.*

th: Dem2 **Theorem 7.7.4.** [Wło03, Theorem 4.6.1] *Let $\overline{K}$ be an algebraically closed field.*

Let σ and τ be two cones of maximal dimension in isomorphic lattices $N_\sigma \simeq N_\tau$. Let S_σ, and S_τ denote the equisingular stratifications on X_σ, and X_τ, and $\Omega_\sigma \subset \overline{\sigma}$, $\Omega_\tau \subset \overline{\tau}$ be the corresponding embedded semifans with the natural bijections $\Omega_\sigma \to S_\sigma$ and $\Omega_\tau \to S_\tau$.

Then the following conditions are equivalent:

(1) $(\overline{\sigma}, \Omega_\sigma)$ and $(\overline{\tau}, \Omega_\sigma)$ are isomorphic over S.

(2) $(\widehat{X}^{\overline{K}}_{\sigma}, \widehat{S}^{\overline{K}}_{\sigma})$ *and* $(\widehat{X}^{\overline{K}}_{\tau}, \widehat{S}^{\overline{K}}_{\tau})$ *are isomorphic over* S *(defining the correspondence between the strata in* $\widehat{S}^{\overline{K}}_{\sigma}$ *and* $\widehat{S}^{\overline{K}}_{\tau}$*).*

Proof. Idea of the proof. The isomorphism $(\widehat{X}_\sigma, \widehat{S}_\sigma) \to (\widehat{X}_\tau, \widehat{S}_\tau)$ over S defines two different toric actions on the same variety $(\widehat{X}_x, \widehat{S}_x) := (\widehat{X}_\sigma, \widehat{S}_\sigma)$ by the relevant tori T_τ and T_σ. The action of these tori determine uniquely (up to constants) the semi-invariant parameters generating the dual cones $\sigma^\vee$, and $\tau^\vee$.

One shows that the tori T_τ and T_σ are maximal and conjugate in the proalgebraic group $\mathrm{Aut}((\widehat{X}_x, \widehat{S}_x)$ of all the automorphisms preserving the stratification. (In fact there is a natural map to the linear group of the tangent space, and its kernel is a unitary proalgebraic group.)

The conjugation defines the desired automorphism of $(\widehat{X}_x, \widehat{S}_x)$ which translates into isomorphism $(\widehat{X}_\sigma, \widehat{S}_\sigma) \to (\widehat{X}_\tau, \widehat{S}_\tau)$. On the other hand the conjugation defines the isomorphism of the cones and their relevant faces, that is the embedded semifans $(\overline{\sigma}, \Omega_\sigma)$ and $(\overline{\tau}, \Omega_\sigma)$. For details see [Wło03]. ♣

th: Dem3

Corollary 7.7.5. *Let* S *be a partially ordered set. Let* σ_1 *and* σ_2 *be two semicones over* S *with isomorphic lattices* $N_1 \simeq N_2$*. Let* S_{σ_1}*, and* S_{σ_2} *denote the equisingular stratifications on* X_{σ_1}*, and* X_{σ_2}*. For* $i = 1, 2$ *denote by* $(\widehat{X}^{\overline{K}}_{\sigma}) = \mathrm{Spec}(\widehat{\mathcal{O}(X^{\overline{K}}_{\sigma_i x_i}}),$ *where* $x_i \in O^{\overline{K}}_{\sigma_i}$ *is a closed point.*

Then the following conditions are equivalent:

(1) *The semicones* (σ_1, N_{σ_1}) *and* (σ_2, N_{σ_2}) *are isomorphic over* S*.*
(2) $(\widehat{X}^{\overline{K}}_{\sigma_1,x_1}, \widehat{S}^{\overline{K}}_{\sigma,x_1})$ *are isomorphic* $(\widehat{X}^{\overline{K}}_{\sigma_2,x_2}, \widehat{S}^{\overline{K}}_{\sigma_2,x_2})$ *over* S*.*

Proof. The theorem is a rephrasing of the previous one. Let $\delta_1 = |\sigma_1| \times \tau_1$, and $\delta_2 = |\sigma_2| \times \tau_2$ be two cones of maximal dimension, in $N_i^{\mathbf{Q}}$, where τ_i are regular.

Then $(\widehat{X}^{\overline{K}}_{\sigma_i,x_i}, \widehat{S}^{\overline{K}}_{\sigma,x_1}) \simeq (\widehat{X}^{\overline{K}}_{\delta_i}, \widehat{S}^{\overline{K}}_{\delta_i})$, and their isomorphism implies the isomorphism of the embedded semifans $(\overline{\delta_1}, \sigma_1) \to (\overline{\delta_2}, \sigma_2)$, which in turn defines the isomorphism of the semicones $(\sigma_1, N_{\sigma_1}) \to (\sigma_2, N_{\sigma_2})$.

♣

7.7.6. *Conical semicomplexes.* Denote by $\mathrm{Aut}(\sigma)$ the groups of automorphisms of the semicone (σ, N_σ).

de: conical semicomplex

Definition 7.7.7. [Wło03, Definition 4.3.1]

Let Σ be a finite collection of semicones σ in $N_\sigma^{\mathbf{Q}} \supset N_\sigma$ with $\dim(\sigma) = \dim(N_\sigma)$. Moreover, suppose that there is a partial ordering "$\leq$" on Σ. We associate with each $\sigma \in \Sigma$ the group of automorphisms $\mathrm{Aut}(\sigma)$.

We call Σ a *semicomplex* if for any pair $\tau \leq \sigma$ in Σ there is an associated linear injection $\imath^\sigma_\tau \colon N^{\mathbb{Q}}_\tau \to N^{\mathbb{Q}}_\sigma$ such that $\imath^\sigma_\tau(\tau) \subset \sigma$. In particular $\imath^\sigma_\tau(|\tau|) \in \sigma$ is a face of the cone $|\sigma|$, $\imath^\sigma_\tau(N_\tau) \subset N_\sigma$ is a saturated sublattice and

(1) $\imath^\sigma_\tau \circ \imath^\tau_\varrho = \imath^\sigma_\varrho \alpha_\rho$, where $\alpha_\rho \in \mathrm{Aut}(\rho)$.
(2) $\imath^\sigma_\varrho(|\varrho|) = \imath^\sigma_\tau(|\tau|)$ implies $\varrho = \tau$,
(3) $\sigma = \bigcup_{\tau \leq \sigma} \imath^\sigma_\tau(\tau) = \{\imath^\sigma_\tau(|\tau|) \mid \tau \leq \sigma\}$.

Remark 7.7.8. [Wło03] The condition (1) is equivalent to the following condition:

$$\imath^\sigma_\tau \circ \imath^\tau_\varrho(\varrho) = \imath^\sigma_\varrho(\varrho).$$

Indeed there is an induced isomorphism $\bar{\imath}^\sigma_\varrho : \rho \to \imath^\sigma_\varrho(\varrho)$, with the inverse $(\bar{\imath}^\sigma_\varrho)^{-1} : \imath^\sigma_\varrho(\varrho) \to \rho$ which defines the automorphism

$$\alpha_\rho := (\bar{\imath}^\sigma_\varrho)^{-1}(\imath^\sigma_\tau \circ \imath^\tau_\varrho).$$

The gluing of the faces and the subdivisions of the semicomplex are defined up to the automorphisms in $\mathrm{Aut}(\sigma)$.

semico **Definition 7.7.9.** We say that the semicomplex Σ is defined over a partially ordered set S if there is an injective map $\Sigma \to S$ respecting the order.

semico3 **Remark 7.7.10.**
1. Any semicomplex Σ can be considered as a semicomplex over itself with the identical map $\mathrm{Id} : \Sigma \to \Sigma$. This approach allows to think of the semicones $\sigma \in \Sigma$ as the ones defined over Σ with the natural injective map $j_\sigma : \sigma \to \Sigma$ associating with faces of σ the relevant faces of Σ. Moreover the maps $\imath^\sigma_\tau$ induce the injective maps of the semicones $j^\sigma_\tau : \tau \to \sigma$ such that $j_\sigma j^\sigma_\tau = j_\tau$.
2. The definition of the semicomplex is equivalent to the properties
   (a) $j_\sigma(\sigma) = \{\tau \in \Sigma \mid \tau \le \sigma\}$
   (b) $j_\sigma j^\sigma_\tau = j_\tau$
3. If a semicomplex Σ is defined over S then the semicones $\sigma \in \Sigma$ are defined over S with natural injective map map $i_\sigma : \sigma \to \Sigma \to S$. Similarly $i_\sigma j^\sigma_\tau = i_\tau$.
4. As we see later the semicomplexes associated with toroidal stratification S are naturally defined over S.
5. If $\Sigma \to S$ is a bijection the properties 2a), 2b) allow to identify semicones in Σ with the subsets of S, of the form $S_a := \{b \in S \mid b \le a\}$ and the induced maps j^σ_τ between them with the natural inclusions.

Definition 7.7.11. [Wło03] By an isomorphism of the semicomplexes $\Sigma \to \Sigma'$ over S we mean a bijection of the sets $\Sigma \to \Sigma'$ commuting with the maps to S and defining the isomorphisms of the corresponding semicones $\phi_{\sigma,\sigma'} : \sigma \to \sigma'$ over S.

Lemma 7.7.12. *If $\phi : \Sigma \to \Sigma'$ is an isomorphism over S then for any $\tau \le \sigma$ there is a unique $\tau' \le \sigma'$. Moreover $\phi_{\sigma,\sigma'}\imath^\sigma_\tau = \imath^{\sigma'}_{\tau'}\phi_{\tau,\tau'}\alpha_\tau$, for some $\alpha_\tau \in \mathrm{Aut}(\tau)$.*

Proof. Note that the semicones $\phi_{\sigma,\sigma'}\imath^\sigma_\tau(\tau)$ and $\imath^{\sigma'}_{\tau'}\phi_{\tau,\tau'}\alpha_\tau(\tau)$ are equal as both are equal to the subset $\imath^{\sigma'}_{\tau'}$ of σ' corresponding to the same subset of S. So they define the same semifan and two different isomorphisms $\tau \to \imath^{\sigma'}_{\tau'}(\tau')$ over S which are different by the automorphism of τ over S so an element $\alpha_\tau \in \mathrm{Aut}(\tau)$.

♣

As a special case of the above notion, we recover the notion of a (conical) complex introduced by Kempf, Knudsen, Mumford and Saint-Donat:

de: conical complex **Definition 7.7.13.** [Wło03, Definition 4.3.2] A *conical complex* is a semicomplex Σ such that every $\sigma \in \Sigma$ is a fan consisting of all the faces of its support cone $|\sigma|$.

Remark 7.7.14. If Σ is a complex then all the semicones are fans consisting of the face of a cone. So they can be identified with the cones without loss of information. The groups $\mathrm{Aut}(\sigma)$ are trivial since if the automorphisms preserve the vertices of σ and thus they are identical on each cone $|\sigma|$. In such a case we obtain the definition of the conical complex. (Definition3.5.1).

7.8. **Associated semicomplexes.**

associated semicomplex

Definition 7.8.1. [Wło03, Definition 4.8.1] Let (X, S) be a stratified toroidal variety.

We say that a semicomplex Σ over S is *associated* to (X, S) if there is a bijection $i : \Sigma \to S$ with the following properties:

Let $\sigma \in \Sigma$ map to $s = \mathrm{strat}_X(\sigma) \in S$. Then any $x \in s$ admits an open neighborhood $U_\sigma \subset X$ and $\varphi_\sigma : U_\sigma \to X_\sigma$ of stratified varieties such that $s \cap U_\sigma$ equals $\varphi_\sigma^{-1}(O_\sigma)$ and the intersections $s' \cap U_\sigma$, $s' \in S$, are precisely the inverse images of the corresponding strata of X_σ (defined by the bijection i).

We call the smooth morphisms $U_\sigma \to X_\sigma$ from the above definition *charts*. A collection of charts satisfying the conditions from the above definition is called an *atlas*.

associated semicomplex

Lemma 7.8.2. [Wło03, Lemma 4.8.2] *For any stratified toroidal variety (X, S) there exists a unique (up to an isomorphism over S) associated semicomplex Σ over S. Moreover (X, S) is a toroidal embedding iff Σ is a complex.*

Proof. For any stratum s consider the associated local étale chart $U \to X_\sigma$ which defines the corresponding *semicone* $\sigma = \sigma_s$ over S, i.e. semifan over S consisting of the faces of σ_s corresponding to strata in Star(s, S). This correspondence defines a natural map $\sigma \to S$ between faces τ in σ the strata $s_\tau \in S$.

The semicone is defined uniquely up to the isomorphism over S. This gives us the correspondence between strata and semicones and defines a bijection $\Sigma \to S$.

If one chart associates to a stratum a semicone σ_s^1 over S , and another associates σ_s^2 over S, then by Demushkin's theorem 7.7.5 σ_s^1, and σ_s^2, are isomorphic over S.

Let $s \leq s'$. Consider $\sigma_{s'}$ defined and σ_s defined by the charts. Then there is a face τ_s of $\sigma_{s'}$ corresponding to the stratum s. So we have a face inclusion $i : \tau_s \to \sigma_{s'}$ over S. Composing it with Demushkin's isomorphism $\alpha_s : \sigma_s \to \tau_s$ over S produces the face inclusion $i_{ss'} : \sigma_s \to \sigma_{s'}$ over S.

The conditions (1) (2) (3) are satisfied. They exactly mean that the face inclusions are defined over S. (See also the equivalent conditions in Remark 7.7.10(2).

By construction and definition the semicomplex Σ is determined uniquely up to isomorphism over S.

For any two such semicomplexes Σ and Σ' defined over S there is a natural bijection $\Sigma \to \Sigma'$. Moreover the corresponding cones σ_s and σ'_s are defined over S and thus related by the Demushkin isomorphism over S.

If a stratified toroidal variety is a toroidal embedding then the semicone σ is the set of all the faces of the cone
sigma|. The semicones can be replaced with cones without loss of data. The associated semicomplex is a usual conical complex.

Conversely if the associated semicomplex of the stratified toroidal variety is a complex. Then locally the strata are defined by toric orbits, so are induced by the intersecting of the divisors (codimension one strata) defining the structure of the toroidal variety. ♣

assoc

Example 7.8.3. Let (X_Σ, S_Ω) be a stratified toric variety associated with the embedded semifan (Σ, Ω) over $S = S_\Omega$. Then the associated semicomplex is given by

$$\Omega^{red} := \{(\omega, N^{\mathbf{Q}}_\omega) \mid \omega \in \Omega\}$$

with the natural face inclusions. It is a semicomplex defined over the stratification S

7.9. Relative semicomplexes and locally toric varieties with a divisor.

Definition 7.9.1. Let (X, S) be a stratified toroidal variety. We say that a subset V of X is *saturated* in (X, S) if it is the union of strata. A divisor of D on X will be called *toroidal* if it is the union of the strata.

The strata defined by the intersecting the components of D will be called *divisorial strata.*

Definition 7.9.2. By a relative semicomplex we mean the pair (Σ, Ω), of a semicomplex Σ and a complex $\Omega \subset \Sigma$.

The following Lemma extends Lemma 5.3.5 to the stratified toroidal varieties.

subcomplex **Lemma 7.9.3.** *Let (X, S) be a stratified toroidal varieties, and Σ be the associated semicomplex. There exists a bijective correspondence between the toroidal divisors D on (X, S) and saturated complexes $\Omega \subset \Sigma$. Any toroidal divisor D on (X, S) defines a unique saturated complex $\Omega := \Omega(D) \subset \Sigma$, such that $D = \overline{D_\Omega}$. In particular Ω is the set of the semicones (in fact cones) $\sigma \in \Sigma$ corresponding to the closed strata* $\operatorname{strat}(\sigma)$ *which are divisorial.*

Conversely any saturated complex $\Omega \subset \Sigma$ defines a unique toroidal divisor $D = \overline{D_\Omega}$.

Proof. The proof is identical to the proof of Lemma 5.3.5. ♣

Definition 7.9.4. By a toroidal divisor D on a stratified toroidal variety (X, S) we mean a divisor whose components are closed strata $\overline{s}$, for some $s \in S$.

subcomplex2 **Lemma 7.9.5.** *Let D be a toroidal divisor on a stratified toroidal variety (X, S). Let Σ be the semicomplex associated the stratifiction $S =$, and $\Omega \subset \Sigma$ be the subcomplex corresponding to D. Denote by $V = X(\Sigma)$ the relevant open subset such that $D = \overline{D_\Omega}$.*

Then

1. *The saturation of V in (X, D) is the toroidal locus $(X, D)^{\mathrm{tor}}$ of (X, D).*
2. *For any chart $\phi : U \to X_\sigma$, with $\sigma \in \Sigma$, and $\Omega_\sigma := \sigma \cap \Omega$ the subset $(U, D \cap U)^{\mathrm{tor}}$ is defined as $\phi^{-1}(X_{\mathrm{Reg}(\sigma, \Omega_\sigma)})$.*

Proof. (1) Since all the strata of D intersect V its saturation is exeactly the toroidal locus.

(2) By Lemma 7.5.6, the closures of the inverse images of the orbit strata of the toric variety $(X_{\Omega_\sigma}, D_{\Omega_\sigma}$ defined the closed strata on $(U, D \cap U)$. The toroidal locus $(U, D \cap U)^{\mathrm{tor}}$ is defined as the inverse image of the saturation of $(X_{\Omega_\sigma}, D_{\Omega_\sigma})$ in (X_σ, S_σ) which is equal to $X_{\mathrm{Reg}(\sigma, \Omega_\sigma)}$.

♣

7.10. **Inverse systems of affine algebraic groups.** We shall give the groups of the automorphisms of the completions of the local rings the structure of proalgebraic groups. All the proalgebraic groups here are considered over an algebraically closed field $\overline{K}$.

ffine proalgebraic group

Definition 7.10.1. [Wło03, Definition 4.4.1] By an *affine proalgebraic group* we mean an affine group scheme that is the limit of an inverse system $(G_i)_{i\in\mathbf{N}}$ of affine algebraic groups and algebraic group homomorphisms $\varphi_{ij}: G_i \to G_j$, for $i \geq j$.

le: epimorphisms

Lemma 7.10.2. [Wło03, Lemma 4.4.2] *Consider the natural morphism $\varphi_i : G \to G_i$. Then $H_i := \varphi_i(G)$ is an algebraic subgroup of G_i, all induced morphisms $H_j \to H_i$ for $i \leq j$ are epimorphisms and $G = \lim_{\leftarrow} G_i = \lim_{\leftarrow} H_i$. In particular $K[H_i] \subset K[H_{i+1}]$ and $K[G] = \bigcup K[H_i]$.*

le: K-points

Lemma 7.10.3. [Wło03, Lemma 4.4.3] *The set G^K of K-rational points of G is an abstract group which is the inverse limit $G^K = \lim_{\leftarrow} G_i^K$ in the category of abstract groups.*

By abuse of notation we shall identify G with G^K.

ex: differential

Example 7.10.4. [Wło03, Example 4.5.4] Let $\varphi_n : \mathrm{Aut}(\widehat{X}_x, S) \to \mathrm{Aut}(X_x^{(n)}, S)$ denote the natural morphisms. For $n = 1$ we get the differential mapping:

$$d = \varphi_1 : \mathrm{Aut}(\widehat{X}_x, S) \longrightarrow \mathrm{Aut}(X_x^{(1)}, S) \subset \mathrm{Gl}(\mathrm{Tan}_{X,x}).$$

de: orientation

Definition 7.10.5. [Wło03, Definition 4.9.1] We shall call a proalgebraic group G *connected* if it is a connected affine scheme. For any proalgebraic group $G = \lim_{\leftarrow} G_i$, denote by G^0 its maximal connected proalgebraic subgroup $G^0 = \lim_{\leftarrow} G_i^0$.

7.11. **Oriented semicomplexes.** Let σ be a semicone in $N_\sigma^{\mathbb{Q}}$. Denote by $\mathrm{Aut}(\sigma)$ the group of automorphisms of the semicone σ. Consider the natural inclusion $\phi : \mathrm{Aut}(\sigma) \to \mathrm{Aut}(\widehat{X}_\sigma)$. Then set

$$\mathrm{Aut}(\sigma)^0 := \phi^{-1}(\mathrm{Aut}(\widehat{X}_\sigma)^0)$$

e: oriented semicomplex

(The definition is equivalent to)

Definition 7.11.1. [Wło03, Definition 4.11.1] By an *oriented semicomplex* we mean a semicomplex Σ together with the associated groups $\mathrm{Aut}(\sigma)^0$ such that for any $\sigma \leq \tau \leq \gamma$ there is $\alpha_\sigma \in \mathrm{Aut}(\sigma)^0$ for which $\imath^\gamma_\tau \imath^\tau_\sigma = \imath^\gamma_\sigma \alpha_\sigma$.

7.12. **Subdivisions of semicomplexes.**

Lemma 7.12.1. [Wło03] *Let σ be a cone in a fan Σ. Let Δ be a subdivision of a fan Σ and $\sigma \in \Sigma$. Then $\Delta|\sigma := \{\delta \in \Delta \mid \delta \subset \sigma\}$ is a subdivision of σ. Recall that we can associate with any semicone σ a fan $\overline{\sigma}$ consisting of all faces of the cone $|\sigma|$.*

an oriented semicomplex

Definition 7.12.2. [Wło03, Definition 4.11.4] A *subdivision* of a semicomplex (respectively an oriented semicomplex) Σ is a collection $\Delta = \{\Delta^\sigma \mid \sigma \in \Sigma\}$ of fans Δ^σ in $N_\sigma^{\mathbb{Q}}$ where $\sigma \in \Sigma$ such that

(1) For any $\sigma \in \Sigma$, Δ^σ is a subdivision of the fan $\overline{\sigma}$ which is $\mathrm{Aut}(\sigma)$-invariant (resp. $\mathrm{Aut}(\sigma)^0$-invariant).

(2) For any $\tau \leq \sigma$, $\Delta^\sigma||\tau| = \imath^\sigma_\tau(\Delta^\tau)$.

Remark 7.12.3. (1) By abuse of terminology we shall understand by a subdivision of a semicone σ a subdivision of the fan $\overline{\sigma}$.

(2) By definition vectors in faces σ of a semicomplex (respectively an oriented semicomplex Σ) are defined up to automorphisms from $\mathrm{Aut}(\sigma)$ (respectively from $\mathrm{Aut}(\sigma)^0$). Consequently, the faces of subdivisions Δ^σ are defined up to automorphisms from $\mathrm{Aut}(\sigma)^0$.

(3) The condition on Δ^σ to be $\mathrm{Aut}(\sigma)$-invariant is for *canonical* subdivisions replaced with a somewhat stronger condition of similar nature which says that the induced morphism $\widehat{X}_{\Delta^\sigma} := X_{\Delta^\sigma} \times_{X_\sigma} \widehat{X}_\sigma \to \widehat{X}_\sigma$ is $\mathrm{Aut}(\widehat{X}_\sigma)$-equivariant.

7.13. **Stable support.**

Remark 7.13.1. It follows from definition of a semicomplex (resp. oriented semicomplex) that vectors in any face σ of a semicomplex (respectively an oriented semicomplex) are defined, in particular up to automorphisms from $\mathrm{Aut}(\sigma)$ (respectively $\mathrm{Aut}(\sigma)^0$). Hence the notion of the support of a semicomplex as a topological space which is the totality of such vectors is not well defined. However if we consider, for instance, vectors (thought as valuations) which are invariant under all such automorphisms and even larger groups of automorphisms like $\mathrm{Aut}(\widetilde{X}_\sigma)$ (respectively $\mathrm{Aut}(\widetilde{X}_\sigma)^0$) then the relevant topological space can be glued to form a topological space called *stable support* [Wło03, Definitions 5.3.1, 5.3.2, 6.1.1]. It plays the key role for oriented semicomplexes and their subdivisions.

The integral vectors in the stable support are called *stable* and the unique valuations they induce on X are also called *stable*. Despite the stable valuations are induced by stable vectors via charts they are independent upon charts. This follows from the fact that the stable vectors are invariant with respect "local analytic automorphisms" preserving the stratification. The notion of stable support of an oriented (or non oriented) semicomplex allows to run the algorithms with admissible- stable centers as in the case of complexes associated with toroidal varieties. The main difference is that only stable centers in the star subdivisions are allowed. In our situation we use minimal vectors or their sums. Since the algorithm in the paper is functorial we do not use explicitly the notion of stable vectors, and refer to more general properties. However all the centers used in the algorithm are stable. (It is implicitly stated in Lemmas 7.17.1, 7.17.3). The properties of the stable centers are used directly in the algorithm when applying Theorem 7.16.4.

7.14. **Orientation.** The notion of *orientation* as well as oriented semicomplexes is not used for the results of this paper since our algorithm of the combinatorial desingularization in Section 1 is functorial with respect to arbitrary automorphisms of semicomplexes. Neverless we briefly mention the idea of the orientation to place the theorems in the more general context. In fact, birational modifications defined for oriented complexes are simply more general and do not require strong coherency or functoriality conditions.

e: the same orientation

Definition 7.14.1. [Wło03, Definition 4.9.5, 4.9.12]

Let $\overline{K}$ be the algebraically closure of the base field K.

We say that two étale morphisms of stratified toroidal varieties over K, $f_1, f_2 : (X,S) \to (Y,T)$, *determine the same orientation* at a geometric point $\overline{x} \to X$ if

$$f_1(\overline{x}) = f_2(\overline{x}) := \overline{y}$$

and

$$(\widehat{f}_{2\overline{x}}^{\overline{K}})^{-1} \circ \widehat{f}_{1\overline{x}}^{\overline{K}} \in \mathrm{Aut}(\widehat{X}_{\overline{x}}^{\overline{K}}, \widehat{S}_{x}^{\overline{K}})^0,$$

[28] where

$$\widehat{f}_i^{\overline{K}} : (\widehat{X}_{\overline{x}}^{\overline{K}}, \widehat{S}_{\overline{x}}^{\overline{K}}) = (X,S) \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K}) \to (\widehat{Y}_{\overline{y}}^{\overline{K}}, \widehat{T}_{\overline{y}}^{\overline{K}}) := (Y,T) \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K}),$$

for $i = 1, 2$ are the induced isomorphisms.

We say that two smooth morphisms of stratified toroidal varieties over K, $g_1, g_2 : (X,S) \to (Y,T)$, *determine the same orientation* at a geometric point $\overline{x} \to X$ over $x \in X$ if $f_1(\overline{x}) = f_2(\overline{x})$, and there is locally a morphism $\phi : X \to \mathbb{A}^n$, such that the induced morphisms $f_i = (g_i, \phi) : (X,S) \to (Y,T) \times \mathbb{A}^n$, are étale at $x \in X$ (preserving strata) and determine the same orientation at $\overline{x}$.

The stratified toroidal variety is said to be *oriented* if it has a collection of charts which determine the same orientation. We say that two charts on a stratified toroidal variety determine the same orientation at a given point if they induce the smooth maps with the same target which determine the same orientation. ([Wło03, Definition 4.10.1]) The clear advantage of the oriented stratified toroidal varieties with oriented associated semicomplexes is that we can construct significantly more modifications (even non functorial ones) as we do have more allowable (canonical) subdivisions. For instance the standard desingularization algorithm for toric varieties can be run on oriented stratified toroidal varieties without essential modifications.

7.15. Toroidal modifications.

: toroidal modification

Definition 7.15.1. [Wło03, Definition 4.12.1] Let (X,S) be a stratified toroidal variety . We say that Y is a *toroidal modification* of (X,S) (respectively oriented toroidal modification) if

(1) There is given a proper morphism $f : Y \to X$ such that for any $x \in s = \mathrm{strat}_X(\sigma)$ there exists a chart $x \in U_\sigma \to X_\sigma$, a subdivision Δ^σ of σ, and a fiber square

$$\begin{array}{ccccc} & & U_\sigma & \xrightarrow{\varphi_\sigma} & X_\sigma \\ & & \uparrow f & & \uparrow \\ U_\sigma \times_{X_\sigma} X_{\Delta^\sigma} & \simeq & f^{-1}(U_\sigma) & \xrightarrow{\varphi_\sigma^f} & X_{\Delta^\sigma} \end{array}$$

(2) For any geometric point $\overline{x} : \mathrm{Spec}(\overline{K}) \to s \subset X$ in a stratum s every automorphism α of $\widehat{X}_{\overline{x}}^{\overline{K}}$ preserving strata (respectively preserving strata and orientation) can be lifted to an automorphism α' of $Y \times_X \widehat{X}_{\overline{x}}^{\overline{K}}$. (Hironaka's condition)

[28] G^0 is a connected component of the proalgebraic group G as in Definition 7.10.5

7.16. **Canonical subdivisions of semicomplexes.**

le: tilde

Lemma 7.16.1. [Wło03, Lemma 4.13.1] *Let (X, S) be any (respectively oriented) stratified toroidal variety of dimension n with associated oriented semicomplex Σ and let $f : Y \to (X, S)$ be a toroidal modification. Let $\overline{x} : \operatorname{Spec}(\overline{K}) \to \{x\} \subset X$ s be a geometric point in the stratum $\operatorname{strat}_X(\sigma) \in S$, $\varphi_\sigma : U \to X_\sigma$ a chart of a neighborhood U of x, and Δ^σ a subdivision of σ for which there is a fiber square*

$$\begin{array}{ccc} U & \stackrel{\varphi_\sigma}{\longrightarrow} & X_\sigma \\ \uparrow f & & \uparrow \\ f^{-1}(U) & \stackrel{\varphi^f_\sigma}{\longrightarrow} & X_{\Delta^\sigma} \end{array}$$

where the horizontal morphisms are smooth. Set

$$\operatorname{reg}(\sigma) := \langle e_1, \ldots, e_{n-\dim(N_\sigma)}\rangle = \langle e_1, \ldots, e_{\dim(\operatorname{strat}_X(\sigma))}\rangle.$$

Then

(1) there is a fiber square of étale extensions

$$\begin{array}{ccc} U & \stackrel{\widetilde{\varphi}_\sigma}{\longrightarrow} & X_{\widetilde{\sigma}} \\ \uparrow f & & \uparrow \\ f^{-1}(U) & \stackrel{\widetilde{\varphi}^f_\sigma}{\longrightarrow} & X_{\widetilde{\Delta}^\sigma} \end{array}$$

where the horizontal morphisms are étale and where

$$\widetilde{\sigma} := \overline{\sigma} \times \operatorname{reg}(\sigma) \text{ and } \widetilde{\Delta}^\sigma := \Delta^\sigma \times \operatorname{reg}(\sigma).$$

(2) $X_{\widetilde{\sigma}}$ is a stratified toric variety with the strata described by the embedded semifan $\sigma \subset \widetilde{\sigma}$. Moreover the strata on U are exactly the inverse images of strata of $X_{\widetilde{\sigma}}$.

(3) There is a fiber square of isomorphisms

$$\begin{array}{ccc} \widehat{X}^{\overline{K}}_x & \stackrel{\widehat{\varphi}_\sigma}{\simeq} & \widetilde{X}_\sigma \\ \uparrow \widehat{f}_x & & \uparrow \\ Y \times_X \widehat{X}^{\overline{K}}_x & \stackrel{\widehat{\varphi}^{f\overline{K}}_\sigma}{\simeq} & \widetilde{X}_{\Delta^\sigma} \end{array}$$

where

$$\widetilde{X}_\sigma := \widehat{X}^{\overline{K}}_{\widetilde{\sigma}} \text{ and } \widetilde{X}_{\Delta^\sigma} := X_{\widetilde{\Delta}^\sigma} \times_{X_{\widetilde{\sigma}}} \widetilde{X}_\sigma.$$

(4) $\widetilde{X}_\sigma$ is a stratified toroidal scheme with the strata described by the embedded semifan $\sigma \subset \widetilde{\sigma}$. The isomorphism $\widehat{\varphi}_\sigma$ preserves strata.

(5) The morphism $\widehat{f}_x : Y \times_X \widehat{X}^{\overline{K}}_{\overline{x}} \to \widehat{X}^{\overline{K}}_{\overline{x}}$ is $\operatorname{Aut}(\widehat{X}^{\overline{K}}_{\overline{x}}, S)$-equivariant .

(6) The morphism $\widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_\sigma$ is $\operatorname{Aut}(\widetilde{X}_\sigma)$-equivariant. ♣

Proof. The Lemma is a reinterpretation of the Hironaka condition and follows from Definition 7.15.1. For more details see [Wło03].

♣

We shall assign to the faces of an oriented semicomplex Σ the collection of connected proalgebraic groups over $\overline{K}$:

$$G_\sigma := \operatorname{Aut}(\widetilde{X}_\sigma).$$

$$G_\sigma^0 := \mathrm{Aut}(\widetilde{X}_\sigma)^0.$$

de: canonical

Definition 7.16.2. [Wło03, Definition 4.13.3] A subdivision $\Delta = \{\Delta^\sigma \mid \sigma \in \Sigma\}$ of a semicomplex (respectively oriented semicomplex) Σ is called *canonical* if for any $\sigma \in \Sigma$, G_σ (respectively G_σ^0) acts on $\widetilde{X}_{\Delta^\sigma}$ (as an abstract group) and the morphism $\widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_\sigma$ is G^σ-equivariant (respectively G_σ^0-equivariant).

le: surjection

Lemma 7.16.3. *(*[Dem82], [Wło03, Lemma 7.3.2]*) Let σ be a semicone. Then* $\mathrm{Aut}(\widetilde{X}_\sigma)^0 \subset \mathrm{Aut}(\widetilde{X}_\sigma)$ *is a normal subgroup and there is a natural surjection* $\mathrm{Aut}(\sigma) \to \mathrm{Aut}(\widetilde{X}_\sigma)/\mathrm{Aut}(\widetilde{X}_\sigma)^0$.

th: modifications

Theorem 7.16.4. *(*[Wło03, Theorem 4.14.1]*) Let (X, S) be stratified toroidal variety (respectively an oriented stratified toroidal variety) with the associated semicomplex (resp. the oriented semicomplex) Σ. There exists a bijective correspondence between the toroidal modifications Y of (X, S) (respectively the oriented toroidal modifications) and the canonical subdivisions Δ of Σ.*

(1) If Δ is a canonical subdivision of Σ then the toroidal modification associated to it is defined locally by

$$\begin{array}{ccccc} & & U_\sigma & \to & X_\sigma \\ & & \uparrow f & & \uparrow \\ U_\sigma \times_{X_\sigma} X_{\Delta^\sigma} & \simeq & f^{-1}(U_\sigma) & \to & X_{\Delta^\sigma} \end{array}$$

(2) If $Y^1 \to X$, $Y^2 \to X$ are toroidal modifications associated to canonical subdivisions Δ_1 and Δ_2 of Σ then the natural birational map $Y^1 \to Y^2$ is a morphism iff Δ_1 is a subdivision of Δ_2.

Proof. Sketch of the proof. The variety Y is obtained by gluing the pieces $U_\sigma \times_{X_\sigma} X_{\Delta^\sigma}$ which are birational to $U_\sigma \subset X$. We need to show that the gluing is independent of charts. Let $x \in s$, where s is a stratum corresponding to the cone σ. First we observe that by shrinking and restricting charts we can reduce the situation to two smooth charts $\phi_1, \phi_2 : U \to X_\sigma$. Then we can further assume that the charts are étale replacing ϕ_i with $\widetilde{\phi}_i : U \to X_\sigma = X_{\widetilde{\sigma}}$ with both charts taking x to the closed orbit $O_{\widetilde{\sigma}}$. Passing to the algebraic closure $\overline{K}$, and to the local rings at a geometric point $\overline{x}$ over x we see that the spaces $U_\sigma^{\overline{K}} \times_{X_{\widetilde{\sigma}}^{\overline{K}}} X_{\Delta_{\widetilde{\sigma}}}^{\overline{K}}$ does not glue over $\overline{x}$ if the induced spaces $\widehat{X}_{\overline{x}}^{\overline{K}} \times_{\widetilde{X}_\sigma^{\overline{K}}} \widetilde{X}_{\Delta^\sigma}^{\overline{K}}$ are not isomorphic for two different isomorphisms $\widehat{\phi}_{i\overline{x}} : \widehat{X}_{\overline{x}}^{\overline{K}} \to \widetilde{X}_\sigma$. But those isomorphisms differ by the automorphisms in $\mathrm{Aut}(\widetilde{X}_\sigma)$ (respectively in $\mathrm{Aut}(\widetilde{X}_\sigma)^0$ for the oriented stratified toroidal varieties), and these automorphisms lift to $\widetilde{X}_{\Delta^\sigma}$ inducing of the isomorphisms of $U_\sigma^{\overline{K}} \times_{X_{\widetilde{\sigma}}^{\overline{K}}} X_{\Delta_{\widetilde{\sigma}}}^{\overline{K}}$. By the faithful flatness descent $U_\sigma \times_{X_\sigma} X_{\Delta^\sigma}$ are also isomorphic.

The converse follows from the Definition and the previous Lemma 7.16.1. For the details see [Wło03].

(2) Follows from the analogous properties of toric varieties.

♣

The above theorem extends the notion of canonical birational modifications for toroidal embeddings. In particular we have le: tembeddings

Lemma 7.16.5. [Wło03, Lemma 6.3.1(2)]. *Let (X,S) be a strict toroidal embedding with the associated complex Σ. Then for any subdivision Δ_σ of σ the induced morphism $\widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_\sigma$ is G_σ-equivariant. Consequently all subdivisions of Σ are canonical. Moreover any morphism $X_{\Delta^\sigma} \to X_\sigma$ associated with the subdivision satisfies the Hironaka condition.*

Proof. Let $\delta \in \Delta^\sigma$. Each automorphism g from G^σ preserves the divisors defined by the rays of σ, hence it multiplies the generating monomials in $(\sigma^\vee)^{\rm integ}$ by invertible functions. This induces the action g of $\widetilde{X}_\sigma$ lifts to an automorphism g' of $\widetilde{X}_\delta = \widetilde{X}_\sigma \times_{X_{\widetilde{\sigma}}} X_{\widetilde{\delta}}$ which also multiplies monomials by suitable invertible functions. As $\mathcal{O}(\widetilde{X}_\delta)$ is a $\mathcal{O}(\widetilde{X}_\sigma)$ module with generators in $(\delta^\vee)^{\rm integ}$ which are the quotient of monomials in $(\sigma^\vee)^{\rm integ}$ Therefore g lifts $\widetilde{X}_\delta$ and to to the scheme $\widetilde{X}_{\Delta^\sigma} = \bigcup_{\delta\in\Delta^\sigma} \widetilde{X}_\delta$. The proof of "moreover part" is identical. ♣

7.17. **Use of minimal vectors.** The following observation is critical as it allows to run the desingularization combinatorial algorithms on oriented stratified toroidal varieties or functorial algorithms non-oriented stratified toroidal varieties.

By abuse of terminology a vector $v \in \sigma$ will be called G_σ-*invariant* (respectively G^0_σ-*invariant* if the corresponding valuation val(v) is G^0_σ-invariant (respectively G^0_σ-*invariant*) on $\widetilde{X}_\sigma$.

There are not too many G_σ-invariant vectors, but quite enough G^0_σ-invariant ones to run the desingularization on the oriented semicomplexes or functorial desingularization on (non-oriented) semicomplexes.

le: minimal vectors

Lemma 7.17.1. [Wło03, Lemma 5.3.15] *Let σ be a semicone and $\widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_\sigma$ be a G^0_σ-equivariant birational morphism induced by a toric morphism $X_{\Delta^\sigma} \to X_\sigma$ associted with subdivision Δ^σ of σ.*

1. [Wło03, Lemma 5.3.15(3,4)] *Let δ be an irreducible face of Δ^σ. Then all minimal internal points of δ are G^0_σ-invariant.*
2. [Wło03, Lemma 5.3.15(5)] *If v is vector in the ray (one dimensional face) in Σ then v is G^0_σ-invariant.*
3. [Wło03, Lemma 6.2.1(1)] *The set of the G^0_σ-invariant vectors in $\sigma$is convex. In particular the sum of minimal vectors or minimal generators of any $\delta \in \Delta^\sigma$ is G^0_σ-invariant.*

le: minimal vectors12

Corollary 7.17.2. *With the above notation*

1. *Let $\delta \in \Delta^\sigma$ be an irreducible face of $\Delta^\sigma, \Omega_\sigma)$. Then the sum of the minimal internal vectors of δ is G^0_σ-invariant. In particular the canonical barycenter of δ is G^0_σ-invariant.*
2. *Let $\delta \in \Delta^\sigma$ be a smiplicial cone then the sum of minimal vectors of (δ,ω) are G^0_σ-invariant*

le: minimal vectors2

Corollary 7.17.3. *Let σ be a semicone containing a complex $\Omega_\sigma \subset \sigma$ and $\widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_\sigma$ be a G^0_σ-equivariant birational morphism induced by a toric morphism $X_{\Delta^\sigma} \to X_\sigma$ associted with subdivision $(\Delta^\sigma,\Omega_\sigma)$ of (σ,Ω_σ).*

1. *All the integral vectors in $|\Omega_\sigma|$ are G^0_σ-invariant.*
2. *Let $\delta \in \Delta^\sigma$ be a relatively irreducible face of $\Delta^\sigma,\Omega_\sigma)$. Then the sum of the minimal internal vectors of* sing(δ) *and the vertices in $\Omega_\delta := \Omega \cap \delta$ is G^0_σ-invariant. In particular the canonical barycenter of (δ,Ω_δ) is G^0_σ-invariant.*

(3) *Let $(\delta,\omega) \in (\Delta^\sigma, \Omega_\sigma)$ be a relative simplicial cone then the the sum of the minimal vectors of (δ,ω) are G^0_σ-invariant*

Proof. (1) The vertices of Ω_σ are G^0_σ-invariant by Lemma 7.17.1(2) so their sums are also G^0_σ-invariant.

(2) Similar as (1).

(3) By Lemma **??** all the minimal vectors of the simplicial cone (δ,ω) are the sums of the minimal internal vectors and some vertices of ω.

♣

Remark 7.17.4. Both corollaries state that the centers used in the desingularization algorithm of the cones (or relative cones) are G^0_σ-invariant.

7.17.5. *Locally toric valuations.* Let X be an algebraic variety and ν be a valuation of the field $K(X)$ of rational functions. By the valuative criterion of separatedness and properness the valuation ring of ν dominates the local ring of a uniquely determined point (in general nonclosed) c_ν on a complete variety X. (If X is not complete such a point may not exist). We call the closure of c_ν the *center of the valuation* ν and denote it by $\mathrm{Z}(\nu)$ or $\mathrm{Z}(\nu, X)$. For any $x \in \mathrm{Z}(\nu)$ and $a \in \mathbf{Z}_{\geq 0}$,

$$I_{\nu,a,x} := \{f \in \mathcal{O}_{X,x} \mid \nu(f) \geq a\}$$

is an ideal in $\mathcal{O}_{X,x}$. For fixed a these ideals define a coherent sheaf of ideals $\mathcal{I}_{\nu,a}$ supported at $\mathrm{Z}(\nu)$.

le: center **Lemma 7.17.6.** *Let v be an integral vector in the support of the fan Σ. Then the toric valuation* $\mathrm{val}(v)$ *is centered on $\overline{O}_\sigma$, where σ is the cone whose relative interior contains v.*

Definition 7.17.7. Let X be a locally toric variety. We say that the valuation ν is locally toric if for any $x \in X$ there exists a neighborhood U of x, and an étale morphism $\phi : U \to X_\sigma$ and a vector $v \in \sigma \cap N_\sigma$ such that $\mathcal{I}_{\nu,a} = \phi^{-1}(I_{\mathrm{val}(v),a})$ for any $a \in \mathbb{N}$.

Smooth morphisms to toric varieties (charts) allow to define valuations locally:

le: ind2 **Lemma 7.17.8.** [Wło03] *Let X_Σ be a toric variety (associated with a fan Σ and $f : U \to X_\Sigma$ be a smooth morphism. Let $v \in \mathrm{int}(\sigma)$, where $\sigma \in \Sigma$, be an integral vector. Assume that the inverse image of $\overline{O}_\sigma$ is irreducible. Then there exists a valuation μ on U such that $\mathcal{I}_{\mu,a} = f^{-1}(\mathcal{I}_{\mathrm{val}(v),a}) \cdot \mathcal{O}_U$.*

Proof. We consider the completion $\widehat{\mathcal{O}_{x,X}}$ of the local ring of a closed point $x \in f^{-1}(O_\sigma)$. The smooth morphism defines locally étale morphism $f : U \to X_\Sigma \times A^n$ in a sufficiently small neighborhood.

We have the induced isomorphism $\widehat{f^*_x} : \widehat{\mathcal{O}_{f(x),X_\sigma}} \to \widehat{\mathcal{O}_{x,X}}$. The vector v defines the valuation on $\widehat{\mathcal{O}_{f(x),X_\sigma}}$ and on $\widehat{\mathcal{O}_{x,X}}$. Its restriction to $\mathcal{O}_{x,X}$ defines a valuation on U.

♣

de: blow **Definition 7.17.9.** [Wło03, Definition 5.2.6] By the *blow-up* $\mathrm{bl}_\nu(X)$ of X at a locally toric valuation ν we mean the normalization of

$$\mathrm{Proj}(\mathcal{O} \oplus \mathcal{I}_{\nu,1} \oplus \mathcal{I}_{\nu,2} \oplus \ldots).$$

pr: blow

Proposition 7.17.10. [Wło03, Proposition 5.2.9] *For any locally toric valuation ν on X there exists an integer d such that*

- $\mathrm{bl}_\nu(X) = \mathrm{bl}_{\mathcal{I}_{\nu,d}}(X)$.
- *The valuation ν is induced by an irreducible* $\mathbf{Q}$*-Cartier divisor on* $\mathrm{bl}_\nu(X)$.

Proof. The proof is identical as the proof of Lemma 6.3.7. Locally the blow-ups are induced by blow-ups of toric valuations so it is defined as $\mathrm{bl}_{\mathcal{I}_{\nu,d}}(X)$. By quasi-compactness of X we can find the same sufficiently divisible d for the open cover with toric chart.

♣

Also we have

divisors2

Corollary 7.17.11. *Let ν be a locally toric valuation on a locally toric variety X, and $\pi : \mathrm{bl}_\nu(X) = \mathrm{bl}_{\mathcal{I}_{\nu,d}}(X) \to X$ be the associated blow-up with the exceptional Weil, $\mathbb{Q}$-Cartier divisor D.*

Then for the ideals $\mathcal{I}_{D,n} = \mathcal{I}_{\mathrm{val}_D,n} = \{f \in \mathcal{O}_X \mid \nu_D(f) \geq a\}$ we have

$$\pi_*(\mathcal{I}_{D,n}) = \mathcal{I}_{\nu,d}$$

Thus the valuation ν is induced by an irreducible exceptional Weil ($\mathbf{Q}$*-Cartier) divisor on the variety* $\mathrm{bl}_{\mathrm{val}(v)}(X)$.

7.17.12. *Stable ideals and functoriality.* As before we represent the centers of the blow-ups of a locally toric valuation ν by the set of ideals $\mathcal{I}_{\nu,a}$.

multiple2

Definition 7.17.13. By the *multiple center* of the blow-up we mean the set of ideals $\{\mathcal{I}_n\}_{n\in\mathbb{N}}$ on a variety X such we have the equality of the blow-ups:

$$\mathrm{Proj}(\mathcal{O} \oplus \mathcal{I}_1 \oplus \mathcal{I}_2 \oplus \ldots) = \mathrm{bl}_{\mathcal{I}_d}(X).$$

for sufficiently divisible d.

7.18. Desingularization theorems.

th: resolution2

Theorem 7.18.1. *For any étale locally binomial (or locally toric) variety X [29] over a field K of any characteristic there exists a canonical resolution of singularities i.e. a birational projective $f : Y \to X$ such that*

1. *(1) f is an isomorphism over the open set of the nonsingular points.*
2. *(2) The inverse image $f^{-1}(\mathrm{Sing}(X)$ of the singular locus $(\mathrm{Sing}(X)$ is a simple normal crossing divisor on Y.*
3. *(3) f is a composition of the normalization and the normalized blow-ups of the locally monomial multiple centers $\{\mathcal{J}_{in}\}_{n\in\mathbb{N}}$ [30].*
4. *(4) f commutes with smooth morphisms and field extensions, in the sense that the centers are transformed functorially, and the trivial blow-ups are omitted.*
5. *(5) Moreover if D is an étale locally toric divisor D on a étale locally toric X[31] then there is functorial desingularization of (X, D) as above such that the strict transform of D has SNC with $f^{-1}(\mathrm{Sing}(X)$.*

---

[29] Definition 7.1.1
[30] Definition 7.17.13
[31] Definition 7.6.6

Proof. First we prove the theorem in the case of Zariski locally binomial varieties. By normalizing X we can further assume that it is locally toric.

Consider the canonical stratification $\mathrm{Sing}(\overline{X})$ on $\overline{X}$ as in Theorem 7.6.7. Then the pair $(\overline{X}, \mathrm{Sing}(\overline{X}))$ is a stratified toroidal variety. In case D is a locally toric divisor we consider the stratification $\mathrm{Sing}(\overline{X})$, with the additional property that the components of D are the closed strata of $\mathrm{Sing}(\overline{X})$ as in Theorem 7.6.7. The rest of the proof is the same for both cases.

Let Σ be the associated (conical) semicomplex. In case of (5) a locally toric divisor D Consider the complex $\overline{\Sigma}$ consisiting of the disjoint union of the cones $\sigma \in \Sigma$.

Let

$$\Delta = V_k \cdot \ldots \cdot V_1 \cdot \overline{\Sigma}$$

be the canonical subdivision of $\overline{\Sigma}$ as in Lemma 4.9.1. It is obtained by a sequence of star subdivisions at sets V_i of the points which are either minimal points or the sums of minimal points in singular irreducible faces. Denote by

$$\Delta_i = V_i \cdot \ldots \cdot V_1 \cdot \overline{\Sigma}$$

the intermediate subdivisions.

The subdivision Δ defines for any face $\sigma \in \Sigma$ the canonical desingularization

$$\Delta^\sigma = V_k^\sigma \cdot \ldots \cdot V_1^\sigma \cdot \sigma,$$

where $V_i^\sigma = V_i \cap |\sigma|$, and the intermediate subdivisions:

$$\Delta_i^\sigma = V_i^\sigma \cdot \ldots \cdot V_1^\sigma \cdot \sigma$$

By Lemma 7.17.1, all the points in the sets V_i^σ define the G^0_σ- invariant valuations on $\widetilde{X_\sigma}$. The action of G^0_σ on $\widetilde{X_\sigma}$ lifts to $\widetilde{X}_{\Delta^\sigma}$, and $\widetilde{X}_{\Delta^\sigma} \to X_\sigma$ is G^0_σ-equivariant. Also, by the canonicity of the algorithm the action of $\mathrm{Aut}(\sigma)$ on σ lifts to Δ_i^σ, so that $\Delta_i^\sigma \to \sigma$ is $\mathrm{Aut}(\sigma)$ invariant. By Lemma 7.16.3, G_σ is generated by G^0_σ and $\mathrm{Aut}(\sigma)$ we deduce that the action of G_σ on $\widetilde{X_\sigma}$ lifts to each scheme $\widetilde{X}_{\Delta_i^\sigma}$. On other hand, the functoriality of the algorithm implies that $(\Delta_i^\sigma)_{|\tau} = \Delta_i^\tau$. These data define a canonical subdivisions $\Delta_i = \{\Delta_i^\sigma \mid \sigma \in \Sigma\}$ of Σ. By Theorem 7.15.1, there exists a unique toroidal modification locally defined by the diagram as in the Theorem.

The canonical subdivisions $\{\Delta_i^\sigma\}$ define the intermediate varieties X_i. Moreover the sets of the vectors V_i^σ in each centers define G_σ-invariant sets of valuations on each $\widetilde{X}_{\Delta_i^\sigma} \to \widetilde{X}_\sigma$ is G_σ-equivariant. The morphism $\pi_i : X_i \to X_{i-1}$, is locally described by the star subdivisions $\Delta^i_\sigma = V_i^\sigma \cdot \Delta^{i-1}_\sigma$ of fans of Δ^{i-1}_σ. Thus the exceptional divisors D_{ij} of each π_i define locally toric valuations, corespond to vectors $v_{ij} \in V_i^\sigma = \{v_{i1}, \ldots, v_{ik_\sigma}\}$.

Consider the corresponding ideals

$$\mathcal{I}_{i,n} := \prod_{v \in V_i} \pi_*(\mathcal{I}_{nD_i}),$$

on X_{i-1}. By Lemma 6.3.8, they are locally described as the pull-backs of the monomial ideals associated with V_i^σ on $X_{\Delta_i^\sigma}$:

$$\mathcal{I}_{V_i,n} := \prod_{v \in V_i^\sigma} \mathcal{I}_{\mathrm{val}(nv_{ij})}$$

Then the morphism $X_i \to X_{i-1}$ is the blow -up of the multiple ideal center $\mathcal{I}_{i,n}$.

The resulting variety $Y = X_k$ defined by $\{\Delta^\sigma \mid \sigma \in \Sigma\}$ is regular. Since the desingularization $\{\Delta^\sigma\}$ does not affect regular cones the nonsingular points remain unaffected. The morphism Y is the composition of the blow-ups of the functorial multiple centers $\mathcal{I}_{i,n}$. The inverse image of the $f^{-1}(\mathrm{Sing}(X)$ of the singular locus $(\mathrm{Sing}(X)$ is defined locally by the toric divisor on a nonsingular toric variety X_{Δ^σ} so it is SNC.

The algorithm commutes with smooth or étale morphisms (or field extensions) since the charts which are pull-backs can be used to describe it.

The desingularization morphism $Y \to X$ of a locally toric variety (X, D) except of V is thus the composition of the blow-ups of the functorial multiple centers $\mathcal{I}_{i,n}$.

Now assume that X is étale locally binomial variety. Consider its normalization. Since the normalization commutes with étale maps we obtain an étale locally toric variety. By the previous case there exist compatible desingularizations on étale locally toric cover. These compatible desingularization descend to to the desingularization Y of X. Moreover they define SNC exceptional divisor E on a strict toroidal cover which descends to an NC divisor on Y. It suffices to apply Proposition 6.2.4 in order to further transform it to an z SNC divisor.

We use identical argument as in the proof of Theorem 6.5.1.

7.19. **Desingularization of étale locally toric varieties except of a toroidal subset.**

th: resolution4

Theorem 7.19.1. *Let X be an étale locally toric variety over a field K with a locally toric Weil divisor D [32]. Let $V \subset X$ be an open toroidal subset [33] of X with the divisor $D_V := D \cap V$. Furthermore assume that D_V has locally ordered components.*[34]

There exists a canonical resolution of singularities of (X, D) except of V i.e. a birational projective toroidal map $f : Y \to X$ such that

1. *f is an isomorphism over the open set V.*
2. *The variety (Y, D_Y) is a strict toroidal embedding, where $D_Y := \overline{D_V}$ is the closure of the divisor D_V in Y.*
3. *The variety (Y, D_Y) is the saturation of (V, D_V).*[35]
4. *The exceptional locus E of f has simple normal crossings with D_Y [36]. So $(Y, D_Y \cup E)$ is a strict toroidal embedding.*
5. *If V is the toroidal locus then $E = Y \setminus V$, and $D_Y \cup E$ is the inverse image of D.*
6. *The set of points where f is an isomorphism is the saturation of (V, D_V) in (X, D_X), where $D_X := \overline{D_V}$ is the closure of the divisor D_V in X.*
7. *In particular, if (V, D_V) is a smooth toroidal subset of (X, D) and D_V is an SNC divisor on V then Y is smooth then E is and SNC divisor and $D_Y \cup E$ is an SNC divisor.*

---

[32]Definitions 7.1.1, 7.6.6
[33]Definition 2.1.5
[34]Definition 2.1.20
[35]Definition 2.1.7
[36]Definition 2.1.14

(8) f is obtained by a sequence of blow-ups at the canonical multiple ideals centers[37].

(9) f commutes with smooth morphisms and field extensions preserving the subset V, and the order of the components D_V, in the sense that the centers are transformed functorially, and the trivial blow-ups are omitted.

Proof. The proof is very similar. First assume that X is locally toric with a locally toric divisor D. If V is the toroidal locus of (X, D) then it intersects all divisorial components which are smooth (so toroidal) at generic points.

In general we can also assume that $D = \overline{D_V}$, and that V is a saturated toroidal subset in (X, D) (by replacing D with $\overline{D_V}$ and saturating V, if necessary).

The locally toric divisor D on a variety X defines a natural stratification $S = \mathrm{Sing}_D(X)$, and the associated semicomplex Σ. Moreover the divisor D defines a subcomplex $\Omega \subset \Sigma$. (see Lemma 7.9.3) For any semicone $\sigma \in \Sigma$ the intersection of σ with Ω defines a subcomplex Ω_σ, which is a complex consisting of some faces of σ. Denote by $X^0_{\Omega_\sigma}$ the saturation of X_{Ω_σ} in X_σ. Consider a local chart $U \to X_\sigma$. The toroidal locus $(U, D \cap U)^{\mathrm{tor}} = V \cap U$ is defined locally as $V \cap U = \phi^{-1}(X_{\mathrm{Reg}(\sigma,\Omega_\sigma})$.

Denote $\mathrm{Reg}(\sigma, \Omega_\sigma)$ by $(\Omega^0_\sigma, \Omega_\sigma)$. The automorphisms $\mathrm{Aut}(\sigma)$ of the cone σ preserve all faces of the semicone σ and are the identical on vertices of ω, and G^0_σ-invariant. (Since both groups preserve vertices of Ω_σ)

In the cononical desingularization algorithm Δ^σ of σ we use centers which are either the sums of the minimal generators in faces σ and vertices in $\sigma \cap \Omega$ or the minimal generators in σ. Such centers are G^0_σ-invariant by Lemma and $\mathrm{Aut}(\sigma)$-invariant.

Repeating the reasoning from the previous proof we construct the variety $Y = X_k$ which is locally defined by a regular fans Δ^σ where $\sigma \in \Sigma$. The desingularization $\{\Delta^\sigma\}$ does not affect relatively regular cones with respect to Ω_σ. Thus , in particular the cones in Ω^0_σ remain unaffected. Consequently the points in V are not modified in the process. Since $(\Delta^\sigma, \Omega_\sigma)$ is an embedded fan we get by Proposition 7.5.6, that:

(1) $(X_{\Delta^\sigma}, \overline{D_{\Omega_\sigma}})$ is a strict toroidal embedding.
(2) The exceptional divisor $E_\sigma \subseteq X_{\Delta^\sigma} \setminus X_{\Omega_\sigma}$ of $X_{\Delta^\sigma} \to X_\sigma$ has SNC.
(3) $(X_{\Delta^\sigma}, \overline{D_{\Omega_\sigma}})$ is the saturation of $(X_{\Omega_\sigma}, D_{\Omega_\sigma})$ so also the saturation of $(X_{\Omega^0_\sigma}, D_{\Omega^0_\sigma})$.

This implies properties (1)-(6).

The rest of the reasoning is the same as in the proof of Theorem 7.19.1. If the variety X is étale locally toric over K then we consider desingularizations at functorial centers on an étale cover X^0 then the centers descend (by the flat descent) to the centers on X.

8. Comparison of the desingualarization algorithm for toroidal embeddings and locally toric varieties

compari

A strict toroidal embedding is a locally toric variety when we forget the toroidal structure. The desingularization algorithm is identical in both cases. If Σ is the

[37] Definition 7.17.13

complex associated with the toroidal structures then $\mathrm{sing}(\Sigma)$ defines the semicomplex associated with the canonical singular stratification $S = \mathrm{Sing}(X)$ on X.

The same algorithm in both situations is induced by the decomposition of the faces of $\mathrm{sing}(\Sigma)$. In the case of toroidal embedding the decomposition of $\mathrm{sing}(\Sigma)$ simply extends uniquely to decomposition of Σ defining the relevant toroidal modification. In the locally toric situation the decomposition of $\mathrm{sing}(\Sigma)$ induces the modification directly via the charts.

A point x on a toroidal embedding belongs to an open neighborhood associated with a face $\sigma \in \Sigma$. The modification of σ is induced by the subdivision of its singular part $\mathrm{sing}(\sigma)$. The very same point on the same variety with a locally toric structure with singular stratification will be associated via a chart with a face $\mathrm{sing}(\sigma)$ of the semicomplex $\mathrm{sing}(\Sigma)$, and the transformation will be again defined by the same subdivision of $\mathrm{sing}(\sigma) \in \mathrm{sing}(\Sigma)$. So both algorithms agree locally and globally.

♣

9. Counterexample to the Hironaka desingularization of locally binomial varieties in positive characteristic

Hir **Example 9.0.1.** Let $X \subset \mathbf{A}^4 = \mathrm{Spec}(K[x,y,z,w])$ be the hypersurface over a field of characteristic p, defined by a single equation $x^p - y^p z$. It admits the action of the group of automorphisms $\mathbf{A}^1 = \mathrm{Spec}(K[t])$:

$$x \mapsto x + ty, \quad y \mapsto y, \quad z \to z - t^p.$$

The locus $\mathrm{Sing}_p(X)$ of the points of maximal order p is given by the smooth subvariety $Z = V(x,y)$. Suppose that there is a maximal contact $u \in \mathcal{O}_X$ such that $V(u) \supset \mathrm{Sing}_p(X)$ and that the property will be preserved after blow-up at smooth centers contained in $\mathrm{Sing}_p(X)$. Thus u has the form

$$u = a(z,w)x + b(z,w)y + f(x,y),$$

where $f(x,y) \in (x,y)^2$. Using automorphisms above we can always assume that $b(0,0) \neq 0$. Applying the blow-up at $x = y = z = w = 0$ in the chart w:

$$(x,y,z,w) \mapsto (xw, yw, zw, w)$$

we transform $x^p - y^p z$, into

$$w^p(x^p - y^p zw).$$

Moreover the equation of u will maintain its form with the condition $b(0,0) \neq 0$.

After p such blow-ups (and factoring the exceptional divisors) we obtain the form

$$x^p - y^p zw^p.$$

Then the order p locus $\mathrm{Sing}_p(X)$ has now two components $V(x,y)$ and $V(x,w)$. The component $V(x,z)$ is not contained in $V(u)$ since $b(0,0) \neq 0$ and the intersection $V(x,w) \cap V(u)$ is equal to $V(x,y,w)$ in a neighborhood $b(z,w) \neq 0$.

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Department of Mathematics, Purdue University, 150 N. University Street,, West Lafayette, IN 47907-2067