

SINGULAR IMPLICIT AND INVERSE FUNCTION THEOREMS. STRONG RESOLUTION WITH NORMALLY FLAT CENTERS

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ABSTRACT. Building upon ideas of Hironaka, Bierstone-Milman, Malgrange and others we generalize the inverse and implicit function theorem (in differential, analytic and algebraic setting) to sets of functions of larger multiplicities (or ideals). This allows one to describe singularities given by a finite set of generators or by ideals in a simpler form. In the special Cohen-Macaulay case we obtain a singular analog of the inverse function theorem. The singular implicit function theorem is closely related to a (proven here) extended version of the Weierstrass-Hironaka-Malgrange division and preparation theorems. The primary motivation for this paper comes from the desingularization problem. As an illustration of the techniques used, we give some applications of our theorems to desingularization extending some results on Hironaka normal flatness, the Samuel stratification and the Hilbert-Samuel function. The notion of the standard basis along Samuel stratum introduced in the paper (inspired by the Bierstone-Milman and Hironaka constructions) allows us to describe singularities along the Samuel stratum in a relatively simple way.

It leads to a canonical reduction of the strong Hironaka desingularization with normally flat centers to a so called resolution of marked ideals. Moreover in characteristic zero the standard basis along Samuel stratum generates a unique canonical Rees algebra along Samuel stratum giving a straightforward proof of the strong desingularization in algebraic and analytic cases.

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0. INTRODUCTION

Suppose that $f(t, x)$ is an analytic function of $t \in \mathbb{C}$ and $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ near the origin, and let k be the smallest integer such that

$$f(0, 0) = 0, \quad \frac{\partial f}{\partial t}(0, 0) = 0, \dots, \frac{\partial^{k-1} f}{\partial t^{k-1}}(0, 0) = 0, \quad \frac{\partial^k f}{\partial t^k}(0, 0) \neq 0.$$

Then the **Weierstrass preparation theorem** states that near the origin, f can be written uniquely as the product of an analytic function c that is nonzero at the origin, and an analytic function that as a function of t is a polynomial of degree k . In other words,

$$f(t, x) = c(t, x)(t^k + a_{k-1}(x)t^{k-1} + \dots + a_0(x))$$

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where the functions c and a_i are analytic and c is nonzero at the origin. The theorem can also be understood as a direct generalization of the implicit function theorem in the analytic situation for a single function. The solution of the equation is given in implicit polynomial form. If $k = 1$ we obtain the solution in the explicit form $t = a(x)$ or equivalently $f(a(x), x) = 0$.

The Weierstrass preparation theorem is closely related to and follows from the **Weierstrass division theorem** which says that if f and k satisfy the conditions above and g is an analytic function near the origin, then we can write

$$g = qf + r,$$

where q and r are analytic, and as a function of t , $r = \sum_{j=0}^{k-1} t^j r_j(x)$ is a polynomial of degree less than k . The theorem plays an important role in analytic and differential geometry. Recall that the analog of the classical Weierstrass preparation theorem for smooth real functions was conjectured by Thom and proved by Malgrange (Weierstrass preparation) and Mather (Weierstrass division). The smooth case is much more difficult and deeper, and of fundamental importance in the theory of singularities.

On the other hand, Hironaka in his proof of desingularization theorem, and many others like Aroca, Vicente, Briançon, Jalambert, Teissier ([2], [20], [49]) studied different versions of division for multiple generators (respectively ideals). The so called **Weierstrass-Hironaka formal division theorem** says that for a set of formal analytic functions f_1, \dots, f_r and the corresponding set of leading exponents there exists division with remainder

$$g = \sum h_i f_i + r,$$

where h_i, r satisfy some combinatorial conditions.

In [38] Hironaka proved a weaker form of so called Henselian division for algebraic functions. A somewhat different approach and language was used in Bierstone-Milman's papers on desingularization ([9], [10]). It allows a better control over singularities exploiting the (simplest) formal analytic version of division and requires passing to formal coordinate charts.

In this paper we use the ideas of Hironaka and Bierstone-Milman and many others. One of our main goals is to establish a generalized version of Weierstrass-Hironaka division for algebraic, analytic, and differentiable functions. As a consequence, we prove a generalized preparation theorem, and the existence of a standard basis (with respect to a monotone order) satisfying some stronger conditions.

One can look at the classical Weierstrass division and preparation from two different perspectives. It can be regarded as an extension of the one-variable implicit function theorem and also as division or preparation with respect to a dominating monomial. The second approach requires introduction of a certain monomial and was exploited by the Hironaka generalization.

The method of monomial order may be easier from the computational point of view but it has severe limitations - as the resulting division and the dominating monomials dramatically change when passing from a point to its neighborhood.

On the other hand, the classical implicit function theorem for several variables allows one to describe subvarieties (submanifolds) in terms of differential (coherent) conditions, and uses no monomial order.

In this paper we generalize the implicit function theorem to the case of several functions of higher multiplicities and to ideals. By using a "resultant-type" extension of standard Jacobian matrices, we get a coherent polynomial description of singularities. The approach gives a great flexibility in selecting "dominating" monomials and controlling the forms of the describing equations in a neighborhood, since no monomial order is used.

In particular, in the simplest situation of singularities defined by k equations of multiplicities d_1, \dots, d_k which are in general position, one can choose the "leading monomials" to be powers of independent coordinates $x_1^{d_1}, \dots, x_k^{d_k}$. This means that the generic intersection of hypersurfaces is somewhat similar to the intersections of transversal hyperplanes with multiplicities, at least in terms of the common zeroes (counted with multiplicities). This gives a certain analogy to Bézout's theorem, already observed by Macaulay. In the case of homogeneous polynomials the resulting matrices were used by Macaulay in his definition of resultant.

Also in the Cohen-Macaulay case one can prove a certain analog of the inverse function theorem (in algebraic, analytic and differential cases). It is closely related, in the analytic case, to the Grauert-Remmert theorem on finite morphisms.

Upon establishing a generalized implicit function theorem we define a more general notion of standard basis (and pre-basis) along Samuel stratum.

Existence of a standard basis quite immediately implies Hironaka's normal flatness theorem and Bennett's theorems on upper semicontinuity of the Hilbert-Samuel functions (also in the differential setting).

As a consequence we show how to reduce in a canonical way a Hironaka desingularization with smooth normally flat centers to resolution of so called marked ideals (or Hironaka's idealistic exponents) which leads to very straightforward proofs of Hironaka resolution theorems for algebraic varieties in its strongest form. (The same method can be also applied to analytic spaces).

The paper is organized as follows. In the first chapter we discuss some extensions of Weierstrass division for coherent sheaves in the algebraic and analytic category. We also show "neighborhood versions" of classical division theorems in the algebraic and analytic cases. We use some results of Grauert and Remmert in the analytic category and some Hironaka's ideas in the algebraic situation. As a consequence we show a certain Cohen-Macaulay analog of the inverse function theorem (in the algebraic and analytic setting ; Theorems 1.1.8, 1.2.4, 1.2.5)

In the second chapter we review the classical results by Malgrange and Mather. We also prove stronger "neighborhood versions" of classical division theorems of Weierstrass and Malgrange-Mather in the differential setting. Also by combining the Grauert-Remmert approach in the analytic situation with the Malgrange-Mather technique in the differential setting we prove a certain Cohen-Macaulay analog of the inverse function theorem in the differential cases; Theorem 1.1.8, 1.2.5). We also define a category of smooth objects. It allows us to treat similarly algebraic, analytic and smooth functions in the subsequent chapters.

In Chapter 3, following the ideas of Hironaka, Galligo, Grauert and Bierstone-Milman we prove a somewhat extended version of Weierstrass-Hironaka-Grauert-Galligo division in the simplest case of formal analytic functions (for any monomial order) (Theorem 3.1.9). We also review basic properties of monotone diagrams and diagrams of finite type, used in the Hironaka and Bierstone-Milman papers. In our paper we combine the quite different approaches of Hironaka and Bierstone-Milman. Then we show some natural decompositions of the diagrams which play an essential role in generalized division and preparation theorems proven in Chapter 5 (Corollary 3.4.5).

In Chapter 4 we introduce a notion of filtered Stanley decomposition (Definitions 4.1.2, 4.1.4). Recall that Stanley decomposition was constructed and studied originally by Stanley, and is a fundamental tool in homological algebra. In this paper we introduce a stronger version of filtered Stanley decomposition which turns out to be a critical tool in the proofs of the implicit function theorem and division-preparation theorem. We prove existence of the filtered Stanley decomposition for any graded modules over polynomial rings (over an infinite field) and for modules over smooth rings, together with some of their direct consequences (Theorems 4.3.5, 4.3.4). The language of filtered Stanley decomposition allows us to reduce, by using the Malgrange theorem for modules, the division of functions to the division of their initial forms in graded rings. The constructions are motivated by ideas of both Hironaka's Henselian division and Bierstone-Milman's stabilization theorems for monotone diagrams (Theorem 4.1.12).

At the beginning of Chapter 5 we establish generalized versions of the Weierstrass-Hironaka division and preparation theorems as a consequence of the results proven in Chapter 4 (Theorems 5.1.1, 5.1.2, 5.1.3). In particular, the division theorem in the algebraic setting extends and strengthens the Hironaka Henselian division theorem. On the other hand, since the results are proven also in differential settings, they extend the Malgrange-Mather preparation and division theorems to the case of multiple generators. As a consequence we give the construction of a standard basis with respect to the monomial order (in either setting).

In the second part of Chapter 5 we prove a generalized implicit function theorem and its consequences (Theorems 5.3.1, 5.3.3, 5.3.3). Upon, introduced here generalized Jacobian conditions, one can represent the singularities in a simpler "reduced" form. The very basic idea goes back to Bierstone-Milman stabilization Theorem and their approach to standard basis ([7]). In the particular case of complete intersections related algebraic conditions were studied by Macauley in the context of his notion of resultant. [57]

In Chapters 6-9 we show some consequences of the singular implicit function theorem proving existence of Hironaka canonical desingularization of algebraic varieties in its strongest version, by the sequence of blow-ups of smooth normally flat centers. Unlike in the weaker version where a nonembedded resolution is achieved merely by a birational projective morphism in Hironaka's original approach the desingularization uses a sequence of smooth normally flat centers (see [38],[7], [68]). The condition of normal flatness means that the normal cones of the varieties along the centers are flat. This gives a certain geometric control over the process.

In Chapter 6 we briefly discuss the notion of resolution of marked ideals. Then using the singular implicit function we construct a coherent notion of standard basis of any ideal along Samuel stratum (Definition 6.3.3), proving Hironaka's normal flatness theorem and Bennett's theorems on the Hilbert-Samuel function (also in the differential setting). The notion of standard basis along Samuel stratum is a counterpart of Hironaka's distinguished data (as in [38]) and Bierstone-Milman's semicoherent presentation of ideals (as in [9],[10]). In our approach it is merely an extension of the classical implicit function theorem to a more complex case of singular subspaces (see also Example 6.3.4). The conditions for the standard basis are closely related to the ones for the Bierstone-Milman's standard basis relative to a diagram for formal analytic functions. On the other hand the constructions are obtained in the language which stems from Hironaka's Henselian division. The notion allows us to reduce strong resolution of singularities to resolution of marked functions with assigned multiplicities (Theorems 6.4.1, 6.5.5, 6.4.3).

In Chapter 7 we show that the constructed standard basis, though not unique, defines a unique associated (multiple) marked ideal- so called canonical Rees Algebra controlling Hilbert-Samuel function (see Theorem 7.2.9).

In Chapter 8 we formulate Hironaka's resolution theorems in algebraic setting and show that (in characteristic zero) they can be deduced via canonical Rees algebra to the canonical resolution of marked ideals (see Theorem 8.2.2).

In Chapter 9 we briefly show existence of canonical resolution of marked ideals in order to complete the proof of the desingularization theorems.

Although one of our main foci is given by the differential perspective of the Malgrange-Mather approach, and the algebraic perspective of Cohen-Macaulay rings, the primary motivation for this paper comes from the desingularization problem. Division theorems are a tool of fundamental importance when studying singularities, especially in positive characteristic. While, in characteristic zero, the technique allows one to prove Hironaka desingularization in its strongest form, it seems to be indispensable in positive characteristic. In fact, different particular versions of Weierstrass division for multiple generators were applied in many recent papers on desingularization (see for instance [12], [40], [4], [46], [73]).

1. WEIERSTRASS DIVISION FOR SHEAVES OVER ALGEBRAIC AND ANALYTIC FUNCTIONS

In this section we shall study generalized versions of Weierstrass division for algebraic and analytic functions which will be exploited in the remaining part of the paper.

1.1. Hironaka's claim and Weierstrass division for algebraic functions. A particular case of the following theorem (for $k = 1$) was used by Hironaka in [38] (with proof omitted) and was referred to as "Hensel's lemma". Here we also prove its general "Malgrange" form (originally proved by Malgrange for smooth functions) and extend it to sheaf "Grauert-Remmert" versions (similar to the versions proven by Grauert-Remmert for holomorphic functions). In the next sections we shall also study the analytic and smooth versions of the theorems proven below and their generalizations.

The Hironaka claim can be derived directly from the properties of Henselian rings via a version of Zariski's main theorem.

Theorem 1.1.1 (Hironaka's claim). *Let R be a local Henselian and Noetherian ring with maximal ideal m , and let $S = R\langle z_1, \dots, z_k \rangle$ denote the Henselianization of the localization $R[z_1, \dots, z_k]_{m_k}$ at the maximal ideal $m_k := m + (z_1, \dots, z_k)$. Suppose M is a finite $R\langle z_1, \dots, z_k \rangle$ -module and $M/(m \cdot M)$ is a finite-dimensional vector space over $K = R/m$. Then M is finite over R .*

Before proving this theorem we shall need a simple Lemma

Lemma 1.1.2. *Let $R \rightarrow S$ be a map (homomorphism) of noetherian local rings. Let $m \subset R$ be a maximal ideal. Suppose M is a finite S -module, and $I := \text{Ann}(M) \subset S$ is the annihilator of M . Then the following conditions are equivalent*

- (1) $M/(m \cdot M)$ is a finite-dimensional vector space over $K = R/m$.
- (2) $S/(I + m \cdot S)$ is finite over $K = R/m$

Proof. Recall that by the support of S -module M we mean

$$\text{supp}(M) := \{p \in \text{Spec}(S) \mid M_p \neq 0\},$$

(where M_p is the localization of M at the prime ideal $p \subset S$). Then, it follows ([66, Tag 080S]) that $\text{supp}(M) = V(I)$, where

$$V(I) = \{p \in \text{Spec}(S) \mid p \supset I\}$$

is the *vanishing locus* of I . Also

$$\text{supp}(M/(m \cdot M)) = V(I) \cap V(m \cdot S) = V(I + m \cdot S) = \text{supp}(S/(I + m \cdot S)).$$

On the other hand $\text{supp}(M/(m \cdot M)) = V(J)$, where $J \subset S$ is the annihilator of $M/(m \cdot M)$. Note that $J \supseteq I + m \cdot S$, and we have a map $R/m \rightarrow S/J$.

Thus S/J acts faithfully on $M/(m \cdot M)$. Since $M/(m \cdot M)$ is finite over $K = R/m$ we get that S/J is finite over K . (Since S/J embeds into a finite K -vector space of endomorphisms $\text{End}_K(M/(m \cdot M))$).

On the other hand $V(J) = V(I + m \cdot S)$, and thus $(I + m \cdot S) \supset J^k$. Since S/\mathcal{J}^k has a filtration \mathcal{J}^i with factors $\mathcal{J}^i/\mathcal{J}^{i+1}$ finite over S/\mathcal{J} we get that each factor is finite over $K = R/m$, and finally S/\mathcal{J}^k and its factor $S/(I + m \cdot S)$ is finite over $K = R/m$.

Conversely $M/(m \cdot M)$ is finite over $S/(I + m \cdot S)$. □

Proof. Let $\mathcal{I} \subset R\langle z_1, \dots, z_k \rangle$ denote the annihilator of M as before. Consider an étale affine neighborhood U of $\mathbf{A}_R^k := \text{Spec}(R[z_1, \dots, z_k])$ containing the generators of \mathcal{I} and preserving the residue field K so that $\mathcal{O}(U) \subset R\langle z_1, \dots, z_k \rangle$. Denote by $\mathcal{I}_U \subset \mathcal{O}(U)$ the ideal determined by these generators. Note that the map $R \rightarrow \mathcal{O}(U)$ is of finite type.

Let $x \in U$ denote the point corresponding to the ideal m_x , let $\mathcal{O}(U)_x$ be the localization at x , set $\mathcal{I}_x := \mathcal{I}_U \cdot \mathcal{O}(U)_x \subset \mathcal{O}(U)_x$, and let $m_x \subset \mathcal{O}(U)_x$ be the maximal ideal of x . The Henselianization of $\mathcal{O}(U)_x/\mathcal{I}_x$ is equal to $R\langle z_1, \dots, z_k \rangle/\mathcal{I}$ since Henselianization commutes with quotients. Likewise the Henselianization of $\mathcal{O}(U)_x/((\mathcal{I}_U + m) \cdot \mathcal{O}(U)_x)$ is $R\langle z_1, \dots, z_k \rangle/(\mathcal{I} + m)$, which is a finite-dimensional vector space over K . Thus $\mathcal{O}(U)_x/((\mathcal{I}_U + m) \cdot \mathcal{O}(U)_x)$ is a finite-dimensional vector space itself. In other words, $R \rightarrow \mathcal{O}(U)/\mathcal{I}_U$ is a map of finite type and the localization of the fiber over m is finite. Thus by shrinking U we can assume that the fiber of $R \rightarrow \mathcal{O}(U)/\mathcal{I}_U$ over m is irreducible and its reduced structure coincides with $\{x\}$, and $m \cdot \mathcal{O}(U) \supset m_x^\ell$ for a certain ℓ .

By a version of Zariski's main theorem due to Raynaud [64] one can factor this map as

$$R \rightarrow \mathcal{O}(U')/\mathcal{I}'_U \rightarrow \mathcal{O}(U)/\mathcal{I}_U$$

into a composition of a finite map $\phi^* : R \rightarrow \mathcal{O}(U')/\mathcal{I}'_U$ followed by the map $\mathcal{O}(U')/\mathcal{I}'_U \rightarrow \mathcal{O}(U)/\mathcal{I}_U$ defining and open inclusion $\text{Spec}(\mathcal{O}(U)/\mathcal{I}_U) \subset \text{Spec}(\mathcal{O}(U')/\mathcal{I}'_U)$. To be more precise, there exists $f \in \mathcal{O}(U') \subset \mathcal{O}(U)$, with $f \notin m_x$, such that $(\mathcal{O}(U)/\mathcal{I}_U)_f = (\mathcal{O}(U')/\mathcal{I}'_U)_f$ with $\mathcal{O}(U')/\mathcal{I}'_U$ a finite R -module. The latter is, by Hensel's lemma, isomorphic to a finite product of local Henselian rings R_i :

$$\mathcal{O}(U')/\mathcal{I}'_U \simeq \bigoplus_{i=1}^k R_i$$

The point $x \in U \subset U'$ defines a maximal ideal of, say, R_1 in the fiber of $R \rightarrow \mathcal{O}(U')/\mathcal{I}'_U \simeq \bigoplus_{i=1}^k R_i$ over m . Consider the function $g \in \mathcal{O}(U')$ such that its class in $\mathcal{O}(U')/\mathcal{I}'_U$ defines the element $(1, 0, \dots, 0) \in \bigoplus_{i=1}^k R_i$. Then $g \notin m_x$ and $\mathcal{O}(U')_g/\mathcal{I}'_{U,g} = (\bigoplus_{i=1}^k R_i)_g = R_1$. Moreover since $f \notin m_x$ we get that $\mathcal{O}(U')_{g,f}/\mathcal{I}'_{U,g,f} = (\bigoplus_{i=1}^k R_i)_{g,f} = (R_1)_f = R_1$

In other words by shrinking U around x to $U_{g,f} = U'_{g,f}$ we can assume that $\mathcal{O}(U)/\mathcal{I}_U \simeq R_1$ is local Henselian itself. Consequently it is isomorphic to its Henselianization $R\langle z_1, \dots, z_k \rangle/\mathcal{I}$. Since M is finite over $R\langle z_1, \dots, z_k \rangle$ and \mathcal{I} acts trivially on M , we find that M is finite over $R\langle z_1, \dots, z_k \rangle/\mathcal{I}$. But $R\langle z_1, \dots, z_k \rangle/\mathcal{I} \simeq \mathcal{O}(U)/\mathcal{I}_U$ is finite over R , and thus M is finite over R . □

The following more general version of Hironaka's claim shall be understood as an algebraic counterpart of Malgrange division for modules over smooth functions.

Theorem 1.1.3. *Let $f : R\langle x_1, \dots, x_k \rangle \rightarrow R\langle y_1, \dots, y_m \rangle$ be a homomorphism of Noetherian R -algebras where R is a local Henselian and Noetherian ring. Suppose that M is a finite $R\langle y_1, \dots, y_m \rangle$ -module and $M/(f(m_k) \cdot M)$ is a finite-dimensional vector space over $K = R\langle y_1, \dots, y_k \rangle/m_k$. Then M is finite over $R\langle x_1, \dots, x_k \rangle$.*

Proof. One can factor f as the composition of the natural inclusion $i := R\langle x_1, \dots, x_k \rangle \hookrightarrow R\langle x_1, \dots, x_k, y_1, \dots, y_m \rangle$ followed by the projection $\pi : R\langle x_1, \dots, x_k, y_1, \dots, y_m \rangle \rightarrow R\langle y_1, \dots, y_m \rangle$, $\pi(x_i) = f(x_i)$, $\pi(y_j) = y_j$. Note that M is a finitely generated $R\langle x_1, \dots, x_k, y_1, \dots, y_m \rangle$ -module, and $M/(f(m_k) \cdot M) = M/(i(m_k) \cdot M)$ is of finite dimension. By the Hironaka claim, M is a finite $R\langle x_1, \dots, x_k \rangle$ -module. \square

Lemma 1.1.4. *Let K be a field. Consider the natural inclusion map*

$$f : R := K\langle z_1, \dots, z_k \rangle \rightarrow S = K\langle z_1, \dots, z_{k+n} \rangle.$$

Let $\mathcal{I} \subset S$ be an ideal such that the map $R \rightarrow S/\mathcal{I}$ is finite. Then there exist smooth affine schemes U_1 and U_2 over $\text{Spec}(K)$ of dimension k and $n+k$ respectively and a smooth map $\phi : U_2 \rightarrow U_1$ of dimension n compatible with f for which:

- (1) $K[z_1, \dots, z_k] \subset \mathcal{O}(U_1) \subset K\langle z_1, \dots, z_k \rangle$,
- (2) $K[z_1, \dots, z_{k+n}] \subset \mathcal{O}(U_2) \subset K\langle z_1, \dots, z_{k+n} \rangle$,
- (3) $\phi^* : \mathcal{O}(U_1) \rightarrow \mathcal{O}(U_2)$ is the restriction of f .
- (4) $\mathcal{O}(U_1) \rightarrow \mathcal{O}(U_2)/\mathcal{I}_2$ is finite, where $\mathcal{I}_2 \subset \mathcal{O}(U_2)$ is an ideal for which $\mathcal{I} = \mathcal{I}_2 \cdot K\langle z_1, \dots, z_{k+n} \rangle$.
- (5) The fiber $\mathcal{O}(U_2)/(m \cdot \mathcal{O}(U_2) + \mathcal{I}_2)$ is irreducible, where $m = (z_1, \dots, z_k) \subset \mathcal{O}(U_1)$.

Proof. Let b_1, \dots, b_s be generators of the module $S/\mathcal{I} = \sum R \cdot b_i$ over R , and $f_1, \dots, f_k \in \mathcal{I}$ be generators of \mathcal{I} . Consider a nonsingular affine V_2 étale neighborhood of $0 \in \mathbf{A}^{n+k} = \text{Spec } K[z_1, \dots, z_{k+n}]$, preserving residue field and with $\mathcal{O}(V_2)$ containing b_i, f_j . Moreover by shrinking V_2 around 0 we can assume that the fiber $K[z_1, \dots, z_n] \rightarrow \mathcal{O}(V_2)/(f_1, \dots, f_k)$ over $m = (z_1, \dots, z_k)$ is irreducible.

Denote by d_1, \dots, d_r generators of the R -algebra $\mathcal{O}(V_2)$ over K . Write

$$b_i b_j \equiv \sum_{l=1}^s c_{ijl} b_l \pmod{\mathcal{I}}, \quad d_i \equiv \sum_{l=1}^r d_{il} b_l \pmod{\mathcal{I}}$$

where $c_{ijl}, d_{il} \in R$.

We shall assume that U_1 is an affine étale neighborhood of $0 \in \text{Spec}(K[z_1, \dots, z_k])$ with the ring of regular functions $\mathcal{O}(U_1) \subset R = K\langle z_1, \dots, z_k \rangle$ containing c_{ijl} and d_{il} . Now let U_2 be a component in $V_2 \times_{\mathbf{A}^k} U_1$ dominated by $\text{Spec}(S)$. Then $U_2 \rightarrow U_1$ is smooth, and by the universal property of the component of the product we get that $\mathcal{O}(U_2)$ is equal to the subring of S generated by $\mathcal{O}(U_1)$ and $\mathcal{O}(V_2)$:

$$\mathcal{O}(U_2) = \mathcal{O}(U_1)\mathcal{O}(V_2) = \mathcal{O}(U_1)[d_1, \dots, d_r] \subset S$$

and it contains b_i and f_j . Then for $\mathcal{I}_2 = \mathcal{I} \cap \mathcal{O}(U_2)$ we have $\mathcal{I}_2 \cdot S = \mathcal{I}$ and

$$\mathcal{O}(U_2)/\mathcal{I}_2 = \sum_{i=1}^k \mathcal{O}(U_1) b_i.$$

Indeed the $\mathcal{O}(U_1)$ -submodule

$$\sum_{i=1}^k \mathcal{O}(U_1) b_i \subset \mathcal{O}(U_2)/\mathcal{I}_2 = \mathcal{O}(U_1)[d_1, \dots, d_r]/\mathcal{I}_2 = \mathcal{O}(U_2)/\mathcal{I}_2$$

is, in fact, a subring since it is closed under multiplication by the relation on $b_i b_j$. Moreover it contains all the generators d_i over $\mathcal{O}(U_1)$ which implies the equality. \square

Corollary 1.1.5. *Let $f : X \rightarrow Y$ be a morphism of schemes of finite type over a field K with $x \in X$ and $y = f(x) \in Y$. Let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules, with a stalk \mathcal{F}_x (which is a finitely generated $\mathcal{O}_{x,X}$ -module). Suppose that the vector space $\mathcal{F}_x/(m_y \cdot \mathcal{F}_x)$ is of finite dimension over the field $K = f^*(\mathcal{O}_{y,Y}/m_{y,Y})$. Then there exist an étale neighborhood $Y' \subset Y$ of y and $\alpha : X' \rightarrow X$ of x , preserving the residue fields with the induced morphism $f' : X' \rightarrow Y'$, and the induced coherent sheaf $\mathcal{F}' = \alpha^*(\mathcal{F})$ such that:*

- (1) $f'_{x*}(\mathcal{F}'_x) \simeq \mathcal{F}'_x$.
- (2) $f'_*(\mathcal{F}')$ is a coherent $\mathcal{O}_{Y'}$ -module.
- (3) In particular $f'_*(\mathcal{O}_{X'}/\text{Ann}(\mathcal{F}'))$ is a coherent $\mathcal{O}_{Y'}$ -module (where $\text{Ann}(\mathcal{F}')$ denotes the annihilator of the sheaf \mathcal{F}').
- (4) If $c_1, \dots, c_k \in \mathcal{F}_x$ generate $\mathcal{F}_x/(m_y \mathcal{F}_x)$ over the field $K = \mathcal{O}_y/m_y$ then X' and Y' can be chosen such that c_1, \dots, c_k are in $\mathcal{F}'(X')$ and generate the sheaf $f'_*(\mathcal{F}')$ over $\mathcal{O}_{Y'}$.

Proof. (2) & (3) One can assume that X and Y are affine schemes and find affine spaces \mathbf{A}^n , \mathbf{A}^m over K containing X and Y respectively with inclusions $i_X : X \hookrightarrow \mathbf{A}^n$, $i_Y : Y \hookrightarrow \mathbf{A}^m$, and an extension $f_A : \mathbf{A}^n \rightarrow \mathbf{A}^m$ of the morphism $f : X \rightarrow Y \subset \mathbf{A}^m$.

Factor $f_A : \mathbf{A}^n \rightarrow \mathbf{A}^m$ into the composition of the closed immersion $i : \mathbf{A}^n \rightarrow \mathbf{A}^{n+m}$ defined by the graph of f_A followed by the natural projection $\pi : \mathbf{A}^{n+m} \rightarrow \mathbf{A}^m$. The coherent sheaf \mathcal{F} on X defines the coherent sheaf $\mathcal{F}_A = j_*(\mathcal{F})$ on \mathbf{A}^{n+m} via the inclusion $j : X \hookrightarrow X \times Y \subset \mathbf{A}^{n+m}$. The annihilator of \mathcal{F}_A contains $\mathcal{I}_{j(X)} \subset \mathcal{O}_{\mathbf{A}^{n+m}}$, and is supported on the closed subset of $j(X) \subset \mathbf{A}^{n+m}$. Moreover, by comparing stalks we see that $\mathcal{O}_{\mathbf{A}^{n+m}}/\text{Ann}(\mathcal{F}_A) = j_*(\mathcal{O}_X/\text{Ann}(\mathcal{F}))$. The latter implies the equality of the vanishing loci $j(V(\text{Ann}(\mathcal{F})) = V(\text{Ann}(\mathcal{F}_A))$.

By Lemma 1.1.2, the restriction

$$\bar{f} : V(\text{Ann}(\mathcal{F})) := \text{Spec}(\mathcal{O}(X)/\text{Ann}(\mathcal{F}(X))) \rightarrow Y$$

of f to the vanishing locus $V(\text{Ann}(\mathcal{F}))$ has a finite fiber $\bar{f}^{-1}(x)$, and likewise the restriction

$$\bar{\pi} : V(\text{Ann}(\mathcal{F}_A)) := \text{Spec}(K[z_1, \dots, z_{n+m}]/\text{Ann}(\mathcal{F}_A(\mathbf{A}^{n+m})) \rightarrow \mathbf{A}^m.$$

The latter defines a map

$$\bar{\pi}^* : K\langle z_1, \dots, z_n \rangle \rightarrow K\langle z_1, \dots, z_{n+m} \rangle / ((\text{Ann}(\mathcal{F}_A) \cdot K\langle z_1, \dots, z_{n+m} \rangle))$$

with finite fiber over the maximal ideal (z_1, \dots, z_{n+m}) . Then, by Lemma 1.1.5, there exist étale affine neighborhoods $j_1 : U_1 \rightarrow \mathbf{A}^n$ of y and $j_2 : U_2 \rightarrow \mathbf{A}^{n+k}$ of x , with induced $\bar{\pi} : U_2 \rightarrow U_1$ such that the map of rings

$$\mathcal{O}(U_1) \rightarrow \mathcal{O}(U_2) / (\text{Ann}(\mathcal{F}_A) \cdot \mathcal{O}(U_2)) = \mathcal{O}(U_2) / \text{Ann}(\mathcal{F}_{U_2})$$

is finite, where $\mathcal{F}_{U_2} := j^*(\mathcal{F}_A)$ is the induced coherent sheaf, and $\text{Ann}(\mathcal{F}_{U_2}) = \text{Ann}(\mathcal{F}_A) \cdot \mathcal{O}(U_2)$ denote its annihilator.

The étale affine neighborhoods U_1 , and U_2 induce étale neighborhoods $X' \subset U_1$ of X and $Y' \subset U_2$ of Y .

The coherent sheaf \mathcal{F} on X defines a unique coherent sheaf \mathcal{F}' on X' . Moreover since $X' \rightarrow X$ is étale, $\text{Ann}(\mathcal{F}') = \text{Ann}(\mathcal{F}) \cdot \mathcal{O}(X')$. By the above, the restriction of $U_2 \rightarrow U_1$ to the support of the annihilator,

$$\bar{f}' : V(\text{Ann}(\mathcal{F}')) \simeq V(\text{Ann}(\mathcal{F}_{U_2})) \rightarrow Y' \subset U_1,$$

is finite.

Since \mathcal{F}' is annihilated by $\text{Ann}(\mathcal{F}')$, we see that the closed immersion $i : V(\text{Ann}(\mathcal{F}')) \subseteq X'$ and the coherent sheaf \mathcal{F}' on X' define a coherent sheaf \mathcal{F}'' on the scheme $V(\text{Ann}(\mathcal{F}'))$, such that $\mathcal{F}' = i_*(\mathcal{F}'')$, which implies that

$$f'_*(\mathcal{F}') = f'_*i_*(\mathcal{F}'') = (\bar{f}')^*(\mathcal{F}'')$$

is a coherent sheaf of $\mathcal{O}_{Y'}$ -modules. This proves (2) and thus (3).

By Lemma 1.1.4, we can assume that the fiber of \bar{f}' over x is irreducible, and thus

$$f'_*(\mathcal{F}')_x = ((\bar{f}')^*(\mathcal{F}''))_x = \mathcal{F}''_x \simeq \mathcal{F}'_x.$$

To prove (4) we shrink X so that \mathcal{F} is generated by $c_1, \dots, c_k \in \mathcal{F}(X)$ over $\mathcal{O}(X)$. This implies that they also generate $\mathcal{F}'(X')$ over $\mathcal{O}(X')$. By the Nakayama lemma, the stalk

$$(f'_*(\mathcal{F}'))_x = \mathcal{F}'_x = \mathcal{F}_x \otimes \mathcal{O}_{X',x}$$

of the coherent sheaf $f'_*(\mathcal{F}')$ is generated over $\mathcal{O}_{Y',y'}$ by c_1, \dots, c_k . This implies that, after shrinking X' and Y' further, we can assume that $f'_*(\mathcal{F}')$ is generated over $\mathcal{O}_{Y'}$ by c_1, \dots, c_k . \square

Both theorems generalize Weierstrass division either locally or in a neighborhood.

Definition 1.1.6. $f(t, x)$ is d -regular with respect to t (where $x = (x_1, \dots, x_n)$) if

$$f(0, 0) = \frac{\partial f}{\partial t}((0, 0) = \dots = \frac{\partial^{d-1} f}{\partial t^{d-1}}(0, 0) = 0, \quad \frac{\partial^d f}{\partial t^d}(0, 0) \neq 0.$$

Theorem 1.1.7 (Weierstrass division of algebraic functions in a neighborhood). *Let X be an affine scheme which is étale over $\mathbb{A}_K^{n+1} = \text{Spec } K[t, x]$, where K is a field. Let $g(t, x) \in \mathcal{O}(X)$ be a d -regular function at $\bar{x} \in X$ over a closed point in $\text{Spec}(K)$. Then there is an étale neighborhood $U_2 \rightarrow X$ of \bar{x} preserving the residue field of \bar{x} such that:*

- (1) There is a smooth morphism $\pi : U_2 \rightarrow U_1$ of dimension one onto an affine scheme U_1 which is smooth over $\text{Spec}(K)$ and a closed embedding $i : U_1 \rightarrow U_2$ such that $\pi \circ i = \text{id}_{U_1} : U_1 \rightarrow U_2$, with $\mathcal{O}(U_1) \subset K\langle x \rangle$ and $\mathcal{O}(U_2) \subset K\langle t, x \rangle$.
- (2) Weierstrass division by f with remainder exists in the ring $\mathcal{O}(U_2)$: For any $g \in \mathcal{O}(U_2)$ there exist $q \in \mathcal{O}(U_2)$ and $r = \sum_{i=0}^{d-1} r_i(x)t^i \in \mathcal{O}(U_2)$ with $r_i \in \mathcal{O}(U_1)$ such that $g = q \cdot f + r$.
- (3) Weierstrass division by f exists for any étale neighborhood $U'_1 \rightarrow U_1$ of \bar{x} preserving the residue field of \bar{x} and for $U'_2 := U_2 \times_{U_1} U'_1$.

Proof. (1) We can assume that X is affine with $\mathcal{O}(X) \subset K\langle t, y_1, \dots, y_n \rangle$. Then the quotient ring $R := \mathcal{O}(X)/(t)$ can be identified with a subring of $K\langle y_1, \dots, y_n \rangle \simeq K\langle t, y_1, \dots, y_n \rangle/(t)$. Let

$$U_1 := \text{Spec}(R) \simeq \{\bar{y} \in U \mid t(\bar{y}) = 0\}, \quad U_2 := \text{Spec}(R \cdot \mathcal{O}(X)).$$

Let $\pi : U_2 \rightarrow U_1$ be the projection defined by the inclusion $R \subset R \cdot \mathcal{O}(X)$. The schemes U_2 and U_1 are smooth over $\text{Spec}(K)$ at \bar{x} and $\pi(\bar{x})$.

By shrinking U_2 and U_1 if necessary, we can assume that π is smooth and $U_2 \rightarrow X$ is étale.

(2) Consider the coherent sheaf $\mathcal{F} := \mathcal{O}_{U_2}/(f)$ and the projection $\pi : U_2 \rightarrow U_1$. Then $\mathcal{F}_x/(m_y \mathcal{F}_x)$ is spanned by $1, t, \dots, t^{d-1}$. Consequently, by using Corollary 1.1.5, after passing to étale neighborhoods of U_1 and U_2 , the sheaf $\pi_*(\mathcal{O}_{U_2}/(f))$ is coherent and generated over \mathcal{O}_{U_1} by the same set of polynomials $1, t, \dots, t^{d-1}$. Since we can assume that all the subschemes are affine, we deduce that the ring of global sections, $\mathcal{O}(U_2)/(f)$, is the $\mathcal{O}(U_1)$ -module generated by $1, t, \dots, t^{d-1}$. This implies existence of Weierstrass division.

(3) The same property is valid when passing to étale neighborhoods. \square

Corollary 1.1.8 (Singular “inverse function” theorem (algebraic version)). *Let $f : X \rightarrow Y$ be a morphism of smooth schemes of finite type over a field K , and let $x \in X$ and $y = f(x) \in Y$ be K -rational points. Assume that $\mathcal{O}_{X,x}/f^*(m_y)$ is of finite dimension d over $K = \mathcal{O}_y/m_y$ generated by $c_1, \dots, c_d \in \mathcal{O}_{X,x}$. Then there exist étale neighborhoods $Y' \rightarrow Y$ of y and $X' \rightarrow X$ of x , preserving the residue field K , with the induced finite morphism $f' : X' \rightarrow Y'$ of degree d . Moreover, if X and Y are of the same dimension, and X is Cohen-Macaulay and Y is regular, then:*

- (1) $f' : X' \rightarrow Y'$ is a finite, flat and surjective morphism (a d -branched cover).
- (2) There is an isomorphism of $\mathcal{O}_{Y'}$ -modules $\mathcal{O}_{X'}^d \xrightarrow{\phi} f'_*(\mathcal{O}_{X'})$, $\phi(a_1, \dots, a_d) = a_1 c_1 + \dots + a_d c_d$.
- (3) The point $y = f'(x) \in Y'$ is in the ramified locus of f' of maximal index d .

Proof. (1) To show finiteness of f' we apply Theorem 1.1.5 for $\mathcal{F} = \mathcal{O}_X$.

(2) $f'_*(\mathcal{O}_{X'})$ is a finite $\mathcal{O}_{Y'}$ -module generated by c_1, \dots, c_d . The fact that ϕ is an isomorphism follows from Theorem 5.4.6, or a result by Eisenbud that local Cohen-Macaulay rings which are finite over regular rings are free modules ([24]). We can represent $X \rightarrow Y$ as the composition of the closed immersion $i : X \rightarrow X \times Y$ followed by the projection $\pi : X \times Y \rightarrow Y$. Then, by Theorem 5.4.6,

$$f'_*(\mathcal{O}_{X'}) = \pi_*(\mathcal{O}_{X' \times Y'} / \mathcal{I}_{i(X')}) \simeq \mathcal{O}_{Y'}^d,$$

is a free $\mathcal{O}_{Y'}$ -module in a neighborhood of y . By shrinking Y' (and X') one can assume that c_1, \dots, c_d is a basis of $f'_*(\mathcal{O}_{X'})$.

(3) follows from the fact that $\{x\} = (f')^{-1}(y)$ is irreducible. \square

Remark. Note that the degree of f is usually greater than d so the result is not valid in the Zariski topology.

1.2. The Grauert-Remmert theorem for finite holomorphic maps. Recall that a map of topological spaces (in particular complex analytic differentiable spaces) is *finite* if it has finite fibers and is closed.

Lemma 1.2.1. *Let \mathcal{F} be a coherent sheaf on a complex space X . Let $f : X \rightarrow Y$ be a holomorphic map of analytic spaces and consider points $x \in U$ and $y = f(x) \in V$. The following conditions are equivalent:*

- (1) x is an isolated point in $f^{-1}(y) \cap V(\text{Ann}(\mathcal{F}))$.
- (2) $\mathcal{F}_x/(m_y \cdot \mathcal{F}_x)$ is a finite-dimensional vector space.

Proof. One can shrink U so that $\{x\} = \mathcal{F}_x/(m_y \cdot \mathcal{F}_x)$; this does not affect condition (2). Consider the coherent sheaf of ideals $\mathcal{I} := \text{Ann}(\mathcal{F})$. Then condition (1) is equivalent to $m_y \cdot \mathcal{O}_X + \mathcal{I} \supset m_x^d$ for some d , or

$\mathcal{O}_{X,x}/(m_y \cdot \mathcal{O}_X + \mathcal{I})$ is of finite dimension. But the module $\mathcal{F}_x/(m_y \cdot \mathcal{F}_x)$ is finitely generated over $\mathcal{O}_{X,x}$, and thus over $\mathcal{O}_{X,x}/(m_y \cdot \mathcal{O}_X + \mathcal{I})$, which in turn is of finite dimension. \square

Remark. Condition (2) was used in particular by Malgrange, for Malgrange-Mather division of moduli over smooth functions, and is very convenient, especially when combined with Nakayama's lemma. Both conditions are equivalent in the algebraic and analytic situation but are different for smooth functions.

The following result is equivalent to the Grauert-Remmert theorem for finite morphisms of complex spaces [30, Theorem 2, p. 54]. (We use here the Malgrange condition on the stalk \mathcal{F}_x .)

Corollary 1.2.2. ([30]) *Let $f : X \rightarrow Y$ be a holomorphic map of analytic spaces, and consider points $x \in U$ and $y = f(x) \in V$. Let \mathcal{F} be a sheaf of \mathcal{O}_X -modules which is coherent, with a stalk \mathcal{F}_x . Suppose $\mathcal{F}_x/(m_y \cdot \mathcal{F}_x)$ is a finite-dimensional vector space over \mathbb{C} . Then there exist neighborhoods $V' \subset V$ of y and U' of x such that:*

- (1) *The sheaf $(f|_{U'})_*(\mathcal{F})$ is a coherent $\mathcal{O}_{V'}$ -module.*
- (2) *The restriction of f to $V(\text{Ann}(\mathcal{F})) \cap U'$ defines a finite map $V(\text{Ann}(\mathcal{F})) \cap U' \rightarrow V'$.*
- (3) *$\mathcal{F}_x \simeq ((f|_{U'})_*(\mathcal{F}))_y$.*
- (4) *If c_1, \dots, c_k generate $\mathcal{F}_x/(m_y \mathcal{F}_x)$ over the field $K = \mathcal{O}_y/m_y$ then $(f|_{U'})_*(\mathcal{F})$ is generated over $\mathcal{O}_{V'}$ by $c_1, \dots, c_k \in \mathcal{O}(V')$.*
- (5) *The module $\mathcal{F}(U')$ is finitely generated over $\mathcal{O}(V')$.*

Proof. The corollary is a consequence of the Grauert-Remmert result and Nakayama's lemma. By the previous lemma we can replace condition (1) with the equivalent condition (2). For part (3) we choose the neighborhood U containing only a single point of the fiber. Part (4) follows from the Nakayama lemma and (1). Part (5) is again a consequence of (4). \square

The corollary also follows from the methods discussed in the next sections. As a particular case of the above we get a neighborhood version of Weierstrass division (which can also be deduced from a result by Hörmander [41, Theorem 6.1.1]).

Theorem 1.2.3 (Weierstrass division of complex analytic functions in a neighborhood). *Let $\mathcal{O}(U)$ denote the ring of holomorphic functions on an open subset $U \subset \mathbb{R}^1 \times \mathbb{R}^n$. Let $f(t, x) \in \mathcal{O}(U)$ be a d -regular function at $(t_0, x_0) \in U$. Then Malgrange-Mather division by f is possible in the ring $\mathcal{O}(U_2)$ for a certain neighborhood $U_2 = W \times U_1 \subset U$ of x . That is, for every $g(t, x) \in \mathcal{O}(U_2)$ there are $q(t, x) \in \mathcal{O}(U_2)$ and $r = \sum_{i=0}^{d-1} r_i(x)t^i \in \mathcal{O}(U_2)$ with $r_i(x) \in \mathcal{O}(U_1)$ such that $g = q \cdot f + r$. This holds for any open subsets $U'_1 \subset U_1$ and $U'_2 = W \times U'_1$.*

Proof. The proof is similar to the proof of the algebraic version (Theorem 1.1.7). We use Theorem 1 of Grauert-Remmert [30, p. 52]. \square

Corollary 1.2.4 (Singular "inverse function" theorem (analytic version)). *Let $f : X \rightarrow Y$ be a holomorphic map of complex analytic spaces, and let $x \in X$ and $y = f(x) \in Y$. Assume $\mathcal{O}_{X,x}/f^*(m_y)$ is of finite dimension d over $\mathbb{C} = \mathcal{O}_y/m_y$ generated by $c_1, \dots, c_d \in \mathcal{O}_{X,x}$. Then there exist open neighborhoods $Y' \subset Y$ of y and $X' \subset X$ of x such that the induced finite morphism $f' : X' \rightarrow Y'$ is of degree d . Moreover, if X is a Cohen-Macaulay complex space and Y is a manifold with $\dim(X) = \dim(Y)$ then:*

- (1) *$f' : X' \rightarrow Y'$ is a finite, flat, open and surjective morphism (a d -branched cover).*
- (2) *There is an isomorphism of $\mathcal{O}_{Y'}$ -modules $\mathcal{O}_{Y'}^d \xrightarrow{\phi} f'_*(\mathcal{O}_{X'})$, $\phi(a_1, \dots, a_d) = a_1 c_1 + \dots + a_d c_d$.*
- (3) *There is an isomorphism of $\mathcal{O}(Y')$ -modules $\phi_{Y'} : \mathcal{O}(Y')^d \simeq \mathcal{O}(X')$.*
- (4) *The point $y = f'(x) \in Y'$ is in the ramified locus of maximal index d .*

Proof. The proof is identical to that for the algebraic version. We use Theorem 1.2.2. \square

Corollary 1.2.5 (Singular "inverse function" theorem 2). *Let $f : X \rightarrow Y$ be a morphism of smooth schemes of finite type over a field K (respectively a map between \mathbb{C} -analytic manifolds) of the same dimension, and let $x \in X$ and $y = f(x) \in Y$ be K -rational points. Let u'_1, \dots, u'_n be a coordinate system on U' at $f(y) = x \in V$, and u_1, \dots, u_n be a coordinate system at x . Suppose that f is given by a finite set of functions $f_i = f^*(u'_i)$ which form a Cohen-Macaulay regular sequence at x ([52]). Then there are étale (respectively open) neighborhoods $X' \rightarrow X$ of x and $Y' \rightarrow Y$ of y such that the induced morphism $f' : X' \rightarrow Y'$ is finite, and there is an isomorphism of $\mathcal{O}_{Y'}$ -modules $\mathcal{O}_{Y'}^d \rightarrow f'_*(\mathcal{O}_{X'})$.*

Example 1.2.6. Let $f : X \rightarrow Y$ be a morphism or a map as above which is given by a finite set of functions $f_i = f^*(u'_i)$ with multiplicities d_i and suppose the initial forms $\text{in}_x(f_1), \dots, \text{in}_x(f_k)$ define a regular sequence. Then the induced morphism $f' : X' \rightarrow Y'$ is finite of degree $d_1 \cdot \dots \cdot d_n$.

Example 1.2.7. If f_1, \dots, f_n is a regular sequence in \mathbf{A}^n , respectively \mathbb{C}^n , then the induced map from the zero set $V(f_1, \dots, f_k) \rightarrow \mathbf{A}^{n-k}$ given by $f_{k+1}, \dots, f_n : \mathbf{A}^n \rightarrow \mathbf{A}^{n-k}$ defines a finite morphism $f : W \rightarrow V$ for some neighborhoods $W \subset V(f_1, \dots, f_k)$ and $U \subset \mathbf{A}^n$, and there is a Weierstrass isomorphism

$$\mathcal{O}_V^d \rightarrow f_*(\mathcal{O}_W/(f_1, \dots, f_k)).$$

2. MALGRANGE-MATHER PREPARATION AND THE INVERSE FUNCTION THEOREM

The proofs of analogous theorems for smooth functions are more involved and will be given in the next few sections of this chapter. For the most part we use the strategy of Malgrange to prove neighborhood Weierstrass division, which is then combined with some ideas of Grauert-Remmert applied to a sheaf version ([30]). The proof of neighborhood Malgrange special division, which is the key technical result, is essentially identical with Milman's proof ([59]) of the Malgrange special division. As a consequence of the methods applied we give an analog of the inverse function theorem (Section 2.5).

We also extend the classical constructions mostly due to Malgrange and Mather in the language of smooth objects (Section 2.6).

2.1. The neighborhood version of the Malgrange special division for smooth functions. The following theorem can be considered as a neighborhood version of Malgrange special division, which is the key technical result in Malgrange's strategy ([55]). We shall extend the method of Milman, slightly modifying his original proof ([59]). It turns out that in order to show the generalized Malgrange-Mather division it suffices to consider only the Malgrange special division by a generic polynomial.

Theorem 2.1.1. *Let $U \subset \mathbb{R}^{n+1}$ be an open neighborhood of $0 \in \mathbb{R}^{n+1}$. Let $\mathcal{O}(U)$ be the ring of smooth (C^∞) functions over U . There exist open neighborhoods $V_1^1 \subset \mathbb{R}^1$, $V_2^d \subset \mathbb{R}^d$ and $W^n \subset \mathbb{R}^n$ of 0 for which $V_1^1 \times W^n \subset U$, and such that for any $f(t, x_1, \dots, x_n) \in \mathcal{O}(V_1^1 \times W^n)$ and the "generic polynomial"*

$$P^d(t, y_1, \dots, y_d) := t^d + y_1 t^{d-1} + \dots + y_d \in \mathcal{O}(\mathbb{R} \times \mathbb{R}^d)$$

there exists "Malgrange special division with remainder":

$$f(t, x) = q^d \cdot P^d + r^d$$

where

$$r^d = r_{d-1}^d(x, y)t^{d-1} + \dots + r_0^d(x, y),$$

and

$$q^d(t, x, y) \in \mathcal{O}(V_1^1 \times V_2^d \times W^n), \quad r_i^d = r_i(x, y) \in \mathcal{O}(V_2^d \times W^n) \subset \mathcal{O}(V_1^1 \times V_2^d \times W^n).$$

Moreover the division exists for any open subset $V_1^1 \times V_2^d \times (W')^n$, with $(W')^n \subset W^n$ an open convex subset of W^n , which is a neighborhood of 0.

Proof. The proof can be obtained by a cosmetic modification of Milman's proof of the Malgrange division theorem. Here we briefly sketch the proof following Milman's original argument and referring to his paper [59] for additional details.

Observe that division by $\bar{P}^1 := t - y$ is very simple. We find neighborhoods $V_1^1 \subset \mathbb{R}$ and $W^n \subset \mathbb{R}^n$ such that $V_1^1 \times W^n \subset U$. Then we set $V_2^1 := V_1^1$ and write

$$(2.1) \quad f(t, x) = \frac{f(t, x) - f(y, x)}{t - y} \cdot (t - y) + f(y, x) = q_1(t - y) + r^1,$$

where $q_1 := \frac{f(t, x) - f(y, x)}{t - y} \in \mathcal{O}(V_1^1 \times V_2^1 \times W^n)$ and $r^1 = r_1^1 = f(y, x) \in \mathcal{O}(V_2^1 \times V_2^1)$. This can be equivalently translated into division by $P^1(t, y) = t + y = \bar{P}^1(t, -y)$ on the same neighborhoods.

Suppose that division by $P^{d-1}(t, y_1, \dots, y_k)$ is defined on $V_1^1 \times V_2^{d-1} \times W^n \subset \mathbb{R}^{n+d}$ in the form

$$f(t, x) = q^{d-1}(t, x, y) \cdot P^{d-1} + r^{d-1}.$$

Applying division by $t - y_d$ to $q^{d-1} = (q')^d(t - y_d) + (r'')^d$, one defines division by $P^{d-1}(t - y_d)$:

$$(2.2) \quad f(t, x) = (q')^d \cdot P^d(t - y_d) + (r')^d,$$

where

$$(2.3) \quad (r')^d := (r'')^d \cdot P^d + r^{d-1} = (r')_{d-1}^d(x, y)t^{d-1} + \dots + (r')_0^d(x, y),$$

which is defined on the open subset $(V_1^1 \times (V_2^{d-1} \times V_2^1) \times W^n) \subset \mathbb{R}^{n+d+1}$ (where $V_2^1 = V_1^1$). The actual division by P^d is done by passing from $P^{d-1}(t - y_d)$ to P^d . Consider the mapping

$$\begin{aligned} \psi_{d-1, n} : \mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}^n &\rightarrow \mathbb{R}^d \times \mathbb{R}^n, \\ (y_1, \dots, y_{d-1}, y_d, x_1, \dots, x_n) &\mapsto (y'_1, \dots, y'_d, x_1, \dots, x_n), \end{aligned}$$

given by the equality of the generic polynomials

$$P^{d-1}(t, y) \cdot (t - y_d) = P^d(t, y').$$

This map can be conveniently represented as a combination of two maps

$$\mathbb{R}^{d+n} \xrightarrow{\phi_1} V^d \times \mathbb{R}^n \xrightarrow{\pi_d} \mathbb{R}^d \times \mathbb{R}^n$$

where $V^d \subset \mathbb{R}^{d+1}$ is the set of zeroes of $P^d(t, y)$ in $\mathbb{R}^{d+1} \times \mathbb{R}^n$, and the isomorphism ϕ_1 is given by

$$(y_1, \dots, y_{d-1}, y_d, x_1, \dots, x_n) \mapsto (t = y_d, y'_1, \dots, y'_d, x_1, \dots, x_n),$$

and is followed by the projection $\pi_d(t, y'_1, \dots, y'_{d-1}, y'_d, x_1, \dots, x_n) = (y'_1, \dots, y'_{d-1}, y'_d, x_1, \dots, x_n)$.

We shall use a modification of Milman's lemma:

Lemma 2.1.2 (Milman). ([59]) *Let $\mathcal{O}_{\pi_d}(V^d \times \mathbb{R}^n)$ be the subspace of $\mathcal{O}(V^d \times \mathbb{R}^n)$ consisting of all functions which are constant on the fibers of π_d . There exists a continuous linear operator $J : \mathcal{O}_{\pi_d}(V^d \times \mathbb{R}^n) \rightarrow \mathcal{O}(\mathbb{R}^d \times \mathbb{R}^n)$ such that for any $f(t, y, x) \in \mathcal{O}_{\pi_d}(V^d \times \mathbb{R}^n)$ there is a function*

$$J(f)(y, x) = f(t, y, x) \in \mathcal{O}(\mathbb{R}^d \times \mathbb{R}^n).$$

Likewise for any open convex neighborhood of 0, say of the form $U^d \times W^n \subset \mathbb{R}^d \times \mathbb{R}^n$, there exists a linear operator $J_d : \mathcal{O}_{\pi_d}(\pi_d^{-1}(U^d \times W^n)) \rightarrow \mathcal{O}(U^d \times W^n)$ for which $J_d(f)(y, x) = f(t, y, x)$. Moreover the operator J_d exists for $U^d \times W^n$ if W^n is replaced with any open convex subset which is a neighborhood of 0.

Proof. The proof remains the same. Roughly speaking, we show that the function $f(t, y, x)$ which is constant on the fibers can be "pushed down" to a unique differentiable function $(\pi_d)_*f(t, y, x)$ on the closed subset $Z := \pi_d(\pi_d^{-1}(U^d \times W^n))$ of $U^d \times W^n$. Since π_d is closed, the function is continuous. It is also differentiable at the points where the Jacobian of π_d does not vanish. It turns out that the Jacobian is equal to $P_d(t)$. By an inductive argument, one can extend differentiability of $(\pi_d)_*f$ to the set where P_d has a root t of multiplicity exactly k , and finally show that it is differentiable on Z .

In the complex analytic situation the mapping is surjective and we are done: the function $J_d(f) = (\pi_d)_*f$ is holomorphic on $Z = U^d \times W^n$. In the differential setting, if the dimension is even the mapping is not surjective and we need to extend it. (This causes nonuniqueness of Malgrange-Mather division.) We extend the function using Stein's (or Whitney's) extension theorem. We use the fact that $(U^d \times W^n) \setminus Z$ is convex. This set can be interpreted as corresponding to the monic polynomials with coefficients in $U^d \times W^n$ having no roots (being positive), and is convex. \square

We briefly sketch Milman's (adjusted) strategy for the remaining part of the proof. We choose a convex W^n and sufficiently small convex V_2^d such that there is an embedding $\phi_1^{-1} : \pi_d^{-1}(V_2^d \times W^n) \rightarrow V_2^1 \times V_2^{d-1} \times W^n$. It follows from Milman's lemma that there is a linear operator

$$J_d : \mathcal{O}_{\psi}(V_2^1 \times V_2^{d-1} \times W^n) \rightarrow \mathcal{O}(V_2^d \times W^n).$$

We define the remainder of division of f by P^d on $V_1^1 \times V_2^d \times W^n$ to be $r^d = \sum r_i^d t^i$, where

$$r_i^d := J_d((r')_i^d) \in \mathcal{O}(V_2^d \times W^n) \quad \text{for} \quad (r')_i^d \in \mathcal{O}(V_2^1 \times V_2^{d-1} \times W^n),$$

and show (as in Milman's proof) that $f - r_d$ is in fact divisible by P^d . (Roughly speaking, the polynomial $P^d = \dots + y_d$ is a coordinate and $f - r_d$ vanishes along its zero locus.) Then there exists a quotient $q^d \in \mathcal{O}(V_1^1 \times V_2^d \times W^n)$ such that $f = q^d P^d + r_d$ on $U_d \times W^n$. The details are the same as in Milman's proof ([59]). Apart from few mentioned modifications the proof is identical with Milman's one. \square

In the real analytic situation the above reasoning proves a local version of Malgrange-Mather special division in the ring \mathcal{E}_n of germs of real analytic functions. It is not clear whether the corresponding neighborhood version remains valid for real analytic functions.

2.2. Malgrange-Mather generalized division. In Sections 2.2, 2.3, 2.4 we shall mean by \mathcal{E}_n the local ring of germs of smooth functions on \mathbb{R}^n at the origin 0.

Definition 2.2.1. $f \in \mathcal{E}_{n+1}(t, x)$ is d -regular with respect to t (where $x = (x_1, \dots, x_n)$) if

$$f(0, 0) = \frac{\partial f}{\partial t}((0, 0) = \dots = \frac{\partial^{d-1} f}{\partial t^{d-1}}(0, 0) = 0, \quad \frac{\partial^d f}{\partial t^d}(0, 0) \neq 0.$$

Theorem 2.2.2 (Malgrange-Mather preparation and division theorem). (1) Let $f(t, x) \in \mathcal{E}_{n+1}$ be a d -regular function. There exists a Weierstrass polynomial

$$P^d = t^d + c_1(x) \cdot t^{d-1} + \dots + c_d(x)$$

and an invertible function $\alpha \in \mathcal{E}_{n+1}$ such that $f(t, x) = P^d \cdot \alpha$.

(2) Let $f(t, x), g(t, x) \in \mathcal{E}_{n+1}$ with f d -regular. Then there exist $q, r \in \mathcal{E}_{n+1}$ and $h_j \in \mathcal{E}_n, j = 0, \dots, d-1$, such that

$$g = qf + r, \quad \text{where } r = r(t, x) = \sum_{j=0}^{d-1} h_j(x)t^j.$$

The above theorem is a consequence of generic Malgrange division via the ‘‘Malgrange trick’’.

Proof. Let $P^d(y, t) = t^d + \dots + y_d \in \mathcal{E}_{n+1}$ be the generic polynomial. Write

$$f(t, x) = h(t, x, y)P(y, t) + r(t, x, y), \quad g(t, x) = h_1(t, x, y)p(y, t) + r_1(t, x, y),$$

where

$$r(t, x, y) = \sum_{i=0}^d A_i(x, y)t^i, \quad r_1(t, x, y) = \sum_{i=0}^d B_i(x, y)t^i$$

with $A_i, B_i \in \mathcal{E}_{n+1}$. One can easily see that $\Psi := (A_1, \dots, A_d, x_1, \dots, x_k)$ is invertible since the Jacobian of Ψ at 0 is upper triangular. Its inverse is given by $\Psi^{-1} = (\phi_1, \dots, \phi_k, x_1, \dots, x_k)$. Then

$$\Psi^{-1}(0, x) = (\phi_1(0, x), \dots, \phi_k(0, x), x_1, \dots, x_k), \quad A_j(\phi_j(0, x), x) = 0.$$

Consequently,

$$f(t, x) = h(t, x, \phi(x))p(\phi, t) + r(t, x, \phi(x)) = h(t, x, \phi(x))p(\phi, t)$$

with $h(0, 0) = c \neq 0$ and $h(t, x, \phi(x))$ invertible. Also

$$g(t, x) = h_1(t, x, \phi)p(y, \phi) + r_1(t, x, \phi) = (h_1(t, x, \phi)h^{-1}(t, x, \phi(x)))f(t, x) + r_1(t, x, \phi)$$

(see the details in [55]). □

The Weierstrass preparation and division theorem was proved by Weierstrass for holomorphic functions. Its extension to convergent power series over a valued field was proved in [53], and for formal power series over a field in [17]. The algebraic case was proven in [48].

Corollary 2.2.3. Let $f(t, x) \in \mathcal{E}_n[t]$ be a polynomial in t , of degree k , which is d -regular in t , where $d \leq k$. Then there exists a factorization into polynomials

$$f(t, x) = g_1(x, t)g_2(x, t),$$

where

$$g_1 = t^d + A_1(x)t^{d-1} + \dots + A_k(x), \quad g_1(t, 0) = t^k,$$

and $g_2(x, t)$ is of degree $k - d$ with $g_2(0, 0) \neq 0$.

Proof. Let $P^d(y, t) = t^d + \dots + y_k \in \mathcal{E}_n[t]$ be the generic polynomial. Consider division with remainder in the polynomial ring $E(2n)[t]$:

$$f(t, x) = h(t, x, y)p(y, t) + r(x, y, t),$$

where $r(x, y, t)$ is a polynomial in t of degree $d - 1$. As before we find functions $\phi_1(x), \dots, \phi_k(x)$ such that after substitution, $r(t, x, \phi) = 0$. The reasoning is the same as in the previous proof. □

As a consequence, all the rings \mathcal{E}_n are Henselian:

Corollary 2.2.4 (Hensel's lemma). *Let $f(t, x) \in \mathcal{E}_n[t]$ be a monic polynomial in t of degree k such that*

$$f(t, 0) = (t - c_1)^{k_1} \cdots (t - c_r)^{k_r}$$

for some $c_i \in K$. Then there exists a factorization

$$f = g_1 \cdots g_r$$

into monic polynomials g_1, \dots, g_r of degree k_1, \dots, k_r such that

$$g_i(t, 0) = (t - c_i)^{k_i}.$$

Proof. The function $f^{c_1}(t) := f(t + c_1, x)$ is k_1 -regular. Hence by the previous result there exists a factorization into the product of polynomials

$$f^{c_1}(t, x) = g_1^{c_1}(x, t) \cdot f_1^{c_1}(x, t),$$

and

$$f(t, x) = g_1(x, t) f_1(x, t),$$

where $g_1(x, t) := g_1^{c_1}(x, t - c_1)$ and $f_1(x, t) := f_1^{c_1}(x, t - c_1)$ are polynomials of degree k_1 and $k - k_1$. Then by uniqueness $g_1(t, 0) = (t - c_1)^{k_1}$ and $f_1(x, 0) = (t - c_2)^{k_2} \cdots (t - c_r)^{k_r}$ and we can proceed by induction. \square

Remark. The fact that the local rings of smooth functions are Henselian was proven in [60].

2.3. Neighborhood version of Malgrange-Mather division. Let X be a topological space and \mathcal{O}_X be a sheaf of rings. We call \mathcal{O}_X the *structural sheaf* and the pair (X, \mathcal{O}_X) a *ringed space*. It is a *locally ringed space* if all the stalks $\mathcal{O}_{X,x}$ are local rings.

We say that a sheaf of \mathcal{O}_X -modules \mathcal{F} is of *finite type* if for any $x \in X$ there is a neighborhood $U \subset X$ and a surjective morphism of sheaves $\mathcal{O}_U^k \rightarrow \mathcal{F}|_U$.

A locally ringed space (Y, \mathcal{O}_Y) is called a closed subspace of X if Y is a closed subset of X , there is an ideal sheaf of finite type $\mathcal{I}_Y \subset \mathcal{O}_X$ for which Y is the support of the sheaf $\mathcal{O}_X/\mathcal{I}_Y$, and \mathcal{O}_Y is the restriction of $\mathcal{O}_X/\mathcal{I}_Y$ to Y . By a *differentiable space* mean a locally ringed space (Y, \mathcal{O}_Y) which is locally a closed subspace of open subset of $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ with smooth functions sheaf $\mathcal{O}_{\mathbb{R}^n}$.

The definition of the sheaf of \mathcal{O}_X -modules of finite type implies immediately that if the stalk \mathcal{F}_x of the sheaf of finite type is an \mathcal{O}_x -module finitely generated by sections $s_1, \dots, s_k \in \mathcal{F}_x$ then there is a neighborhood U of x such that $s_1, \dots, s_k \in \mathcal{F}(U)$ generate the $\mathcal{O}(U)$ -module $\mathcal{F}(U)$.

We shall use the following lemmas:

Lemma 2.3.1. *Let X be an open subset of \mathbb{R}^n and \mathcal{O}_X be a sheaf of smooth functions on X . Let \mathcal{F} be a sheaf of \mathcal{O}_X -modules of finite type generated over X by finitely many sections $s_1, \dots, s_k \in \mathcal{F}(X)$. Then for an open $U \subset X$, the $\mathcal{O}(U)$ -module $\mathcal{F}(U)$ is generated by s_1, \dots, s_k over $\mathcal{O}(U)$.*

Proof. If $t \in \mathcal{F}(U)$, then for any $x \in U$ the germ of t at x can be written as a finite sum $t_x = \sum a_{ix} s_{ix}$, where $a_{ix} \in \mathcal{O}(V_x) \subset \mathcal{O}_x$, for a sufficiently small neighborhood V_x of x . One can consider a locally finite subcover $\{V_j\}$ of $\{V_x\}$ and a subordinate partition of unity $1 = \sum b_j$. Then consider $1 \cdot t = (\sum b_j) \cdot t$, and $t|_{V_j} = (\sum_{V_j \subset U_i} b_j a_i)|_{V_j} s_i|_{V_j}$. But $c_i := (\sum_{V_j \subset U_i} b_j a_i) \in \mathcal{O}(U)$ is a function defined on U , and the sections t and $\sum c_i s_i$ coincide on each V_j and thus $t = \sum c_i s_i$. \square

Lemma 2.3.2. *Let X be an open subset of \mathbb{R}^n . If \mathcal{F} is a sheaf of \mathcal{O}_X -modules of finite type generated by global s_1, \dots, s_k , and $\mathcal{G} \subset \mathcal{F}$ is its \mathcal{O}_X -submodule also of finite type generated by global sections t_1, \dots, t_s , then*

$$(\mathcal{F}/\mathcal{G})(U) = \mathcal{F}(U)/\mathcal{G}(U)$$

for any open subset U .

Proof. Consider the quotient morphism of sheaves $f : \mathcal{F} \rightarrow \mathcal{F}/\mathcal{G}$. The sheaf \mathcal{F}/\mathcal{G} is of finite type generated by s_1, \dots, s_k . Then $(\mathcal{F}/\mathcal{G})(U)$ is also generated by s_1, \dots, s_k , and $f_U : \mathcal{F}(U) \rightarrow (\mathcal{F}/\mathcal{G})(U)$ is an epimorphism. On the other hand, $\mathcal{G}(U)$ is in the kernel $K(U)$ of f_U . For any $t \in K(U) \subset \mathcal{F}(U)$ the germs of t are generated by finitely many sections t_1, \dots, t_s of $\mathcal{G}(X)$. Then by the same reasoning $t = \sum c_i s_i$. \square

Also it follows from the definition that

Lemma 2.3.3. *If $f : Y \rightarrow X$ is a closed embedding of differentiable spaces then for any sheaf of finite type \mathcal{F} , the direct image $f_*(\mathcal{F})$ is of finite type*

Proof. Suppose \mathcal{F} is generated by sections $s_1, \dots, s_k \in \mathcal{F}(U)$, where $U \subset Y$. For any $y \in U$ and $f \in \mathcal{O}(U)$ exist $g_x \in \mathcal{O}_{X,x}$ which restricts to $f_x \in \mathcal{O}_{U,x}$. Each g_x is defined on open neighborhood $V_x \subset X$. By using the partition of unity argument we extend $f \in \mathcal{O}(U)$ to a certain $g \in \mathcal{O}(V)$ which is an open subset $V \subset Y$ such that $V \cap X = U$, and which is common for all f . Thus $\mathcal{O}(V) \rightarrow \mathcal{O}(U)$ is the surjection, and $s_1, \dots, s_k \in f_*(\mathcal{F})(V)$ generate $f_*(\mathcal{F})$ on V over \mathcal{O}_V . \square

Theorem 2.3.4 (Generalized Malgrange-Mather division in a neighborhood). *Let $\mathcal{O}(U)$ denote the ring of smooth functions on an open subset $U \subset \mathbb{R}^1 \times \mathbb{R}^n$. Let $f(t, x) \in \mathcal{O}(U)$ be a d -regular function at $y = (t_0, x_0) \in U$. Then there is Malgrange-Mather division by f in the ring $\mathcal{O}(U_2)$ for a certain neighborhood $U_2 = V_1 \times U_1 \subset U$ of x . That is, for every $g(t, x) \in \mathcal{O}(U_2)$ there are $q(t, x) \in \mathcal{O}(U_2)$ and $r = \sum_{i=0}^{d-1} r_i(x)t^i \in \mathcal{O}(U_2)$ with $r_i(x) \in \mathcal{O}(U_1)$ such that $g = q \cdot f + r$.*

Proof. First, by applying Malgrange preparation and shrinking U , we can assume that $f(t, x) = \alpha(t, x) \cdot \overline{P}^d(t, x)$ on U where $\overline{P}^d(t, x) = t^d + c_1(x) \cdot t^{d-1} + \dots + c_k(x)$. Then there is generic division by $P^d(t, y) = t^d + y_1 \cdot t^{d-1} + \dots + y_k$ in some neighborhood $V_1 \times V_2 \times W$, where $W = U_1$. Consider the maps

$$c : U_1 \rightarrow \mathbb{R}^d, \quad c_1 := (c, \text{id}_{U_1}) : U_1 \rightarrow \mathbb{R}^d \times U_1 \quad \text{and} \quad c_2 := \text{id}_{V_1} \times c_1 : V_1 \times U_1 \rightarrow V_1 \times \mathbb{R}^d \times U_1, \quad t \mapsto t, \quad y_i \mapsto c_i.$$

By shrinking U_1 we can assume that $c(U_1) \subset V_2$ and thus

$$c_2(V_1 \times U_1) \subset V_1 \times V_2 \times U_1.$$

Applying special division in $V_1 \times V_2 \times U_1$ to $g \in \mathcal{O}(V_1 \times U_1)$ we can write

$$g = q^d P^d + r^d$$

with $q^d, r^d \in \mathcal{O}(V_1 \times V_2 \times U_1)$ and $r_i^d \in \mathcal{O}(V_2 \times U_1)$. Then $\overline{P}^d(t, x) = P^d(t, x) \circ c_2 \in \mathcal{O}(V_1 \times U_1)$ and for $\overline{r}(t, x) := r^d \circ c_2 \in \mathcal{O}(V_1 \times U_1)$, and $\overline{q} := q^d \circ c_2 \in \mathcal{O}(V_1 \times U_1)$ we have the division

$$f = \overline{q} \overline{P}^d + \overline{r},$$

which implies $g = q \cdot f + r$ with $q = \overline{q}/\alpha$ and $r = \overline{r}$. \square

The results below are extensions of the Grauert-Remmert ‘‘Weierstrass isomorphism’’ [30, Theorem 1.2.3] in the holomorphic case to the differential setting with a similar proof.

Lemma 2.3.5. *If $f : X \rightarrow Y$ is finite map of subsets in \mathbb{R}^n then for any $y \in Y$ with the fiber $f^{-1}(y) = \{x_1, \dots, x_k\}$ and $\epsilon > 0$ there exists $\delta > 0$ such that if $f^{-1}(B_\delta(y)) \subset \bigcup B_\epsilon(x_i)$. (Here $B_\delta(y)$ is an open ball of radius δ with the center x .)*

Proof. Suppose otherwise. Then for each $n \in \mathbb{N}$, there exists $z_n \notin \bigcup B_\epsilon(x_i)$ with $f(z_n) - y < 1/n$. If the sequence (z_n) has an accumulation point $z = \lim z_n$ then $f(z) = \lim(f(z_n)) = y$. But $z \notin \{x_1, \dots, x_k\} = f^{-1}(y)$ which is a contradiction. If (z_n) does not have an accumulation points and defines a closed subset of X with non closed image $\{f(z_n)\}$ with accumulation point $y = \lim f(z_n)$. First prove the lemmas \square

Lemma 2.3.6. *If $f : X \rightarrow Y$ is a finite map of subsets in \mathbb{R}^n with the fiber $f^{-1}(y) = \{x_1, \dots, x_k\}$ of $y \in Y$, and \mathcal{F} is a sheaf on X then there is an isomorphism of $\mathcal{O}_{Y,y}$ -modules $(f_*(\mathcal{F}))_y \simeq \bigoplus_i \mathcal{F}_{x_i}$.*

Proof. By the previous lemma for $n \gg 0$ and $U_n = B_{1/n}(y)$ the preimage is a union of disjoint neighborhoods W_{ni} of x_i with $\bigcap_n W_{ni} = \{x_i\}$. We conclude that

$$\pi_*(\mathcal{F})_y = \lim_{U_n \ni y} \mathcal{F}(\pi_i^{-1}(U_n)) = \bigoplus \lim_{W_{ni} \ni x_i} \mathcal{F}(W_{ni}) = \bigoplus \mathcal{F}_{x_i}.$$

\square

Corollary 2.3.7 (Grauert-Remmert ‘‘Weierstrass isomorphism’’). *Let $\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ be the standard projection. For any open subset $U \subset \mathbb{R}^{n+1}$ denote by \mathcal{O}_U the sheaf of smooth functions on U . Consider a function $f(t, x) \in \mathcal{O}(U)$ which is d -regular for $\overline{x} \in U$. Then there is a convex open neighborhood $U_2 := W \times U_1$ of \overline{x} such that:*

- (1) $\pi_{0*}(\mathcal{O}(U_2)/(f(t, x)))$ is a finitely generated $\mathcal{O}(U_1)$ -module, where $\pi_0 : U_2 = W \times U_1 \rightarrow U_1$ is the natural projection.

(2) There are surjections of \mathcal{O}_{U_1} -modules

$$\phi : \bigoplus_{i=0}^{d-1} \mathcal{O}_{U_1} t^i \rightarrow \pi_*(\mathcal{O}_{U_2}/(f(t, x))),$$

and of the rings of global sections

$$\phi_{U_2} : \bigoplus_{i=0}^{d-1} \mathcal{O}(U_1) t^i \rightarrow \mathcal{O}(U_2)/(f(t, x)).$$

- (3) The restriction $\pi_1 : V(f) \rightarrow U_1$ to the zero set $V(f)$ of f on U_2 is a finite (thus closed and proper) map.
- (4) If \mathcal{F} is a \mathcal{O}_U -sheaf of finite type over U which is annihilated by the function f and generated by the global sections on U_2 then $\pi_{1*}(\mathcal{F}|_{U_2})$ is of finite type on U .
- (5) $\mathcal{F}_x \simeq (\pi_{1*}(\mathcal{F}|_{U_2}))_x$.
- (6) U_1 can be replaced, in particular, with any open convex subset U'_1 containing $\pi(x)$, and the above conditions will be satisfied.

Proof. (3) Let us first show that the map $\pi_1 : V(f) \rightarrow U_1$ is closed and has finite fibers. One can replace f with the Weierstrass polynomial $P_d = t^d + c_1 t^{d-1} + \dots + c_d$ defined on the open set $U \subset \mathbb{C}^k$ (or \mathbb{R}^n). Then the points in the fibers correspond exactly to the roots of the polynomial P_d , and thus are finite and of cardinality $\leq d$. Now, if the sequence of points $y_n := \pi_1(x_n)$ converges to y then it defines a converging sequence of polynomials P_{d, y_i} with coefficients $c_i(y_n) \mapsto c_i(y)$. This implies that the coefficients of the polynomials P_{d, y_i} are bounded, as also are their roots (see computation below). Thus there is a convergent subsequence $x_{n_k} \rightarrow x$, and $y := \pi(x)$ is the limit of y_n . This shows that π_1 is closed and finite. (See proof of [30, Theorem 1.2.3] for details)

(5) By Lemma 2.3.6, for any $p \in U_1$ there is an isomorphism of stalks

$$\pi_*(\mathcal{O}_{U_2}/(f))_p \simeq \bigoplus_{y \in \pi_1^{-1}(p)} (\mathcal{O}_{U_2}/(f))_y,$$

and in general if \mathcal{F} is annihilated by f then it can be considered as a direct image of a sheaf \mathcal{F}' on the vanishing locus $V(f) \subset U_2$, and again using Lemma 2.3.6, we get

$$\pi_*(\mathcal{F})_p \simeq \bigoplus_{y \in \pi_1^{-1}(p)} \mathcal{F}_y.$$

(1) & (2) By Theorem 2.3.4, there is an epimorphism

$$\phi_{U_1} : \bigoplus_{i=0}^{d-1} \mathcal{O}(U_1) t^i \rightarrow \mathcal{O}(U_2)/(f(t, x))$$

of the rings of global sections. By a partition of unity argument, any germ of $\mathcal{O}_{U_2, y}$ or $(\mathcal{O}_{U_2}/(f))_y$ at a point extends to a global section over U_2 . Using the epimorphism of global sections we find that any element in $(\mathcal{O}_{U_2}/(f))_y$ is in the image of $\bigoplus_{i=0}^{d-1} \mathcal{O}_{U_1} t^i$. Likewise any element in $\bigoplus_{y \in f^{-1}(p)} (\mathcal{O}_{U_2}/(f))_y \simeq \pi_*(\mathcal{O}_{U_2}/(f))_p$ extends to a global section in $\mathcal{O}_{U_2}/(f)$, and thus in $\bigoplus_{i=0}^{d-1} \mathcal{O}_{U_1} t^i$. This implies that ϕ is an epimorphism on the sheaves.

(4) The restriction $\mathcal{F}|_{U_2}$ of \mathcal{F} to U_2 can be considered as a sheaf of $\mathcal{O}_{U_2}/(f)$ -modules of finite type. The module $\pi_{1*}(\mathcal{O}_{U_2}/(f))$ is a finite \mathcal{O}_{U_1} -module. We need to show that $\pi_{1*}(\mathcal{F})$ is a finite $\pi_{1*}(\mathcal{O}_{U_2}/(f))$ -module. The stalk $\pi_*(\mathcal{F})_y$ is, as before, isomorphic to $\bigoplus_{y \in f^{-1}(p)} \mathcal{F}_y$. The epimorphism of sheaves $\mathcal{O}_{U_2}^k \rightarrow \mathcal{F}$ factors through $\mathcal{O}_{U_2}^k/(f) \rightarrow \mathcal{F}$, which gives an epimorphism of stalks $(\mathcal{O}_{U_2}^k/(f))_y \rightarrow \mathcal{F}_y$.

Since $\pi_{1*}(\mathcal{F})_p \simeq \bigoplus_{y \in f^{-1}(p)} \mathcal{F}_y$ and $\pi_{1*}(\mathcal{O}_{U_2}^k/(f))_p \simeq \bigoplus_{y \in f^{-1}(p)} (\mathcal{O}_{U_2}^k/(f))_y$, we get a surjection on the stalks $\pi_{1*}(\mathcal{O}_{U_2}^k/(f))_y \rightarrow (\pi_{1*}(\mathcal{F}))_y$. \square

2.4. Malgrange preparation for modules. The Malgrange-Mather division theorem generalizes to the following theorem on moduli (see also [16]).

Theorem 2.4.1 (Malgrange). *Denote by $\mathcal{E}_k := C_0^\infty(\mathbb{R}^k)$ the local ring of smooth functions at $0 \in \mathbb{R}^k$. Let $\phi_k^* : \mathcal{E}_k \rightarrow \mathcal{E}_m$ be any homomorphism induced by a smooth map $\phi_k : U \rightarrow \mathbb{R}^k$ defined in a neighborhood U of $0 \in \mathbb{R}^m$, and M be any finitely generated \mathcal{E}_n -module. Then the following conditions are equivalent:*

- (1) M is finitely generated over \mathcal{E}_k .
- (2) The dimension of the vector space $M/(\phi_k^*(m_k) \cdot M)$ over $\mathbb{R} = \mathcal{E}_k/m_k$ is finite.

Proof. The proof is identical to Malgrange's original proof. As we need some elements of the proof later, we shall briefly present it here.

(1) \Rightarrow (2). Obvious.

(2) \Rightarrow (1). Consider the case where $n = k + 1$ and $\phi_k^* = \pi_{n,k}^* : \mathcal{E}_k(x) \rightarrow \mathcal{E}_n(t, x)$ is defined by the natural projection onto the second factor, and for simplicity of notation let us identify $\mathcal{E}_k(x)$ with the subring of $\mathcal{E}_{k+1}(t, x)$. Let a_1, \dots, a_r be elements of M whose classes modulo m_n form a basis of the vector space $M/(m_n \cdot M)$ over $\mathbb{R} = \mathcal{E}_n/m_n$. Then, by Nakayama, a_1, \dots, a_r generate M over \mathcal{E}_n and their classes modulo m_k generate $M/(m_k \cdot M)$ over $\mathbb{R} = \mathcal{E}_k/m_k$.

Write

$$ta_j = \sum \alpha_{ij} a_i + \sum_{s \in S_j} \beta'_{sj} \cdot b_{sj}, \quad \text{where } \alpha_{ij} \in \mathbb{R}, \quad \beta'_{sj} \in m_k, \quad b_{sj} \in M,$$

and the sets of indices are finite. Since a_1, \dots, a_r generate M , we have

$$b_{sj} = \sum \gamma_{sij} a_j$$

with $\gamma_{sij} \in \mathcal{E}_n = \mathcal{E}_{k+1}$. This yields

$$ta_j = \sum (\alpha_{ij} + \beta_{ij}) a_i,$$

where $\beta_{ij} = \sum_i \sum_s \beta'_{sj} \gamma_{sij} \in m_k \cdot \mathcal{E}_n$.

Consider the square matrices $A := [\alpha_{ij}]$, $B = [\beta_{ij}]$ and the vector $\bar{a} := (a_1, \dots, a_r)$. Then we write the above in matrix form

$$(tI - A - B)\bar{a} = 0.$$

Then

$$\text{adj}(tI - A - B) \cdot (tI - A - B)\bar{a} = \det(tI - A - B)\bar{a} = 0.$$

This implies that $\Delta := \det(tI - A - B) \in \mathcal{E}_n$ annihilates M , that is, $\Delta \cdot M = 0$.

Then $\Delta(t, 0) = t^k + c_1(0)t^{k-1} + \dots + c_k(0) \in \mathbb{R}[t]$ is a monic polynomial in t of degree k . Let $q \leq k$ denote the multiplicity of $\Delta(t, 0)$ at 0. Then we can write $\Delta(t, 0)$ as a product of two polynomials $\Delta(t, 0) = t^q \cdot \alpha$, where $\alpha(0, 0) \neq 0$ and Δ is q -regular at $(t, 0)$.

The module M is a finitely generated $\mathcal{E}_{k+1}/(\Delta \cdot \mathcal{E}_{k+1})$ -module. By the division theorem, $\mathcal{E}_{k+1}/(\Delta \cdot \mathcal{E}_{k+1})$ is generated by $\{1, t, \dots, t_{q-1}\}$ over \mathcal{E}_k . This implies that M is finitely generated over \mathcal{E}_k .

Assume now that $k \geq n$ is arbitrary, and $\text{rank}(f) = n$. Then in certain coordinates, f is an embedding onto the first n coordinates, $f : \mathbb{R}^n \rightarrow \mathbb{R}^k$, with

$$f : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, \dots, 0),$$

and $f^* : \mathcal{E}_k \rightarrow \mathcal{E}_n$ is surjective, that is, $f^*(\mathcal{E}_k) = \mathcal{E}_n$ and $f^*(m_k) = m_n$, and the conclusion is valid. In general any map $f : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is the composite of the immersion

$$(id, f) : \mathbb{R}^n \rightarrow \mathbb{R}^k \rightarrow \mathbb{R}^{n+k}$$

followed by a sequence of n projections

$$g : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^k$$

as in the first case. It suffices to notice that if the theorem is true for two maps f and g then it is true for their composition. \square

The above theorems can be extended to their neighborhood versions by using Weierstrass division proven before. The following results can be related to [30, Theorem 1.3.1].

Lemma 2.4.2. *Let $V \times U \subset \mathbb{R}^k \times \mathbb{R}^n$ be an open neighborhood of the point $z = (x, y)$, and $\pi : V \times U \rightarrow U$ be the restriction of the standard projection $\pi : \mathbb{R}^{k+n} \rightarrow \mathbb{R}^k$. Let \mathcal{F} be a sheaf of $\mathcal{O}_{V \times U}$ -modules of finite type, with a stalk \mathcal{F}_z . Suppose $\mathcal{F}_x/(m_y \cdot \mathcal{F}_z)$ is a finite-dimensional vector space over \mathbb{R} . Then there exists an open convex neighborhood $V' \times U'$ of $z = (x, y)$ such that:*

- (1) The sheaf $(\pi|_{V' \times U'})_*(\mathcal{F}|_{V' \times U'})$ is of finite type (in the differential setting).
- (2) The restriction of π to the vanishing locus of the annihilator $V(\text{Ann}(\mathcal{F})) \cap U'$ is a finite map (thus closed).
- (3) $((\pi|_{U'})_*(\mathcal{F}))_y \simeq \mathcal{F}_z$.

(4) If U'' is a convex open neighborhood of y then the above conditions are satisfied for $V' \times U''$.

Proof. The first part of the reasoning is the same as in the proof of Theorem 2.4.1. We settle the case of $k = 1$. Let $a_j \in \mathcal{F}_z/(m_y \cdot \mathcal{F}_z)$ be generators as in the proof of Theorem 2.4.1. Find convex subsets $V_1 \times U_1 \subset U$ with $V_1 \subset \mathbb{R}$, $U_1 \subset \mathbb{R}^n$ such that:

- (1) The elements a_j generate $\mathcal{F}(V_1 \times U_1)$ over $\mathcal{O}(V_1 \times U_1)$.
- (2) $ta_j = \sum(\alpha_{ij} + \beta_{ij})a_i$, where $\beta_{ij} \in m_y \cdot \mathcal{O}(V_1 \times U_1)$, $\alpha_{ij} \in \mathbb{R}$, and $m_y \subset \mathcal{O}(U_1)$ is the maximal ideal of y .
- (3) There is a q -regular function $\Delta \in \mathcal{O}(V_1 \times U_1)$ at $(0, 0)$ which annihilates $\mathcal{F}|_{V_1 \times U_1}$.
- (4) $\mathcal{O}(V_1 \times U_1)/\Delta$ is a finitely generated $\mathcal{O}(U_1)$ -module.

The polynomial Δ annihilates the generators a_i of the module $\mathcal{F}(V_1 \times U_1)$. Hence $\mathcal{F}(V_1 \times U_1)$ is a finite $\mathcal{O}(V_1 \times U_1)/(\Delta)$ -module and thus a finite $\mathcal{O}(U_1)$ -module. Thus the case follows from Theorem 2.3.7. Also if we replace U_1 with an open neighborhood U'_1 of y then the above conditions will be satisfied.

Then we show the general case by induction on n . We can find an open convex neighborhood $V = V_1 \times \dots \times V_k \subset \mathbb{R}^k$ of (x_1, \dots, x_k) such that π is a composition of the codimension one projections

$$\pi_i : U_i := V_i \times \dots \times V_k \times U \rightarrow U_{i+1} := V_{i+1} \times \dots \times V_k \times U.$$

Set inductively $\bar{x}_i := (x_i, \dots, x_k)$ and $\mathcal{F}_0 := \mathcal{F}_{U_1}$, $\mathcal{F}_{i+1} = \pi_{i|U_i}(\mathcal{F}_i)$. By the above we can assume that:

- (1) For any projection $\pi_i : U_i = V_i \times U_{i+1} \rightarrow U_{i+1}$ there is a q_i -regular function $\Delta_i \in \mathcal{O}(U \times W)$ at $z_i := (\bar{x}_i, y)$ which annihilates \mathcal{F}_i .
- (2) $\mathcal{O}(U_i)/\Delta_i$ is a finitely generated $\mathcal{O}(U_{i+1})$ -module.
- (3) The restriction of the projection π_i to $V(\Delta_i) \subset U_i$ defines a finite map $V(\Delta_i) \rightarrow U_{i+1}$, likewise its restriction $V(\text{Ann}(\mathcal{F}_i)) \rightarrow V(\text{Ann}(\mathcal{F}_{i+1}))$ to $V(\text{Ann}(\mathcal{F}_i))$ is a finite map .
- (4) \mathcal{F}_i is of finite type.
- (5) $(\mathcal{F}_i)_{z_i} \simeq (\mathcal{F}_{i+1})_{z_{i+1}}$ is an isomorphism of $\mathcal{O}_{z_{i+1}, U_{i+1}}$ -modules.

□

The following result is very similar to Theorem 1.2.2.

Corollary 2.4.3. *Let $f : X \rightarrow Y$ be a differentiable map of differentiable spaces, and consider points $x \in U$ and $y = f(x) \in V$. Let \mathcal{F} be a sheaf of \mathcal{O}_X -modules which is of finite type, with a stalk \mathcal{F}_x . Suppose $\mathcal{F}_x/(m_y \cdot \mathcal{F}_x)$ is a finite-dimensional vector space over \mathbb{R} with a basis defined by $c_1, \dots, c_k \in \mathcal{F}_x$. Then there exist neighborhoods $V' \subset V$ of y and U' of x such that:*

- (1) *The sheaf $(f|_{U'})_*(\mathcal{F})$ is an $\mathcal{O}_{V'}$ -module of finite type.*
- (2) *The restriction of f to $V(\text{Ann}(\mathcal{F})) \cap U' \rightarrow V'$ is a finite map.*
- (3) *$\mathcal{F}_x \simeq ((f|_{U'})_*(\mathcal{F}))_y$.*
- (4) *The sheaf $(f|_{U'})_*(\mathcal{F})$ is generated over $\mathcal{O}_{V'}$ by $c_1, \dots, c_k \in \mathcal{O}(V')$*
- (5) *The module $\mathcal{F}(U')$ is generated over $\mathcal{O}(V')$ by $c_1, \dots, c_k \in \mathcal{O}(V')$.*

Proof. (1) and (2) The situation is local so we may assume that all subsets are closed subspaces of domains in \mathbb{R}^n . Let us first consider the case when Y is a domain in \mathbb{R}^n , and X is a subspace of a domain $B \subset \mathbb{R}^k$. The map $f : X \rightarrow Y \subset \mathbb{R}^n$ can be written as the composition $f = \pi \circ \alpha$ of the closed embedding

$$\alpha := (id, f) : X \subset B \times V$$

followed by the projection

$$\pi : B \times V.$$

Let $z = \alpha(x)$ with $\pi(z) = y$. Note that $m_z \supset \pi^*(m_y)$ and $\alpha^*(m_z) \supset f^*(m_y)$. Then $\mathcal{F}_x/(\alpha^*(m_z))$ is of finite dimension, and the sheaf $\mathcal{F}' := \alpha_*(\mathcal{F})$ is of finite type, with $\mathcal{F}'_z = \mathcal{F}_x$, and thus $\mathcal{F}_x/f^*(m_y) = \mathcal{F}'_z/\pi^*(m_y)$ is finitely generated. By the previous result we can find neighborhoods V of x and U of y such that for the restrictions $f|_V : V \rightarrow U$ and $\pi|_{V \times U} : V \times U \rightarrow U$, $f|_{V*}(\mathcal{F}) = (\pi|_{V \times U})_*(\mathcal{F}')$ is of finite type. Moreover $\alpha(V(\text{Ann}(\mathcal{F})) \subset (V(\text{Ann}(\alpha_*(\mathcal{F}))) \subset B \times V$. By Theorem 2.4.1, the restriction of projection $\pi : B \times V \rightarrow V$ to $(V(\text{Ann}(\alpha_*(\mathcal{F})))$ is finite. Since α defines an inclusion of $V(\text{Ann}(\mathcal{F}))$ into $(V(\text{Ann}(\alpha_*(\mathcal{F})))$ and thus is finite we get that the restriction of the composition $f = \pi \alpha$ of two finite maps to $V(\text{Ann}(\mathcal{F}))$ is finite

Now in the general case if $Y \subset D$ is a complex subspace of the domain $D \subset \mathbb{R}^n$, then we consider the induced map $\bar{f} : X \rightarrow D$. Then $\bar{f}|_{U*}(\mathcal{F}|_U)$ is an \mathcal{O}_D -module of finite type supported on Y , and annihilated

by \mathcal{I}_Y . It is the trivial extension of the sheaf $f_{|U*}(\mathcal{F}_{|U})$. This implies that the sheaf $f_{|U*}(\mathcal{F}_{|U})$ is an $\mathcal{O}_{V \cap Y}$ -module of finite type. Again the annihilator of \mathcal{F} has the same vanishing locus as the annihilator of $f_{|U*}(\mathcal{F}_{|U})$, and by the special case considered before the restriction of f to the vanishing locus $V(\text{Ann}(\mathcal{F})) = V(f_{|U*}(\mathcal{F}_{|U}))$ is finite.

(3) Since α is an embedding $\alpha_*(\mathcal{F}_x) \simeq \mathcal{F}_x$, and $\alpha_*(\mathcal{F})$ is of finite type by Lemma 2.3.3. Also, by Lemma 2.4.2 and the above:

$$f_*(\mathcal{F}_x) = \pi_*(\alpha_*(\mathcal{F}_x)) = \pi_*(\alpha_*(\mathcal{F}))_x \simeq (\alpha_*(\mathcal{F}))_x \simeq \mathcal{F}_x$$

(4) Since \mathcal{F}_x is finitely generated over \mathcal{O}_y and c_1, \dots, c_k generate $\mathcal{F}_x/(m_y \mathcal{F}_x)$, by the Nakayama lemma they generate it over \mathcal{O}_y . It follows that they generate $(f_{|U'})_*(\mathcal{F})$ over \mathcal{O}_U in a certain neighborhood of y .

(5) The natural map $i : \mathcal{F}(U') \rightarrow (f_{|U'})_*(\mathcal{F}_{|U'})$ is injective since, by Lemma 2.3.6, $((f_{|U'})_*(\mathcal{F}_{|U'}))_y = \bigoplus \mathcal{F}_{x_i}$. Thus if $i(s)_y = 0$ for $s \in \mathcal{F}(U')$, and all $y \in V'$ then $s_x = 0$ for all $x \in U'$, and thus $s = 0$. By shrinking U' we can assume that $c_1, \dots, c_k \in \mathcal{F}(U')$ which is $\mathcal{O}(V')$ -submodule of the module of the global sections $\Gamma((f_{|U'})_*(\mathcal{F}_{|U'}))$. By (4) we see that they generate all the stalks $(f_{|U'})_*(\mathcal{F}_{|U'})$. Lemma 2.3.1 implies that c_1, \dots, c_k generate the $\mathcal{O}(U)$ -module $\Gamma((f_{|U'})_*(\mathcal{F}_{|U'}))$ and thus its submodule $\mathcal{F}(U') = \Gamma((f_{|U'})_*(\mathcal{F}_{|U'}))$. \square

2.5. Singular inverse function theorem. If $f : X \rightarrow Y$ is a finite differentiable map of differential spaces then we define the locus $Y_d \subset Y$ (resp. $Y_{\geq d} \subset Y$ of points where f has degree d , that is,

$$Y_d := \left\{ y \in Y \mid \sum_{x \in f^{-1}(y)} \dim(\mathcal{O}_x/m_y) = d \right\}.$$

The set $Y_{\geq d}$ is closed, as we will see below. For any subset $Z \subset X$ denote by m_Z^∞ the ideal of all the functions flat along Z such that $\partial^k f|_Z = 0$.

Corollary 2.5.1 (Singular “inverse function” theorem (differential version)). *Let $f : X \rightarrow Y$ be a differentiable map of differentiable spaces, and let $x \in X$ and $y = f(x) \in Y$. Assume $\mathcal{O}_{X,x}/f^*(m_y)$ is of finite dimension d over $\mathbb{R} = \mathcal{O}_y/m_y$. Then there exist neighborhoods $Y' \subset Y$ of y and $X' \subset X$ of x such that the induced finite thus proper and closed morphism $f' : X' \rightarrow Y'$ is of degree d . If $\mathcal{O}_{X,x}/f^*(m_y)$ is generated by c_1, \dots, c_d over \mathbb{R} then the sections generate $f'_*(\mathcal{O}_{X'})$ in the neighborhood of X' over $\mathcal{O}_{Y'}$, defining an epimorphism of sheaves of $\mathcal{O}_{Y'}$ -modules*

$$\phi : \mathcal{O}_{Y'}^d \rightarrow f'_*(\mathcal{O}_{X'}), \quad \phi(a_1, \dots, a_d) = a_1 c_1 + \dots + a_d c_d,$$

and the corresponding epimorphism of $\mathcal{O}(Y')$ -modules

$$\phi_{Y'} : \mathcal{O}(Y')^d \rightarrow \mathcal{O}(X').$$

Moreover, if X is Cohen-Macaulay and Y is a manifold of the same dimension then the kernels of ϕ and $\phi_{Y'}$ are contained in $m_{Y'}^\infty \cdot \mathcal{O}_{Y'}^d$, (respectively in $m_{Y'}^\infty \cdot \mathcal{O}(Y')^d$). Moreover the point $y = f'(x) \in Y'$ is in the ramified locus of maximal index d , and $y \in Y'_d$.

Proof. The proof is similar to one before. We apply Corollary 2.4.3 to $\mathcal{F} = \mathcal{O}_X$. This shows surjectivity of ϕ in a neighborhood of x .

For the “moreover” part, for any $y' \in Y'_d$ consider the fiber $f^{-1}(y') = \{x_1, \dots, x_k\}$ with $k \leq d$, $d_i = \dim(\mathcal{O}_{x_i}/m_{y'})$, and $\sum d_i = d$. As before we represent $X' \rightarrow Y'$ as the composition of the closed immersion $i : X' \rightarrow X' \times Y'$ followed by the projection $\pi : X' \times Y' \rightarrow Y'$. The sheaf

$$f'_*(\mathcal{O}_{X'}) = \pi_*(i_*(\mathcal{O}_{X'})) = \pi_*(\mathcal{O}_{X' \times Y'} / \mathcal{I}_{i(X')})$$

can be written by Theorem 5.4.6, as the image of the free $\mathcal{O}_{Y'}$ -module $\mathcal{O}_{Y'}^d$, in a neighborhood of y . By shrinking Y' (and X') one can assume that c_1, \dots, c_d generate the $\mathcal{O}_{Y'}$ -module $f'_*(\mathcal{O}_{X'})$.

Since $X' \rightarrow Y'$ is finite, by Lemma 2.3.6, there is an isomorphism of $\mathcal{O}_{Y',y'}$ -modules

$$f'_*(\mathcal{O}_{X'})_{y'} \simeq \pi_*(\mathcal{O}_{X' \times Y'} / \mathcal{I}_{i(X')})_{y'} \simeq \bigoplus (\mathcal{O}_{X' \times Y'} / \mathcal{I}_{i(X')})_{x_i} \simeq \bigoplus \mathcal{O}_{X,x_i}.$$

Applying Theorem 5.4.5 to each point x_i we obtain surjections $\mathcal{O}_{Y',y'}^{d_i} \rightarrow \mathcal{O}_{X,x_i}$ with kernel contained in $m_{y'}^\infty$. Thus there is such a surjection

$$\psi_{y'} : \mathcal{O}_{Y',y'}^d = \bigoplus_i \mathcal{O}_{Y',y'}^{d_i} \rightarrow f'_*(\mathcal{O}_{X'})_{y'} = \bigoplus_i \mathcal{O}_{X,x_i}$$

with the kernel contained in $m_{y'}^\infty$. Consider the induced generators $e_1 := \psi_{y'}(1, 0, \dots, 0), \dots, e_d := \psi_{y'}(0, \dots, 0, 1)$ of $f'_*(\mathcal{O}_{X'})_{y'}$. Since c_1, \dots, c_d is another set of generators we can represent the generators e_i as $e_i = \sum a_{ij}c_j$ for some invertible matrix $[a_{ij}(y')]$. The matrix $[a_{ij}]$ defines an isomorphism α of $\mathcal{O}(Y')^d$ in a neighborhood of y' . Locally we get the relation $\phi_{y'} = \alpha_{y'}\psi_{y'} : \mathcal{O}_{Y', y'}^d \rightarrow f'_*(\mathcal{O}_{X'})_{y'}$ with kernel contained in $m_{y'}^\infty$.

The homomorphism of $\mathcal{O}(Y')$ -modules

$$\phi_{Y'} : \mathcal{O}(Y')^d \rightarrow \mathcal{O}(X').$$

is an epimorphism since the sections $c_1, \dots, c_d \in \mathcal{O}(X')$ generate all the stalks □

Remark. Observe that in a neighborhood of each $y \in Y_d$, the number $\sum_{x \in f^{-1}(y)} \dim(\mathcal{O}_x/m_y)$ is at most d , since there is an epimorphism of vector spaces

$$(\mathcal{O}_y/m_y)^d \rightarrow \bigoplus_{x \in f^{-1}(y)} \mathcal{O}_x/m_y.$$

In the holomorphic or algebraic setting, $Y'_d = Y'$ and $m_{Y'_d}^\infty = 0$.

2.6. Smooth objects. Summarizing the results from the previous sections we introduce the category of *smooth objects* \mathcal{R}^n over a field K modeled, in particular, on the local rings of smooth functions on \mathbb{R}^n over \mathbb{R} . This approach allows us to treat algebraic, analytic, and smooth functions in the same way.

Each \mathcal{R}^n is a triple

$$\mathcal{R}^n = (\mathcal{E}_n, \{x_1, \dots, x_n\}, m_n),$$

such that:

- (1) $\mathcal{E}_0 = K$, $m_0 = (0)$.
- (2) \mathcal{E}_n is a local ring with maximal ideal m_n , and containing its residue field K .
- (3) x_1, \dots, x_n are elements of m_n and their classes in m_n/m_n^2 form a basis of a free module over K .
- (4) There exists a homomorphism

$$T_n : \mathcal{E}_n \rightarrow \widehat{\mathcal{E}_n} = \lim_k \mathcal{E}_n/m_n^k \simeq K[[x_1, \dots, x_n]]$$

of K -algebras, whose kernel is equal to m_n^∞ , and which transforms $x_i \in \mathcal{E}_n$ to $x_i \in K[[x_1, \dots, x_n]]$.

A map $f : \mathcal{R}^n \rightarrow \mathcal{R}^m$ is given by any sequence of functions $f_1, \dots, f_m \in m_n$, which defines a unique ring homomorphism

$$f^* : \mathcal{E}_m \rightarrow \mathcal{E}_n, \quad f^*(x_i) = f_i,$$

commuting with T_i . We denote the map defined by the sequence (f_i) by $f = (f_1, \dots, f_m)$. By abuse of notation we write

$$f^*(g) = g(f_1, \dots, f_m)$$

for any $g \in \mathcal{E}_m$. We assume the following conditions hold:

- (5) The map $\sigma := (x_1, \dots, x_n) : \mathcal{R}^n \rightarrow \mathcal{R}^n$ defines the identity map, that is, the endomorphism $\sigma^* : \mathcal{E}_n \rightarrow \mathcal{E}_n$ is the identity.
- (6) The composition of maps $\phi_1 : \mathcal{R}^n \rightarrow \mathcal{R}^m$ and $\phi_2 : \mathcal{R}^m \rightarrow \mathcal{R}^k$ is a map $\phi_2 \circ \phi_1 : \mathcal{R}^n \rightarrow \mathcal{R}^k$, given by the composition of the ring homomorphisms $\phi_1^* \phi_2^* : \mathcal{E}_k \rightarrow \mathcal{E}_n$.
- (7) There exists a differentiation $\frac{\partial}{\partial x_i}$ of \mathcal{E}_n commuting with T_n and defining the standard derivation $\frac{\partial}{\partial x_i}$ on $K[[x_1, \dots, x_n]]$.
- (8) For any $k \leq n$ there is the natural projection $p_{n,k} : \mathcal{R}^n \rightarrow \mathcal{R}^k$, given by (x_1, \dots, x_k) . It defines the inclusions $\mathcal{E}_k \subset \mathcal{E}_n$ and $K[[x_1, \dots, x_k]] \subset K[[x_1, \dots, x_n]]$ commuting with the T_n . Moreover

$$\mathcal{E}_k = \left\{ f \in \mathcal{E}_n \mid \frac{\partial}{\partial x_i}(f) = 0 \text{ for } i > k \right\}$$

and for $i \leq n$, the restriction of the differentiation $\frac{\partial}{\partial x_i}$ on \mathcal{E}_n to \mathcal{E}_k coincides with that on \mathcal{E}_n .

- (9) For any $k \leq n$ the map $(x_1, \dots, x_k, 0, \dots, 0) = i_{k,n} : \mathcal{R}^k \rightarrow \mathcal{R}^n$ defines a ring surjection $\mathcal{E}_n \rightarrow \mathcal{E}_k$ whose kernel is the ideal (x_{k+1}, \dots, x_n) .
- (10) (Inverse function theorem) For any functions $u_1, \dots, u_n \in \mathcal{E}_n$ for which $\det\left[\frac{\partial u_i}{\partial x_j}\right](0) \neq 0$ the map $(u_1, \dots, u_n) : \mathcal{R}^n \rightarrow \mathcal{R}^n$ is invertible.

- (11) (Malgrange-Mather special division) Let \mathcal{E}_{n+k+1} be the ring of smooth objects with coordinates $(t, x, y) := (t, x_1, \dots, x_n, y_1, \dots, y_k)$. For any $g(t, x_1, \dots, x_n) \in \mathcal{E}_{n+1}$ and the “generic polynomial”

$$P^d = t^d + y_1 t^{d-1} + \dots + y_d \in \mathcal{E}_{n+d+1}$$

there exists “special Malgrange-Mather division”:

$$f(t, x) = h^d(t, x, y) \cdot P^d + r^d,$$

where

$$r^d = \sum r_{d-1}^d(x, y) t^{d-1} + \dots + r_0^d(x, y),$$

and

$$h^d(t, x, y) \in \mathcal{E}_{n+d+1}, \quad r_i^d = r_i^d(x, y) \in \mathcal{E}_{n+d}.$$

- (12) If $m_n^\infty = 0$ then (\mathcal{E}_n) will be called *reduced*.

Remark. In positive characteristic we assume the existence of Hasse derivatives $\frac{\partial}{\partial x_i^{p^j}}$ commuting with T_n in condition (4). Also condition (8) is slightly modified:

$$\mathcal{E}_k = \left\{ f \in \mathcal{E}_n \mid \frac{\partial}{\partial x_i^{p^j}}(f) = 0 \text{ for } i > k, j > 0 \right\}.$$

Example 2.6.1. Examples of categories of smooth objects include:

- (1) The germs $C_x^\infty(\mathbb{R}^n)$ of smooth functions on \mathbb{R}^n over $K = \mathbb{R}$.
- (2) The germs $\mathcal{O}_x(K^n)$ of analytic functions on K^n over $K = \mathbb{C}$ or $K = \mathbb{R}$.
- (3) The germs $K\langle x_1, \dots, x_n \rangle$ of algebraic functions on smooth algebraic varieties of dimension n over a field K . (Here $K\langle x_1, \dots, x_n \rangle$ denotes the Henselianization of the localization $K[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$ of $K[x_1, \dots, x_n]$ at the maximal ideal (x_1, \dots, x_n) .) In this case $K\langle x_1, \dots, x_n \rangle$ is the direct image of étale extensions of $K[x_1, \dots, x_n]$, preserving the residue field K , and it is a subring of the Henselian ring $K[[x_1, \dots, x_n]]$.
- (4) The rings of formal analytic functions $K[[u_1, \dots, u_k]]$ over a field K . (This follows in particular from more general “Hironaka formal division”) (Theorem 3.1.9).

Malgrange’s strategy presented in the previous sections utilizes only the algebraic properties defined for the category of smooth functions, and thus can be extended to the more general situation. In particular this implies the following:

Theorem 2.6.2 (Malgrange). *Let $\phi_k^* : \mathcal{E}_k \rightarrow \mathcal{E}_m$ be any homomorphism (in the smooth category), and let M be any finitely generated \mathcal{E}_n -module. Then the following conditions are equivalent:*

- (1) M is finitely generated over \mathcal{E}_k .
- (2) The dimension of the vector space $M/(\phi_k^*(m_k) \cdot M)$ over $K = \mathcal{E}_k/m_k$ is finite.

Proof. The proof is the same as the proof of Theorem 2.4.1. □

We are going to use the following theorems due to Malgrange (with some modifications) (see also [16]).

Corollary 2.6.3 (Malgrange). *Given a finite \mathcal{E}_n -module M and a homomorphism $f^* : \mathcal{E}_k \rightarrow \mathcal{E}_n$, the set $\{b_1, \dots, b_r\}$ generates M as an \mathcal{E}_k -module if it generates the K -vector space $M/(f^*(m_k)M)$.*

Proof. (\Rightarrow) By Malgrange preparation, $M/(f^*(m_k) \cdot M) = \langle b_1, \dots, b_r \rangle R$ is finite over \mathcal{E}_k . Hence

$$M = \langle b_1, \dots, b_r \rangle \mathcal{E}_k + m_k \cdot A.$$

By the Nakayama lemma we get $M = \langle b_1, \dots, b_r \rangle \mathcal{E}_k$. □

For a finite \mathcal{E}_n -module M define its *completion*

$$\widehat{M} = \lim_{\leftarrow} M/(m_n^i \cdot M),$$

which is a module over the completion of the ring

$\widehat{\mathcal{E}}_n := \lim_{\leftarrow} \mathcal{E}_n/m_n^i = K[[x_1, \dots, x_n]]$. These rings define the category of smooth objects over K . There is a natural homomorphism $M \rightarrow \widehat{M}$ with kernel defined by $m^\infty \cdot M$. Any ring homomorphism $\phi : \mathcal{E}_k \rightarrow \mathcal{E}_n$ induces a unique homomorphism $\widehat{\phi} : \widehat{\mathcal{E}}_k \rightarrow \widehat{\mathcal{E}}_n$.

Corollary 2.6.4 (Preparation theorem (in Malgrange form)). *Given a finite \mathcal{E}_n -module M and a map $f : \mathcal{R}^k \rightarrow \mathcal{R}^n$, the following statements are equivalent for $a_1, \dots, a_r \in M$:*

- (1) $a_1, \dots, a_r \in M$ generate M as an $f^*(\mathcal{E}_k)$ -module.
- (2) $a_1, \dots, a_r \in M$ span $M/(f^*(m_k) \cdot M)$ as a $K = f^*(\mathcal{E}_k/m_k)$ -vector space.
- (3) $\widehat{a}_1, \dots, \widehat{a}_r \in \widehat{M}$ generate \widehat{M} as an $\widehat{f}^*(\widehat{\mathcal{E}}_k)$ -module.
- (4) $\widehat{a}_1, \dots, \widehat{a}_r \in \widehat{M}$ generate $\widehat{M}/(\widehat{f}^*(\widehat{\mathcal{E}}_k/\widehat{m}_k) \cdot \widehat{M})$ as a $K = \widehat{f}^*(\widehat{\mathcal{E}}_k/\widehat{m}_k)$ -vector space.

Proof. The rings \mathcal{E}_n and $\widehat{\mathcal{E}}_n$ define categories of smooth objects. Then by Malgrange preparation and Nakayama's lemma we get the equivalences (1) \Leftrightarrow (2) and (3) \Leftrightarrow (4).

(2) \Rightarrow (4). There exists a natural epimorphism

$$M/(m_k \cdot M) \rightarrow \widehat{M}/(m_k \cdot \widehat{M}) = \lim_{\leftarrow} M/(m_k \cdot M + m_n^d \cdot M).$$

(4) \Rightarrow (2). By the assumption, a_1, \dots, a_r generate the finite-dimensional vector space

$$\widehat{M}/(m_k \cdot \widehat{M}) = \lim_{\leftarrow} M/(m_k \cdot M + m_n^d \cdot M).$$

Then there exists d_0 such that for $d \geq d_0$ we have the natural epimorphism

$$M/(m_k \cdot M + m_n^d \cdot M) \rightarrow M/(m_k \cdot M + m_n^{d_0} \cdot M),$$

which implies that

$$m_k \cdot M + m_n^{d_0} \cdot M = m_k \cdot M + m_n^{d_0+1} \cdot M = \dots$$

Let

$$A := m_k \cdot M + m_n^{d_0} \cdot M, \quad B := m_k \cdot M.$$

Then by the above

$$A = B + m_n \cdot A,$$

which by Nakayama's lemma yields $A = B$, that is,

$$m_k \cdot M = m_k \cdot M + m_n^{d_0} \cdot M,$$

and consequently

$$\widehat{M}/(m_k \cdot \widehat{M}) = \lim_{\leftarrow} M/(m_k \cdot M + m_n^d \cdot M) = M/(m_k \cdot M).$$

□

3. DIAGRAMS OF INITIAL EXPONENTS

3.1. Weierstrass-Hironaka division for formal analytic functions. Consider the ring

$$K[[u]] = K[[u_1, \dots, u_n]]$$

of formal power series over any field K . The monomials u^α can be naturally identified with the elements of \mathbb{N}^n , where \mathbb{N} denotes the set of natural numbers and zero. For any nonzero function $f \in K[[u]]$, $f = \sum c_\alpha u^\alpha$, define the *support* of f to be

$$\text{supp}(f) := \{\alpha \in \mathbb{N}^n \mid c_\alpha \neq 0\}.$$

By the *differential support* of f we mean

$$\text{supd}(f) := \{\alpha \mid D_{u^\alpha}(f) \neq 0\},$$

where $D_{u^\alpha} = \frac{1}{\alpha!} \frac{\partial}{\partial u^\alpha}$. Note that the latter makes sense in positive characteristic and is called the *Hasse derivative*.

This notion of differential support can be extended to regular functions on a smooth variety. It also better reflects properties of the functions. It is coherent and is not defined pointwise like support, thus allowing one to control singularities in a neighborhood.

Example 3.1.1. If $\text{char}(K) = 0$ and $f = u^d$ is a function on \mathbf{A}^1 then $\text{supp}(f) = \{d\}$ at 0, while $\text{supd}(f) = \{0, 1, \dots, d\}$. However, in the neighborhood of 0, $f = (u+t)^d = u^d + tu^{d-1} + \dots + t^d$ has the same support and differential support. If $\text{char}(K) = p$ and $f = u^p$ then $\text{supp}(f) = \{p\}$ at 0, while $\text{supd}(f) = \{0, p\}$ for $u \neq 0$, and $\text{supd}(f) = \{0, p\}$.

The following lemma is an immediate consequence of the definition:

Lemma 3.1.2. (1) $\text{supp}(f) \subseteq \text{supd}(f)$.

(2) If $\alpha \in \text{supd}(f)$ then $\alpha + \beta \in \text{supd}(f)$ for some $\beta \in \mathbb{N}^n$.

The definition immediately yields:

Lemma 3.1.3. *Let X be a smooth scheme over a field K (respectively an analytic or differentiable manifold), with a coordinate system u_1, \dots, u_n , and let $\Gamma \subset \mathbb{N}^n$. Then the sheaf*

$$\mathcal{O}_X^\Gamma := \{f \in \mathcal{O}_X \mid \text{supd}(f) \subset \Gamma\} = \{f \in \mathcal{O}_X \mid D_{u^\alpha}(f) \equiv 0, \alpha \notin \Gamma\} \subset \mathcal{O}_X$$

is a subsheaf of groups of \mathcal{O}_X .

For any n -tuple $\alpha = (a_1, \dots, a_n)$ of nonnegative integers set $|\alpha| := a_1 + \dots + a_n$. Then the *multiplicity* of $f = \sum c_\alpha u^\alpha$ is defined as

$$\text{ord}(f) = \min\{|\alpha| \mid c_\alpha \neq 0\}.$$

It follows immediately from the definition that

$$\text{ord}(f_1 \cdot f_2) \geq \text{ord}(f_1) + \text{ord}(f_2), \quad \text{ord}(u^\alpha \cdot f) = |\alpha| + \text{ord}(f).$$

Any ordered set of exponents $\alpha^1, \dots, \alpha^k \in \mathbb{N}^n$ defines a unique decomposition of

$$\mathbb{N}^n = \Gamma \cup \Delta_1 \cup \dots \cup \Delta_k,$$

where

$$\begin{aligned} \Delta_1 &:= a_1 + \mathbb{N}^n, \dots, \Delta_j := a_j + \mathbb{N}^n \setminus \bigcup_{i=1}^{j-1} a_i + \mathbb{N}^n = a_j + \mathbb{N}^n \setminus \bigcup_{i=1}^{j-1} \Delta_i, \\ \Gamma = \Gamma_0 &:= \mathbb{N}^n \setminus \bigcup_{i=1}^k a_i + \mathbb{N}^n = a_i + \mathbb{N}^n \setminus \bigcup_{i=1}^k \Delta_i, \quad \Delta = \bigcup_{i=1}^k \Delta_i = \bigcup_{i=1}^k a_i + \mathbb{N}^n \end{aligned}$$

For $i = 1, \dots, k$, $\Gamma_i := \Delta_1 - \alpha^i \subset \mathbb{N}^n$ is defined to be the set satisfying

$$\Delta_i = a_i + \Gamma_i,$$

Note that $\Gamma \cup \bigcup_{i=1}^j \Delta_i = \Gamma + \Delta = \mathbb{N}^n$ and $\Delta + \mathbb{N}^n \subseteq \Delta$.

Definition 3.1.4. We call $\Delta = \bigcup_{i=1}^k a_i + \mathbb{N}^n$ the *diagram* defined for the set $\{a_1, \dots, a_k\}$.

Lemma 3.1.5. *If $\alpha \notin \Gamma_i$ then $\alpha + \beta \notin \Gamma_i$ for all $\beta \in \mathbb{N}^n$. In particular the following conditions are equivalent:*

- (1) $\text{supp}(f) \subset \Gamma_i$.
- (2) $\text{supd}(f) \subset \Gamma_i$.

Proof. The condition $\alpha \notin \Gamma_i$ is equivalent to $\alpha + \alpha_i \notin \Delta_i = (a_i + \mathbb{N}^n) \setminus \bigcup_{j < i} (a_j + \mathbb{N}^n)$. The latter can be stated as $\alpha + \alpha_i \in \bigcup_{j < i} (a_j + \mathbb{N}^n)$. Obviously if $\alpha + \alpha_i \in \bigcup_{j < i} (a_j + \mathbb{N}^n)$ then $\alpha + \beta + \alpha_i \in \bigcup_{j < i} (a_j + \mathbb{N}^n)$.

Now suppose $\text{supp}(f) \subset \Gamma_i$ and $\alpha \in \text{supd}(f)$. This implies that $\alpha + \beta \in \text{supp}(f) \subset \Gamma_i$ for some $\beta \in \mathbb{N}^n$ and $\alpha \in \Gamma_i$. Consequently, $\text{supd}(f) \subset \Gamma_i$. The other implication is obvious: $\text{supp}(f) \subseteq \text{supd}(f) \subset \Gamma_i$. \square

Definition 3.1.6. We call a linear form

$$L = a_1 x_1 + \dots + a_n x_n : \mathbb{R}^n \rightarrow \mathbb{R}$$

positive (respectively *nonnegative*) if $a_i > 0$ (resp. $a_i \geq 0$) for all i . Any k -tuple $\bar{T} = (T_1, \dots, T_k)$ of nonnegative linear forms is called *positive* if for any $a \in \mathbb{N}^n \subset \mathbb{R}^n$ there exists at least one i such that $T_i(a) > 0$.

Any positive k -tuple \bar{T} defines the *monomial grading* $\bar{T} : \mathbb{N}^n \rightarrow \mathbb{R}^k$, of \mathbb{N}^n and thus a (partial) *monomial order* on \mathbb{N}^n induced by the lexicographic order on \mathbb{R}^k :

$$\alpha \leq_{\bar{T}} \beta \quad \text{if} \quad \bar{T}(\alpha) \leq_{\text{lex}} \bar{T}(\beta).$$

We shall call this grading *total* if $\bar{T} : \mathbb{N}^n \rightarrow \mathbb{R}^k$ is injective. \bar{T} will be called *normalized* if $T_1 = x_1 + \dots + x_n$.

The definition immediately yields

Lemma 3.1.7. (1) *If \bar{T} is positive then for any increasing sequence $\bar{T}(\alpha_1) < \bar{T}(\alpha_2) < \dots$, the sequence of $|\alpha_i|$, where $i = 1, 2, \dots$, diverges to infinity, and consequently $\{\bar{T}(\alpha_i)\} \subset \mathbb{R}^k$ is not bounded. And vice versa, if $\{\bar{T}(\alpha_i)\}$ is bounded then (α_i) is finite.*

(2) If \bar{T} is total, the order defined by \bar{T} on \mathbb{N}^k is total.

Proof. It follows that there exists i such that the i -th component $T_i(\alpha_j)$ diverges to infinity, as does α_i . \square

For any $f \in R[[u]]$, $f = \sum c_\alpha u^\alpha$, we call

$$\beta = \exp_{\bar{T}}(f)$$

its *initial exponent* if $\beta = \min_{\bar{T}}(\text{supp}(f))$ is a unique minimal element with respect to the \bar{T} -order. Also define the \bar{T} -multiplicity of any $f = \sum c_\alpha u^\alpha$ to be

$$\text{ord}_{\bar{T}}(f) = \min\{\bar{T}(\beta) \mid \beta \in \text{supp}(f)\}.$$

Observe that we have

Lemma 3.1.8. (1) If \bar{T} is normalized then $\text{ord}_{\bar{T}}(f) \leq \text{ord}_{\bar{T}}(g)$ implies $\text{ord}(f) \leq \text{ord}(g)$. In particular if $\alpha = \exp_{\bar{T}}(f)$ then $\text{ord}(f) = |\alpha|$.
 (2) If T is total then $\exp_{\bar{T}}(f)$ exists for any f .

The following theorem extends Weierstrass-Hironaka formal division in Grauert-Galligo form to any monomial order.

Theorem 3.1.9 (Weierstrass-Hironaka formal division theorem [2], [28], [9]). *Consider any monomial order defined by a positive r -tuple $\bar{T} = (T_1, \dots, T_r)$ on \mathbb{N}^n . Let*

$f_1, \dots, f_k \in K[[u, v]] = K[[u_1, \dots, u_n, v_1, \dots, v_m]]$ be formal analytic functions. Assume there exist exponents

$$\alpha_1 := \exp_{\bar{T}}(f_1(u, 0)), \dots, \alpha_r := \exp_{\bar{T}}(f_r(u, 0)) \in \mathbb{N}^n.$$

Let Δ be the diagram defined for the exponents $\alpha_1, \dots, \alpha_k \in \mathbb{N}^n$. Then for every $g \in K[[u, v]]$, there exist unique $h_i \in K[[u, v]]$ and $r(g) \in K[[u, v]]$ such that $\text{supd}(h_i) \subset \Gamma_i$, $\text{supd}(r(g)) \subset \Gamma$, and

$$g = \sum h_i f_i + r(g).$$

Moreover, if $\text{ord}(f_i) = |\alpha_i|$ for any i then

$$\text{ord}(r(g)) \geq \text{ord}(g), \quad \text{ord}(h_i) \geq \text{ord}(g) - |\alpha_i|.$$

Proof. First note that the order \bar{T} defined on \mathbb{N}^n extends to an order on \mathbb{N}^{n+k} .

We can assume that the coefficient $c_{i\alpha_i}$ of f_i is 1, by replacing f_i with $c_{i\alpha_i}^{-1} f_i$ if necessary.

Observe that any function $g \in K[[u]]$ can be uniquely written as $g = \sum h_i u^{\alpha_i} + r(g)$, where $\text{supd}(h_i) \subset \Gamma_i$ and $\text{supd}(r(g)) \subset \Gamma$. Define a K -linear transformation $\Phi : K[[u]] \rightarrow K[[u]]$ as

$$g = \sum h_i u^{\alpha_i} + r(g) \mapsto \Phi(g) = \sum h_i f_i + r(g).$$

We show that Φ is invertible. We can write $\Phi = I + U$, where

$$U(g) = \sum h_i (f_i - u^{\alpha_i})$$

with

$$\text{ord}_{\bar{T}}(f_i - u^{\alpha_i}) > \text{ord}_{\bar{T}}(u^{\alpha_i}), \quad \text{ord}_{\bar{T}}(g) \leq \text{ord}_{\bar{T}}(h_i u^{\alpha_i}), \quad \text{ord}_{\bar{T}}(g) \leq \text{ord}_{\bar{T}}(r(f)).$$

This implies that $\text{ord}_{\bar{T}}(U(g)) > \text{ord}_{\bar{T}}(g)$, and we get an increasing sequence $\text{ord}_{\bar{T}}(U^i(g))$, which implies, by Lemma 3.1.7, that $\text{ord}(U^i(g)) \rightarrow +\infty$, and $\Phi^{-1} = I - U + U^2 + \dots$ is well defined for any g . Also $\text{ord}_{\bar{T}}(\Phi(g)) = \text{ord}_{\bar{T}}(g)$, and if $g = \sum h_i u^{\alpha_i} + r(g)$ then

$$\text{ord}_{\bar{T}}(g) = \min\{\text{ord}_{\bar{T}}(h_i) + \text{ord}_{\bar{T}}(\alpha_i)\}.$$

Analogous considerations imply the ‘‘moreover’’ part of the theorem. \square

3.2. Diagrams of initial exponents. Let \mathcal{I} be any ideal in $K[[x_1, \dots, x_n]]$ (or any homogeneous ideal in $K[x_1, \dots, x_n]$). Consider a total normalized order \bar{T} on \mathbb{N}^n . Then the corresponding diagram

$$\Delta = \Delta(\mathcal{I}) = \exp_{\bar{T}}(\mathcal{I}) := \{\exp_{\bar{T}}(f) \mid f \in \mathcal{I}\} \subset \mathbb{N}^n$$

will be called the *diagram of initial exponents* of \mathcal{I} with respect to \bar{T} . Again we see that $\exp(\mathcal{I}) + \mathbb{N}^n \subseteq \exp(\mathcal{I})$ and $\exp(\mathcal{I})$ is finitely generated in the sense that $\exp(\mathcal{I}) = \bigcup_{i=1}^n \{\alpha^i + \mathbb{N}^n\}$ for a certain finite set of exponents $\alpha_1, \dots, \alpha_k$ called *vertices* of $\exp(\mathcal{I})$, characterized by the property

$$\alpha_i \notin \Delta \setminus (\alpha_i + \mathbb{N}^n).$$

Lemma 3.2.1 ([9]). *If $f_i \in \mathcal{I}$ for which $\exp(f_i) = \alpha_i$ are vertices of $\exp(\mathcal{I})$ then the elements f_i generate \mathcal{I} . Moreover, there is a uniquely determined set of generators $\bar{f}_i := x^{\alpha_i} + r_i$, called the Hironaka standard basis, such that $\text{supp}(r_i) \subset \Delta$ and $\text{mon}(\bar{f}_i) = x^{\alpha_i}$.*

Proof. By Hironaka formal division, for any $g \in \mathcal{I}$ we can write $g = \sum h_i f_i + r(g)$. Since $g, f_i \in \mathcal{I}$, we get $r(g) \in \mathcal{I}$ and consequently $\text{supp}(r(g)) \in \exp(\mathcal{I}) = \Delta$. But again by the Hironaka division theorem, $\text{supp}(r(g)) \in \Gamma$. Both conditions imply that $\text{supp}(r(g)) \subset \Gamma \cap \Delta = \emptyset$. Thus $r(g)$ is constant and eventually 0, and $g = \sum h_i f_i$.

For the second part apply the formal division algorithm directly to the set of functions x^{α_i} to get functions r_i with $\text{supp}(r_i) \subset \Gamma$. Set $\bar{f}_i := x^{\alpha_i} + r_i$. □

Denote by $H_{\mathcal{I}}$ the Hilbert-Samuel function of $K[[x_1, \dots, x_n]]/\mathcal{I}$, defined as

$$H_{\mathcal{I}}(s) = \dim_K(K[[x_1, \dots, x_n]]/(\mathcal{I} + m^{s+1})), \quad s \in \mathbb{N}.$$

Similarly for any diagram Δ we set

$$H(\Delta)(s) := \{\alpha \notin \Delta \mid |\alpha| \leq s\}.$$

Corollary 3.2.2. *There is a natural isomorphism of vector spaces over K given by remainder and preserving filtration by (m^s) , where $m = (x_1, \dots, x_n)$,*

$$\bar{r} : K[[x_1, \dots, x_n]]^{\Gamma} = \{f \in K[[x_1, \dots, x_n]] \mid \text{supp}(f) \subset \Gamma\} \rightarrow K[[x_1, \dots, x_n]]/\mathcal{I}.$$

In particular the following functions are equal:

$$H_{\mathcal{I}} = H(\Delta(\mathcal{I})).$$

3.3. Diagrams of finite type.

Definition 3.3.1 ([28], [38], [9], [10]). A diagram Δ of initial exponents is *monotone* if for any $i < j$ and any element $\alpha = (\alpha_1, \dots, \alpha_i, \dots, \alpha_j, \dots, \alpha_n) \in \Delta$ we have that the element $R_{ij}(\alpha) := (\alpha_1, \dots, \alpha_i + \alpha_j, \dots, 0_j, \dots, \alpha_n)$ is in Δ .

Monotone diagrams were introduced by Hironaka in his proof of the Henselian division theorem [38]. They also played an important role in the Bierstone-Milman proof of the Hironaka strong desingularization theorem [9]. The monotonicity of diagrams will be considered as an analog to the regularity condition in the Weierstrass preparation and division theorems. As observed by Weierstrass, by the generic change of coordinates one can make any analytic function into a d -regular one. A similar approach was used by Galligo and Grauert who proved the analogous result for diagrams in generic coordinates [28]. One should mention that the initial condition for the ‘‘generic diagrams’’ considered by Grauert and Galligo was stronger than the Hironaka condition above, but was valid only in characteristic zero. Still the Galligo argument works for monotone diagrams in any characteristic.

Definition 3.3.2. We shall call a monomial order \bar{T} *monotone* if it is total, normalized and $\bar{T}(R_{ij}(\alpha)) \leq \bar{T}(\alpha)$ for any $\alpha \in \mathbb{N}^n$ and $i < j$.

Example 3.3.3. It follows from the definition that the monomial order

$$\bar{T} = (x_1 + \dots + x_n, x_2 + \dots + x_n, x_3 + \dots + x_n, \dots, x_n)$$

is total, normalized and monotone.

Let \mathcal{I} be any ideal in \mathcal{E}_n (or any homogeneous ideal in $K[x_1, \dots, x_n]$). For a coordinate system $u_1, \dots, u_n \in \mathcal{E}_n$ and a total monotone monomial order \bar{T} consider the corresponding diagram

$$\Delta = \Delta(\mathcal{I}) = \exp_{\bar{T}}(\mathcal{I}) = \{\exp_{\bar{T}}(f) \mid f \in \mathcal{I}\}.$$

Then $\Delta(\mathcal{I})$ clearly depends upon the choice of the coordinate system. One can represent the diagram as an infinite sequence $\alpha(\Delta) = (\alpha_1, \dots, \alpha_k, \dots)$ of all its elements $\alpha_i \in \Delta$ put in ascending order. We can then order the set of all possible diagrams corresponding to the ideal \mathcal{I} by introducing the lexicographic order on the set of values $\alpha(\Delta)$. Fix a total, normalized and monotone order \bar{T} .

Theorem 3.3.4 (Galligo-Grauert). ([28]) *If K is an infinite field then the minimal diagram $\Delta(\mathcal{I})$ corresponding to \mathcal{I} with respect to a monotone order and a generic coordinate system is unique and monotone.* \square

Proof. The proof in [28] can be easily adapted to monotone diagrams. \square

Corollary 3.3.5. *Let K be any field. Then there exists a finite separable extension K' of K and a coordinate change defined over K' such that $\Delta(\mathcal{I})$ is monotone.*

Proof. Let K^s be a separable closure of K . Then K^s is infinite and we can apply the previous theorem. \square

The definition of monotone diagram is closely related to a generic linear transformation which takes an element $x^a y^b$ to the polynomial with initial exponent x^{a+b} . It gives however unnecessary constraints on the diagrams.

Definition 3.3.6. A diagram Δ of initial exponents is of *finite type* if for any $i < j$ and any element $\alpha = (\alpha_1, \dots, \alpha_i, \dots, \alpha_j, \dots, \alpha_n) \in \Delta$ there exists an element

$$S_\alpha(\alpha) := (\alpha_1, \dots, \alpha_i + \alpha'_i, \dots, 0_j, \dots, \alpha_n) \in \Delta$$

for some $\alpha'_i \in \mathbb{N}$. A diagram is *finite* if $\Gamma = \mathbb{N}^n \setminus \Delta$ is finite.

It follows from the definition that finite and monotone diagrams are of finite type. However, not all finite diagrams are monotone.

Now, for any $i \leq n$ let us identify the set \mathbb{N}^i with the subset $\mathbb{N}^i \times \{0\}$ of \mathbb{N}^n of elements with the last $n-i$ components zero. Similarly by $\mathbb{N}^{*,i,n}$ will mean the subset $\{0\} \times \mathbb{N}^i$ of \mathbb{N}^n of elements with the first $n-i$ components zero.

For any $i < j \leq n$ denote by π_{ji} the natural projection $\pi_{ji} : \mathbb{N}^j \rightarrow \mathbb{N}^i$. It follows from the definition that

$$\pi_{ji}\pi_{kj} = \pi_{ki}.$$

Lemma 3.3.7. (1) *If $\Delta \subset \mathbb{N}^n$ is of finite type then so is $\Delta \cap \mathbb{N}^i \times \{0\} \subset \mathbb{N}^i \times \{0\}$.*

(2) *If $\Delta \subset \mathbb{N}^n$ is of finite type then so is $\pi_i(\Delta) \subset \mathbb{N}^i \times \{0\}$.*

Proof. The properties are simple consequences of the definition. \square

Let $\Gamma^0 := \Gamma = \mathbb{N}^n \setminus \Delta$. Set $\bar{\Gamma}^i := \mathbb{N}^i \setminus \pi_{ni}(\Delta)$ and $\Gamma^i := \bar{\Gamma}^i \times \mathbb{N}^{n-i} \subset \Gamma$. This defines the natural filtration of sets

$$\Gamma^0 := \emptyset \subset \Gamma^1 \subset \dots \subset \Gamma^i \subset \dots \subset \Gamma^n = \Gamma.$$

Then

$$\Gamma^i \setminus \Gamma^{i-1} = (\bar{\Gamma}^i \setminus (\bar{\Gamma}^{i-1} \times \mathbb{N})) \times \mathbb{N}^{n-i} = A_i \times \mathbb{N}^{n-i},$$

where the set

$$A_i := \bar{\Gamma}^i \setminus (\bar{\Gamma}^{i-1} \times \mathbb{N}) = \pi_{i,i-1}^{-1}(\pi_{n,i-1}(\Delta) \setminus \pi_{ni}(\Delta))$$

is a subset of $\mathbb{N}^i \setminus \mathbb{N}^{i-1}$.

Here is a slight enhancement of an important observation of Hironaka.

Lemma 3.3.8 ([38]). *For any diagram Δ there is a finite decomposition*

$$\Gamma = \bigcup_{i=0}^n A_i \times \mathbb{N}^{n-i},$$

where $A_i \subset \mathbb{N}^i \setminus \mathbb{N}^{i-1}$. Moreover Δ is of finite type if and only if all A_i are finite.

Proof. By the above,

$$\bigcup_{i=0}^n A_i \times \mathbb{N}^{n-i} = \bigcup_{i=0}^n \Gamma^i \setminus \Gamma^{i-1} = \Gamma^n = \Gamma.$$

Suppose now Δ is of finite type. It suffices to show the conclusion for $i = n - 1$ and $\pi_i = \pi$. Then since $\pi_i(\Delta) \in \mathbb{N}^i$ is of finite type, we reduce the case of π_i to the situation of codimension one. Denote $\overline{\Delta}_s := \alpha^s + \mathbb{N}^n \subset \Delta$. Then $\pi(\Delta) = \bigcup \pi(\overline{\Delta}_s)$. We show that

$(\pi(\overline{\Delta}_s) \times \mathbb{N}) \setminus \Delta$ is finite. Then the set

$$A_n = \pi^{-1}\pi(\Delta) \setminus \Delta = \bigcup_s ((\pi(\overline{\Delta}_s) \times \mathbb{N}) \setminus \Delta) \subset \mathbb{N}^n \setminus \Delta = \Gamma$$

is finite as well. To this end, write $\alpha^s = (\alpha_1, \dots, \alpha_n)$ and $\pi(\alpha^s) = (\alpha_1, \dots, \alpha_{n-1}, 0)$. If Δ is of finite type then for any $i = 1, \dots, n - 1$,

$$\pi(\alpha^s) + \alpha'_i e_i = (\alpha_1, \dots, \alpha_i + \alpha'_i, \alpha_{n-1}, 0)$$

are in Δ for suitable α'_i . In other words,

$(\pi(\overline{\Delta}_s) \times \mathbb{N}) \setminus \Delta$ contains only vectors of the form $\gamma = \pi(\alpha^s) + (\beta_1, \dots, \beta_n)$, where $\beta_i < \alpha'_i$ for $i = 1, \dots, n - 1$ and $\beta_n < \alpha_n$. This implies the finiteness of each set

$$(\pi(\overline{\Delta}_s) \times \mathbb{N}) \setminus \Delta.$$

Conversely, assume all A_i are finite. Let $\alpha = (\alpha_1, \dots, \alpha_i, \dots, \alpha_j, \dots, \alpha_n) \in \Delta$, and suppose all $(\alpha_1, \dots, \alpha_i + k, \dots, 0_j, \dots, \alpha_n)$ are in $\Gamma = \bigcup_{i=0}^n A_i \times \mathbb{N}^{n-i}$ for any natural k . Then it follows from finiteness of A_i that for sufficiently large k the elements $(\alpha_1, \dots, \alpha_i + k, \dots, 0_j, \dots, \alpha_n)$ are in $A_{i_0} \times \mathbb{N}^{n-j}$ for $i_0 < i$. But then $(\alpha_1, \dots, \alpha_{i_0}, 0, \dots, 0) \in A_j$ and $\alpha \in A_j \times \mathbb{N}^{n-j} \subset \Gamma$, which contradicts the assumption. \square

3.4. Decomposition of diagrams of finite type. Our next goal will be to find a similar finite decomposition for the set Δ . For that purpose consider the reverse lexicographic order $<_r$ on \mathbb{N}^n , that is, the one corresponding to the linear map

$$\overline{T}_r = (x_n, x_{n-1}, \dots, x_1).$$

Note that if $\alpha \in \mathbb{N}^{i+1} \setminus \mathbb{N}^i$ and $\beta \in \mathbb{N}^{j+1} \setminus \mathbb{N}^j$ with $i < j$ then $\alpha <_r \beta$.

Remark. We are going to use the reverse lexicographic order only to determine the desired subdivision of Δ and not to determine the initial exponents of functions.

Lemma 3.4.1. *If $\Delta = \bigcup_{s=1}^k \Delta_s$ is of finite type then for any $r \leq k$ the diagram $\Delta^s = \bigcup_{r=1}^s \Delta_r$ is also of finite type.*

Proof. Note that if $\alpha \in \Delta_s = \alpha^s + \mathbb{N}^n$ then $S_\alpha(\alpha) <_r \alpha$. On the other hand, $S_\alpha(\alpha) \in \Delta_{s'} = \alpha_{s'} + \mathbb{N}^n$, where $\alpha'_{s'} \leq S_\alpha(\alpha_{s'}) \leq_r \alpha_{s'}$. Finally, $S_\alpha(\alpha) \in \Delta^{s'} \subset \Delta^s$. \square

Using the reverse lexicographic order, we rewrite the sequence of vertices of the diagram Δ of finite type as

$$\alpha_1 < \alpha_{2,1} < \dots < \alpha_{2,k_2} < \alpha_{3,1} < \dots < \alpha_{r,k_r} = \alpha_k,$$

where $\alpha_{i,j} \in \mathbb{N}^i \setminus \mathbb{N}^{i-1}$. Then as in Section 3.1, we define a subdivision of \mathbb{N}^n into the disjoint union of the sets

$$\Delta_{i,j} = \alpha_{i,j} + \mathbb{N}^n \setminus \bigcup_{\alpha_{i,j'} <_r \alpha_{i,j}} (\alpha_{i,j'} + \mathbb{N}^n) = \alpha_{i,j} + \mathbb{N}^n \setminus \bigcup_{\alpha_{i,j'} <_r \alpha_{i,j}} \Delta_{i,j'},$$

where

$$\Delta_{i,j} = \alpha_{i,j} + \Gamma_{i,j}, \quad \Gamma = \mathbb{N}^n \setminus \Delta.$$

Consider the filtration of subsets of Δ analogous to the above:

$$\overline{\Delta}^0 := \{0\} \subset \overline{\Delta}^1 := (\Delta \cap \mathbb{N}^1) \times \mathbb{N}^{n-1} \subset \overline{\Delta}^2 := (\Delta \cap \mathbb{N}^2) \times \mathbb{N}^{n-2} \subset \dots \subset \overline{\Delta}^n := \Delta \cap \mathbb{N}^n = \Delta,$$

and similarly set

$$\Delta^i := \overline{\Delta}^i \times \mathbb{N}^{n-i}$$

to get the filtration

$$\Delta^0 = \{0\} \subset \Delta^1 \subset \dots \subset \Delta^n = \Delta.$$

Then

$$\Delta^i \setminus \Delta^{i-1} = (\Delta \cap \mathbb{N}^i) \setminus ((\Delta \cap \mathbb{N}^{i-1}) \times \mathbb{N}) \times \mathbb{N}^{n-i} = \overline{B}_i \times \mathbb{N}^{n-i},$$

where

$$B_i := (\Delta \cap \mathbb{N}^i) \setminus ((\Delta \cap \mathbb{N}^{i-1}) \times \mathbb{N})$$

is a subset of $\mathbb{N}^i \setminus \mathbb{N}^{i-1}$. So we have

$$\Delta = \bigcup B_i \times \mathbb{N}^{n-i}.$$

Note however that the sets B_i are usually not finite and thus they will be subsequently decomposed.

Lemma 3.4.2. $\pi_{n,i-1}(\Delta) = \pi_{n,i-1}(\Delta \cap \mathbb{N}^i)$.

Proof. Let $\bar{a} = \bar{a}^n := (a_1, \dots, a_n) \in \Delta$. By the finite type assumption the elements

$$\begin{aligned} \bar{a}^{n-1} &:= (a_1, \dots, a_{n-1} + a'_n, 0) \in \mathbb{N}^{n-1}, \\ \bar{a}^{n-2} &:= (a_1, \dots, a_{n-2} + a'_{n-1} + a'_n, 0, 0) \in \mathbb{N}^{n-2}, \quad \dots, \\ \bar{a}^i &:= (a_1, \dots, a'_i + \dots + a'_n, 0, \dots, 0) \in \mathbb{N}^i \end{aligned}$$

are all in Δ for some $a'_i \in \mathbb{N}$. But $\pi_{i,i-1}(\bar{a}^i) = \pi_{n,i-1}(\bar{a})$, which finishes the proof. \square

Write

$$C_1 := \mathbb{N}^1 \setminus (A_1 \cup B_1), \quad C_1 \times \mathbb{N} := \mathbb{N}^2 \setminus (A_1 \times \mathbb{N}^1 \cup B_1 \times \mathbb{N}^1).$$

By definition $A_2, B_2 \subset C_1 \times \mathbb{N}$. Set

$$C_2 := (C_1 \times \mathbb{N}) \setminus (A_2 \cup B_2).$$

Again $A_3, B_3 \subset C_2 \times \mathbb{N}$, and we set

$$C_3 := (C_2 \times \mathbb{N}) \setminus (A_3 \cup B_3).$$

We define C_k recursively by

$$C_n := (C_{n-1} \times \mathbb{N}) \setminus (A_n \cup B_n).$$

This gives us the decomposition of \mathbb{N}^i into a union of disjoint subsets:

$$\mathbb{N}^i = \left(\bigcup_{j=1}^i A_j \times \mathbb{N}^{i-j} \right) \cup \left(\bigcup_{j=1}^i B_j \times \mathbb{N}^{i-j} \right) \cup C_i,$$

where $A_j, C_j \subset \Gamma \cap (\mathbb{N}^j \setminus \mathbb{N}^{j-1})$, $B_j \subset \Delta \cap (\mathbb{N}^j \setminus \mathbb{N}^{j-1})$ are finite, $\pi_{j,j-1}(A_j) \subset \Gamma$, $\pi_{j,j-1}(C_j) \subset \Delta$, and

$$\left(\bigcup_{j=1}^i A_j \times \mathbb{N}^{i-j} \right) \cup C_i = \Gamma \cap \mathbb{N}^i, \quad \bigcup_{j=1}^i B_j \times \mathbb{N}^{i-j} = \Delta \cap \mathbb{N}^i.$$

In particular

$$\mathbb{N}^n = \left(\bigcup_{j=1}^n A_j \times \mathbb{N}^{n-j} \right) \cup \left(\bigcup_{j=1}^n B_j \times \mathbb{N}^{n-j} \right) \cup C_n = \Gamma \cup \Delta \cup C_n = \mathbb{N}^n \cup C_n.$$

So $C_n = \emptyset$, and we get

$$(3.1) \quad \mathbb{N}^n = \left(\bigcup_{j=1}^n A_j \times \mathbb{N}^{n-j} \right) \cup \left(\bigcup_{j=1}^n B_j \times \mathbb{N}^{n-j} \right).$$

Lemma 3.4.3. (1) C_i is finite.

(2) $\pi_{i+1,i} : B_{i+1} \rightarrow C_i$ is surjective.

(3) $B_{i+1} = \bigcup_j \Delta_{i+1,j} \cap \mathbb{N}^{i+1}$.

(4) $B_{i+1} \times \mathbb{N}^{n-i-1} = \bigcup_j \Delta_{i+1,j}$.

(5) The sets $C_{ij} := \pi_{i+1,i}(\Delta_{i+1,j})$ are disjoint and define a subdivision of $C_i = \bigcup C_{ij}$.

(6) Write $C_{ij} := \pi_{i+1,i}(\alpha_{i+1,j}) + \bar{\Gamma}_{i+1,j}$. Then $\Gamma_{i+1,j} = \bar{\Gamma}_{i+1,j} \times \mathbb{N}^{n-i}$.

Proof. (1) By Lemma 3.3.7, the diagram $\Delta \cap \mathbb{N}^i$ is monotone. Then, by Lemma 3.3.8,

$$\pi_{i,i-1}^{-1} \pi_{i,i-1}(\Delta \cap \mathbb{N}^i) \setminus (\Delta \cap \mathbb{N}^i)$$

is finite. Intersecting this set with \mathbb{N}^{i-1} we see that

$$C_i := \pi_{i,i-1}(\Delta \cap \mathbb{N}^i) \setminus (\Delta \cap \mathbb{N}^{i-1})$$

is finite.

(2) We have $B_{i+1} = (\Delta \cap \mathbb{N}^{i+1}) \setminus \pi_{i+1,i}^{-1}(\Delta_{i+1} \cap \mathbb{N}^{i+1})$ and thus

$$\pi_{i+1,i}(B_{i+1}) = \pi_{i+1,i}(\Delta \cap \mathbb{N}^{i+1}) \setminus (\Delta \cap \mathbb{N}^i).$$

By Lemma 3.4.2, this gives $\pi_{i+1,i}(B_{i+1}) = \pi_{n,i}(\Delta) \setminus B_i = \mathbb{N}^i \setminus A_i \setminus B_i = C_i$.

(3) We have

$$B_{i+1} = (\Delta \cap \mathbb{N}^{i+1}) \setminus \pi_{i+1,i}^{-1}(\Delta_{i+1} \cap \mathbb{N}^{i+1}) = \bigcup_{i' \leq i,j} \Delta_{i',j} \cap \mathbb{N}^i \setminus \bigcup_{i' < i,j} \Delta_{i',j} \cap \mathbb{N}^i = \bigcup_j \Delta_{i,j} \cap \mathbb{N}^i.$$

(4) Follows from (3).

(5) Suppose $\bar{\beta} = (\beta_1, \dots, \beta_i) \in \pi_{i+1,i}(\Delta_{i+1,j_1}) \cap \pi_{i+1,i}(\Delta_{i+1,j_2})$, where $j_1 < j_2$. Let $\beta = (\beta_1, \dots, \beta_{i+1})$ be the smallest element in $\Delta_{i+1,j_1} \cap \mathbb{N}^{i+1}$, with respect to the reverse lexicographic order, such that $\pi_{i+1,i}(\beta) = \bar{\beta}$. Then $\beta = \alpha_{i+1,j_1} + \beta'$, where $\beta' \in \mathbb{N}^i \subset \mathbb{N}^{i+1}$. Since $\alpha_{i+1,j_1} <_r \alpha_{i+1,j_2}$, and the last $n - i - 1$ coordinates (in \mathbb{N}^n) of both are zero the $i + 1$ -coordinate $(\alpha_{i+1,j_1})_{i+1}$ of α_{i+1,j_1} is not greater than the one of α_{i+1,j_2} .

Also by definition, $\beta_{i+1} = (\alpha_{i+1,j_1})_{i+1} + \beta'_{i+1} = (\alpha_{i+1,j_1})_{i+1}$, and all the elements $(\beta_1, \dots, \beta_i, \beta'_{i+1})$ with $\beta'_{i+1} \geq \beta_{i+1}$ are in $\Delta_{i+1,j_1} \cap \mathbb{N}^{i+1}$. (I)

On the other hand, if $\beta'' \in \Delta_{i+1,j_2}$, and $\pi_{i+1,i}(\beta'') = \bar{\beta}$ then β'' has a form $\beta'' = (\beta_1, \dots, \beta_i, \beta''_{i+1})$ with $\beta''_{i+1} \geq \alpha_{i+1,j_2}$. This implies

$$\beta''_{i+1} \geq (\alpha_{i+1,j_2})_{i+1} \geq (\alpha_{i+1,j_1})_{i+1} \geq \beta_{i+1}.$$

By (I) we conclude that $\beta'' \in \Delta_{i+1,j_1}$ and thus is not an element of Δ_{i+1,j_2} , which contradicts the definition of β'' .

(6) follows from (5). □

Let

$$\bar{B}_{i,j} := \{\alpha_{i,j} + \bar{\Gamma}_{i,j}\}, \quad \bar{B}_i := \bigcup_j \bar{B}_{i,j}.$$

It follows from the above considerations that the following decompositions of sets are finite:

- Corollary 3.4.4.** (1) $\Delta_{i,j} \cap \mathbb{N}^i = \bigcup_{\beta \in \bar{B}_{i,j}} (\beta + \mathbb{N}^{*1,n-i})$,
 (2) $\Delta_{i,j} = \bigcup_{\beta \in \bar{B}_{i,j}} (\beta + \mathbb{N}^{*n-i+1,n})$,
 (3) $B_i = \bigcup_{\beta \in \bar{B}_i} (\beta + \mathbb{N}^{*1,n-i})$,
 (4) $B_i \times \mathbb{N}^{n-i} = \bigcup_{\beta \in \bar{B}_i} (\beta + (\mathbb{N}^{*n-i+1,n}))$.

Summarizing we get from formula 3.1 and the above:

$$(3.2) \quad \mathbb{N}^n = \Gamma + \Delta = \left(\bigcup_{j=1}^n A_j \times \mathbb{N}^{n-j} \right) \cup \bigcup_{j=1}^n \bigcup_{\beta \in \bar{B}_j} (\beta + (\mathbb{N}^{*n-j+1,n})).$$

Consider the natural filtration of rings

$$R_0 = K \subset R_1 = K[x_n] \subset \dots \subset R_n = R = K[x_1, \dots, x_n].$$

We rewrite some of the relations between the sets A_i , B_j , and C_s in the language of modules. For $i = 1, \dots, n$, set $M_0^A = 0$, $M_{n+1}^A = M_{n+1}^B = 0$ and

$$M_i^A := \bigoplus_{\alpha \in A_i} R_{n-i} \cdot x^\alpha, \quad M_i^C := \bigoplus_{\alpha \in C_i} R_{n-i} \cdot x^\alpha, \quad M_i^B := \sum_{\beta \in \bar{B}_i} R_{n-i+1} \cdot x^\beta.$$

There exists a decomposition of R_n as a group into modules

$$\begin{aligned} R_n &= (M_1^A \oplus M_1^C) \oplus M_1^B \\ (M_2^A \oplus M_2^C) \oplus M_2^B &= M_1^C \\ &\dots \\ (M_n^A \oplus M_n^C) \oplus M_n^B &= M_{n-1}^C \end{aligned}$$

or alternatively

$$\begin{aligned} R_n &= (M_1^A \oplus M_1^C) \oplus (M_0^A \oplus M_1^B) \\ (M_2^A \oplus M_2^C) \oplus (M_1^A \oplus M_2^B) &= (M_1^A \oplus M_1^C) \\ &\dots \end{aligned}$$

$$(M_n^A \oplus M_n^C) \oplus (M_{n-1}^A \oplus M_n^B) = (M_{n-1}^A \oplus M_{n-1}^C) \oplus \{0\} \oplus (M_n^A \oplus \{0\}) = (M_n^A \oplus 0),$$

where $M_n^C = M_n^B = 0$. The latter sequence yields the following

Corollary 3.4.5. *There is a decomposition*

$$(3.3) \quad R_n = \bigoplus_{i=1}^{n+1} (M_{i-1}^A \oplus M_i^B) = R_n^\Delta \oplus R_n^\Gamma$$

into a direct sum of free R_{n-i+1} -modules $M_{i-1}^A \oplus M_i^B$, where

$$R_n^\Delta = \{f \in R_n \mid \text{supp}(f) \subset \Delta\} = \bigoplus_{i=1}^{n+1} M_i^B,$$

$$R_n^\Gamma = \{f \in R_n \mid \text{supp}(f) \subset \Gamma\} = \bigoplus_{i=1}^{n+1} M_{i-1}^A.$$

4. FILTERED STANLEY DECOMPOSITION

4.1. Properties of filtered Stanley decomposition. The classical Weierstrass theorem and its Malgrange-Mather extension allow us to write the \mathcal{E}_n -module \mathcal{E}_n as the image of a direct sum of groups (more specifically modules over different rings), namely

$$(4.1) \quad \mathcal{E}_n \cdot f \oplus \bigoplus_{i=0}^{k-1} \mathcal{E}_{n-1} \cdot x^i \rightarrow \mathcal{E}_n$$

for any k -regular function f . The above (group) homomorphism is an isomorphism in a reduced category. This decomposition can be viewed as a modification or perturbation of the decomposition

$$(4.2) \quad \mathcal{E}_n = \mathcal{E}_n \cdot x^k \oplus \bigoplus_{i=0}^{k-1} \mathcal{E}_{n-1} \cdot x^i.$$

We are going to extend this approach to several functions and ideals. Note that the generalization of decomposition (4.2) for graded rings has been worked out in the previous section in formula (3.3).

Definition 4.1.1. A homomorphism of finite \mathcal{E}_n -modules

$$\Psi : M \rightarrow N$$

will be called a *quasi-isomorphism* if Ψ is an epimorphism and its kernel is contained in $m_n^\infty \cdot M$. We will call a finite \mathcal{E}_n -module M *quasi-free* if there exists a quasi-isomorphism $\mathcal{E}_n^k \rightarrow M$.

Definition 4.1.2. Let M be a finitely generated \mathcal{E}_n -module with filtration (M_s) of \mathcal{E}_n -modules satisfying

$$(4.3) \quad m_n^r \cdot M_s \subset M_{r+s}.$$

We say that the \mathcal{E}_n -module M admits a *filtered Stanley decomposition* over $(\mathcal{E}_i)_{i=0}^n$ if there exist free finitely generated \mathcal{E}_i -modules N_i for $i = 0, \dots, n$ and a homomorphism of K -spaces

$$\Psi : \bigoplus N_i \rightarrow M$$

such that:

- (1) The restriction $\Psi|_{N_i} : N_i \rightarrow M$ of Ψ to N_i is a homomorphism of \mathcal{E}_i -modules.
- (2) Ψ is surjective.
- (3) The kernel of Ψ is contained in $\bigoplus m_i^\infty \cdot N_i$.
- (4) There exist bases $\{e_{i1}, \dots, e_{ik_i}\}$ of N_i for $i = 0, \dots, n$ such that

$$\Psi^{-1}(M_s) = \bigoplus m_i^{s-d_{ij}} e_{ij},$$

where $d_{ij} = \text{ord}(\Psi(e_{ij}))$.

The set $\{b_{ij}\} := \{\Psi(e_{ij})\}_{i,j}$ will be called a *basis* of M over $(\mathcal{E}_i)_{i=0}^n$. The homomorphism Ψ will be called a *quasi-isomorphism over $(\mathcal{E}_i)_{i=0}^n$* .

Any \mathcal{E}_n -module M admits a filtration $m_n^{s+1} \cdot M$, which we refer to as the *standard filtration*. While this filtration is of primary concern, it would be useful to consider in certain situations a more general, positive linear grading $T : \mathbb{N}^n \rightarrow \mathbb{R}$ with real values, given by a single positive linear form T . Observe that in that case, by Lemma 3.1.7, each subset of the set of values $S_n := T(\mathbb{N}^n)$ that is bounded above is finite. We consider the natural filtration on \mathcal{E}_n defined by T , indexed by $a \in S_n = T(\mathbb{N}^n)$:

$$m_{n,T,a} := \{f \in \mathcal{E}_n \mid \text{ord}_T(f) \geq a\}.$$

Then $\text{ord}_T(f) := \max\{a \in S_n \mid f \in m_{n,T,a}\}$. We shall call an \mathcal{E}_n -module T -filtered if it satisfies the condition

$$(4.4) \quad m_{n,T,r} \cdot M_s \subset M_{r+s}.$$

Definition 4.1.3. We say that any T -filtered \mathcal{E}_n -module M with T -filtration M_s admits a *T -filtered Stanley decomposition* over $(\mathcal{E}_i)_{i=0}^n$ if there is a homomorphism ϕ as in the definition above satisfying conditions (1)–(3) and the condition

(4') There exist bases $\{e_{i_1}, \dots, e_{i_{k_i}}\}$ of N_i for $i = 0, \dots, n$ such that

$$\Psi^{-1}(M_s) = \bigoplus m_{i,T,s-d_{ij}} e_{ij},$$

where $d_{ij} = \text{ord}_T(\Psi(e_{ij}))$.

One can extend this immediately to graded modules over graded rings. Observe that the graded ring of \mathcal{E}_n defined by the filtration $m_n^k \subset \mathcal{E}_n$ is equal to

$$R := \text{gr}(\mathcal{E}_n) = \text{gr}(\widehat{\mathcal{E}_n}) = \text{gr}(K[[x_1, \dots, x_n]]) = \bigoplus_i m^i / m^{i+1} = K[x_1, \dots, x_n].$$

For any ring K consider the natural filtration of rings

$$R_0 = K \subset R_1 = K[x_n] \subset R_2 = K[x_n, x_{n-1}] \subset \dots \subset R_n = R = K[x_1, \dots, x_n].$$

For any module \mathcal{E}_n -module M with filtration (M_s) satisfying condition (4) from the definition above, we consider the associated graded R_n -module $\text{gr}(M) := \bigoplus M_s / M_{s+1}$.

Similarly for any positive linear grading $T : \mathbb{N}^n \rightarrow \mathbb{R}$ one can consider a ring $R_T = K[x_1, \dots, x_n]_T$ with gradation defined by T , and the filtration R_{iT} .

Definition 4.1.4. Let K be any (commutative) ring with 1, and M be a finitely generated (or simply finite) graded $K[x_1, \dots, x_n]$ -module. By a *filtered Stanley decomposition*, or simply a *Stanley decomposition*, of M over $(R_i)_{i=0}^n$ we mean a graded group decomposition

$$M = \bigoplus N_i,$$

where all N_i are free finite R_i -modules.

We shall call a finite set $\{g_{ij}\}_{i=1, \dots, n, j \in S_j}$ of homogeneous elements in M a *generating system* of M if $\{g_{ij}\}_{j \in S_j} \subset N_i$ generate the R_i -module N_i for any i and

$$\sum N_i = M.$$

We shall call a set $\{g_{ij}\}_{i=1, \dots, n, j \in S_j}$ of homogeneous elements in M *independent* over $(R_i)_{i=0}^n$ if

$$\sum c_{ij} g_{ij} = 0, \quad c_{ij} \in R_i,$$

implies that $c_{ij} = 0$. A set $\{g_{ij}\}_{i=1, \dots, n, j \in S_j}$ of homogeneous elements is a *Stanley basis*, or simply a *basis*, of R if $M = \bigoplus N_i$ and each N_i is a free R_i -module generated by a basis $\{g_{ij}\}_{j \in S_j}$.

Stanley decomposition provides an effective tool for computing Hilbert (or Hilbert-Samuel) functions for finite \mathcal{E}_n -modules.

Let us define a function $\phi(n, k) : \mathbb{Z} \rightarrow \mathbb{N}$ by

$$\phi(n, k) = \begin{cases} \binom{n+k}{k} & \text{if } n \geq 0; \\ 0 & \text{if } n < 0. \end{cases}$$

Denote by $K[x_1, \dots, x_n]_s$ the s -gradation of the ring $K[x_1, \dots, x_n]$. Then

$$\text{rank}_K(K[x_1, \dots, x_n]_s) = \phi(n-1, s), \quad \text{rank}_K\left(\bigoplus_{i \leq s} K[x_1, \dots, x_n]_i\right) = \phi(n, s).$$

Corollary 4.1.5. *Let M be any finite graded R -module over R with a (filtered) Stanley decomposition. Let g_{ij} be its Stanley basis, and let $d_{ij} \in \mathbb{N}$ denote the degrees of the generators. Then each module M_s/M_{s+1} and*

M/M_{s+1} is free over K , and the Hilbert function

$$H_M(s) = \text{rank}_K(M_s/M_{s+1})$$

is equal to

$$H_M(s) = \sum_{i,j} \phi(s - d_{ij}, i).$$

In particular $H_M(s)$ is a polynomial for

$$s \geq d(M) = \max\{d_{ij}\}.$$

Proof. We have

$$M_s/M_{s+1} \simeq \psi^{-1}(M_s)/\psi^{-1}(M_{s+1}) \simeq \bigoplus m_i^{s-d_{ij}}/m_i^{s+1-d_{ij}} e_{ij}.$$

□

Remark. Existence of a Stanley decomposition was proven by Stanley and it is a tool of fundamental importance in homological algebra. In this paper we show existence of a filtered Stanley decomposition of any graded ring over an infinite field, likewise over the smooth category of rings \mathcal{E}_n over K . The stronger conditions imposed on Stanley decompositions are critical for this paper, in view of, in particular, the stabilization theorem for graded rings (Theorem 4.3.4).

The relation between $(\mathcal{E}_i)_{i=0}^n$ -modules and their graded $(R_i)_{i=0}^n$ -modules is useful in view of the following observation.

Theorem 4.1.6. *Let M be a finitely generated \mathcal{E}_n -module with filtration M_i such that $m_n^j \cdot M_i \subset M_{i+j}$, and with the associated finite graded R_n -module $\text{gr}(M) = \bigoplus M_k/M_{k+1}$. (Respectively let \widehat{M} be any T -filtered module with the associated finite T -graded R_n -module $\text{gr}(\widehat{M})$.) Denote by $\widehat{M} := \varprojlim M/M_s$ the completion of M , which is an $\widehat{\mathcal{E}}_n$ -module. Let $b_{11}, \dots, b_{n,k_n} \in M$ be a finite set of elements. Then the following conditions are equivalent:*

- (1) $b_{11}, \dots, b_{n,k_n} \in M$ is a basis of M over $(\mathcal{E}_i)_{i=0}^n$.
- (2) $\widehat{b}_{11}, \dots, \widehat{b}_{n,k_n} \in \widehat{M}$ is a basis of \widehat{M} over $(\widehat{\mathcal{E}}_i)_{i=0}^n$.
- (3) $\text{in}(b_{11}), \dots, \text{in}(b_{n,k_n})$ form a basis of $\text{gr}(M)$ over $(R_i)_{i=0}^n = (\text{gr}(\mathcal{E}_i))$.

Proof. For simplicity of notation we consider only the case of the standard filtration. The case of the T -filtration is identical.

Write $\text{gr}(M) = \bigoplus_{i=0}^k \text{gr}(N_i)$, where $\text{gr}(N_i)$ are free R_i -modules generated by b_{ij} , $j \in A_i$. We shall induct on n . If $n = 0$, $M = \widehat{M} = \text{gr}(M) = \text{gr}(N_0)$ is a K -vector space of finite dimension generated by b_1, \dots, b_k . So the three conditions are equivalent. The inductive assumption will be only used for the implication (2) \Rightarrow (1).

(3) \Rightarrow (2). Suppose that $\sum c_{ij} b_{ij} \in M_s$ defines an element of order s . If $\min(\text{ord}(c_{ij} b_{ij})) = s_0 < s$ then

$$\text{ord}\left(\sum_{\text{ord}(c_{ij} b_{ij})=s_0} c_{ij} b_{ij}\right) > s_0$$

and

$$\sum_{\text{ord}(c_{ij} b_{ij})=s_0} \text{in}(c_{ij}) \text{in}(b_{i,j}) = 0,$$

which is impossible since $\text{in}(b_{i,j})$ is a basis of $\text{gr}(N)$. Thus $\text{ord}(c_{ij}) = s - d_{ij}$. In particular if $\sum c_{ij} b_{ij} \in M_\infty := \bigcap_{s=1}^\infty M_s$ then $c_{ij} \in m_i^\infty$. Also, if $\sum c_{ij} b_{ij} = 0$ then $c_{ij} \in m_i^\infty$. This implies that there exists a gradation preserving epimorphism

$$(4.5) \quad \bigoplus m_i^{s-d_{ij}} \cdot b_{ij} \rightarrow M_s \cap \sum \mathcal{E}_i \cdot b_{ij}$$

and its kernel is contained in $\bigoplus m_i^\infty \cdot b_{ij}$.

By the assumption for any $a \in M_s$ one can find $c_{ij} \in m_i^{s-d_{ij}}$ such that $a = \sum c_{ij} b_{ij} \pmod{M_s}$. Then consider $a_1 := a - \sum c_{ij} b_{ij} \in M_{s+1}$ and repeat the procedure. This allows representing any element in

M_s/M_{s+d} as $\sum c_{ij}b_{ij}$, where the classes of the elements $c_{ij} \in m_i^{s-d_{ij}}/m_i^{s+d-d_{ij}}$ are uniquely determined. This implies that

$$(4.6) \quad M/M_s = \left(\bigoplus \mathcal{E}_i b_{ij} + M_s \right) / M_s = \bigoplus (\mathcal{E}_i / m_k^{s-d_{ij}}) \cdot b_{ij}$$

Consequently,

$$(4.7) \quad \widehat{M} = \bigoplus_{i \leq n} \widehat{\mathcal{E}}_i b_{ij}, \quad \widehat{M}_s = \bigoplus_{i \leq n} \widehat{m}_k^{s-d_{ij}} \cdot b_{ij}.$$

(2) \Rightarrow (3). Obvious.

(2) \Rightarrow (1). Let $N := N_n$ be the submodule generated over \mathcal{E}_n by $b_{n,j}$. Then $\overline{M} := M/N$ is an \mathcal{E}_n -module. The graded module $\text{gr}(\overline{M}) = \text{gr}(M/N) = \text{gr}(M)/\text{gr}(N)$ is generated by a basis $\text{in}(b_{ij})$, $i \leq n-1$.

The module M/N is a finitely generated \mathcal{E}_n -module. By the formula (4.6),

$$(4.8) \quad M/(M_s + N) = \left(\bigoplus_{i < n} \mathcal{E}_i b_{ij} + M_s + N \right) / (M_s + N) = \bigoplus_{i < n} (\mathcal{E}_i / m_k^{s-d_{ij}}) \cdot b_{ij}$$

and

$$\widehat{M/\overline{N}} = \bigoplus_{i < n} \widehat{\mathcal{E}}_i b_{ij} = \widehat{M}/\widehat{N}$$

is finitely generated over $\widehat{\mathcal{E}}_{n-1}$ by $(b_{ij})_{i < n}$. By Malgrange division (Corollary 2.6.4), the module M/N is a finitely generated \mathcal{E}_{n-1} -module. By the inductive assumption $\{b_{ij}\}_{i \leq n-1}$ is a Stanley's basis of M/N . In particular, $\bigoplus_{i=0}^{n-1} N_i \rightarrow M/N$ is surjective, as also is $\phi : \bigoplus_{i=0}^n N_i \rightarrow M$. Since $\widehat{\phi} : \bigoplus_{i=0}^n \widehat{N}_i \rightarrow \widehat{M}$ is an isomorphism, the kernel of ϕ is contained in $\bigoplus_{i=0}^n m_i^\infty \cdot N_i$. Moreover, since $M = \sum_{i \in \text{eqn}} \mathcal{E}_i \cdot b_{ij}$ by the formula 4.5, there exists a surjection

$$(4.9) \quad \bigoplus m_i^{s-d_{ij}} \cdot b_{ij} \rightarrow M_s \cap \sum \mathcal{E}_i \cdot b_{ij} = M_s$$

and its kernel is contained in $\bigoplus m_i^\infty \cdot b_{ij}$.

(1) \Rightarrow (3). Let b_{ij} be a basis of M . Then for any $a = \sum c_{ij}b_{ij}$ we get a unique decomposition $\text{in}(a) = \sum \text{in}(c_{ij}) \text{in}(b_{ij})$, where $\text{in}(c_{ij}) \in R_i$ (as before). That is, $(\text{in}(b_{ij}))$ is a basis of $\text{gr}(M)$ over (R_i) . \square

Lemma 4.1.7. *Assume M is a finitely generated graded $R_n = K[x]$ -module. Let $b_{11}, \dots, b_{n,k_n} \in M$ be homogeneous elements in M . Then the following conditions are equivalent:*

- (1) $b_{11}, \dots, b_{n,k_n} \in M$ is a basis of M over $(R_i)_{i=0}^n$.
- (2) $b_{11}, \dots, b_{n,k_n} \in M$ is a basis of $M \otimes_{R_n} R_{n+m}$ over $(R_{i+m})_{i=0}^n$ with respect to any grading T on $M \otimes_{R_n} R_{n+m}$ extending the standard grading.

Moreover condition (1) implies

- (3) $b_{11}, \dots, b_{n,k_n} \in M$ is a basis of $M \otimes_K K'$ over $(R_i \otimes_K K')_{i=0}^n$, where $K' \supset K$ is a commutative ring with 1.

Proof. This follows from the definition of basis.

(1) \Rightarrow (2). If $M = \bigoplus_{i,j} R_i \cdot b_{ij}$ then $M \otimes_{R_n} R_{n+m} = M \otimes_K R_m = \bigoplus_{i,j} R_{i+m} \cdot b_{ij}$.

(2) \Rightarrow (1). If $M \otimes_{R_n} R_{n+m} = M \otimes_K R_m = \bigoplus_{i,j} R_{i+m} \cdot b_{ij}$ with $b_{ij} \in M$ then there is an epimorphism

$$\bigoplus_{i,j} R_i \cdot b_{ij} \rightarrow \bigoplus_{i,j} R_{i+m} / ((x_{n+1}, \dots, x_{n+m}) \cdot b_{ij}) \rightarrow (M \otimes_K R_m) / (x_{n+1}, \dots, x_{n+m}) \simeq M.$$

On the other hand, $\bigoplus_{i,j} R_i \cdot b_{ij} \rightarrow M$ is also a monomorphism since it is a restriction of $\bigoplus_{i,j} R_{i+m} \cdot b_{ij} \rightarrow M \otimes_{R_n} R_{n+m} \supset M$.

(1) \Rightarrow (3). Obvious. \square

Lemma 4.1.8. *Consider the graded R_{n+m} -module $M = R_{n+m} = K[x, y]$ (with the standard gradation). Let $b_{11}, \dots, b_{n,k_n} \in M$ be homogeneous elements in M with $\deg(b_{ij}) = \deg(b_{ij})(x, 0)$. If $b_{11}(x, 0), \dots, b_{n,k_n}(x, 0) \in R_n \subset R_{n+m}$ is a basis of M over $(R_{i+m})_{i=0}^n$ then $b_{11}, \dots, b_{n,k_n} \in M$ is a basis of M over $(R_{i+m})_{i=0}^n$.*

Proof. We apply double induction on m and the grading s . Suppose the theorem is valid for $m - 1$ and any n . Let $b_{ij}^1 \in R^{n+m}$ denote the restrictions of the elements b_{ij} to $R_{n+1} = R_{n+m}/(y_2, \dots, y_m)$.

We show, by induction on the degree s , that (b_{ij}^1) generate M and are independent. Note that $b_{ij}^1(x, 0) = b_{ij}(x, 0)$. Consider any element $a(x, y)$ in the s -grading of R_{n+1} . Then by the assumption we can write $a(x, 0) = \sum c_{ij} b_{ij}(x, 0)$, where $c_{ij} \in R_i = K[x_{n-i+1}, \dots, x_n]$. Consequently, $a - \sum c_{ij} b_{ij}^1 = y_1 a'$. By induction $a' = \sum c'_{ij} b_{ij}^1$, where $c'_{ij} \in R_i[y_1]$ and

$$a = \sum (c_{ij} + y_1 c'_{ij}) b_{ij}^1$$

with $y_1 c'_{ij} \in R_i[y_1]$.

Now suppose $\sum c_{ij} b_{ij}^1 = 0$ with $c_{ij} \in R_i[y_1]$ is a relation of the smallest degree. Taking it modulo y_1 we get $\sum c_{ij}(x, 0) b_{ij}^1(x, 0) = 0$. By the assumption $b_{ij}^1(x, 0) = b_{ij}(x, 0)$ is a basis of R_n . This implies $c_{ij}(x, 0) = 0$ and $c_{ij} = y_1 c'_{ij}$. Consequently, $\sum c'_{ij} b_{ij} = 0$, with the smallest degree of c'_{ij} . We have shown that the restrictions of b_{ij} to R_{n+1} are a basis of R_{n+1} . By the inductive assumption on $m - 1$ (or by repeating the argument $m - 1$ times) the elements b_{ij} form a basis of $R_{n+1+(m-1)} = R_{n+m}$. \square

Lemma 4.1.9. *Consider the \mathcal{E}_{n+m} -module $M = \mathcal{E}_{n+m}(x, y)$ -module with standard filtration $M_i = m_n^j \cdot \mathcal{E}_{n+m}$ over \mathcal{E}_m such that $m_n^j \cdot M_s \subset M_{s+j}$, and each M_s/M_{s+1} is a free $\mathcal{E}_m(y)$ -module. If*

$$\text{in}(b_{11})(x, 0), \dots, \text{in}(b_{n, k_n})(x, 0) \in \text{gr}(M/m_m) = R_n$$

is a basis of $\text{gr}(M/m_m)$ over $(R_i)_{i=0}^n$ then

$$b_{11}, \dots, b_{n, k_n} \in M$$

is a basis of M over $(\mathcal{E}_{i+m})_{i=0}^n$ with respect to a certain filtration M_T on $\mathcal{E}_{n+m}(x, y)$ extending the standard filtration on \mathcal{E}_n .

Moreover, if $\deg(\text{in}(b_{ij})(x, 0)) = \text{ord}(b_{ij})$ then $b_{11}, \dots, b_{n, k_n} \in M$ is a basis of M over $(\mathcal{E}_{i+m})_{i=0}^n$ with respect to the standard filtration on \mathcal{E}_{m+n} .

Proof. There exists a monomial grading $T = T_M$, where

$$T_M(x_1, \dots, x_n, y_1, \dots, y_m) = x_1 + \dots + x_n + M y_1 + \dots + M y_m,$$

for sufficiently large $M \gg 0$ such that $\text{in}_T(b_{ij}) = \text{in}_T(b_{ij})(x, 0) = \text{in}(b_{ij})(x, 0)$. Then $\text{in}_T(b_{ij}) = \text{in}(b_{ij})(x, 0)$ is a basis of R_n over $(R_i)_{i=0}^n$, and, by the previous lemma, it is a basis of R_{n+m} over $(R_{i+m})_{i=0}^n$. Consequently, $\text{in}_T(b_{ij}) = \text{in}_T(b_{ij})(x, 0)$ is a basis of R_{n+m} over $(R_{i+m})_{i=0}^n$, and by Theorem 4.1.6, b_{ij} is a basis of \mathcal{E}_{m+n} with respect to the T -filtration.

For the “moreover” part, observe that if $\text{in}(b_{ij})(x, 0)$ is a basis of R_{n+m} over $(R_{i+m})_{i=0}^m$ then $\text{in}(b_{ij})$ is a basis of R_{n+m} over $(R_{i+m})_{i=0}^m$ by Lemma 4.1.8. Thus, by Theorem 4.1.6, b_{ij} is a basis of \mathcal{E}_{m+n} with respect to the standard filtration. \square

Corollary 4.1.10. *Let M be any filtered \mathcal{E}_n -module, $M_0 \subset M$ an \mathcal{E}_n -module with induced filtration, and $\text{gr}(M)$ the induced graded module with graded submodule $\text{gr}(M_0)$. Assume there exists a basis $(\bar{b}_{ij})_{(i,j) \in S}$ of $\text{gr}(M)$ and a decomposition $S = S_1 \cup S_2$ such that $(\bar{b}_{ij})_{(i,j) \in S_1}$ is a basis of $\text{gr}(M_0)$. Then there exists a basis $(b_{ij})_{(i,j) \in S}$ of M such that:*

- (1) $\text{in}(b_{ij}) = \bar{b}_{ij}$.
- (2) $(b_{ij})_{(i,j) \in S_1}$ is a basis of M_0 .
- (3) $(b_{ij})_{(i,j) \in S_2}$ is a basis of M/M_0 .

Proof. It suffices to find elements $(b_{ij})_{(i,j) \in S}$ in M such that $\text{in}(b_{ij}) = \bar{b}_{ij}$, and $b_{ij} \in M_0$ whenever $(i, j) \in S_1$, and apply Theorem 4.1.6. \square

Definition 4.1.11. Let M be a finite graded module and let a_{ij} be its basis over $(R_i)_{i=0}^n$. We say that a_{ij} *majorizes* (respectively *majorizes up to degree d*) a set of homogeneous elements $b_{ij} \in M$ if there is a bijective correspondence $a_{ij} \leftrightarrow b_{ij}$, preserving degrees $\deg(a_{ij}) = \deg(b_{ij})$ and multiplication rings R_i , such that the elements b_{ij} generate M over $(R_i)_{i=0}^n$ (respectively generate all M_s , where $s \leq d$).

The following result generalizes the stabilization theorem for monotone diagrams [9], which plays a critical role in the Bierstone-Milman approach to desingularization and their use of the Hilbert-Samuel function.

Theorem 4.1.12 (Stabilization for modules). *Let R be a ring of polynomials over a commutative ring K with identity and let $M = \bigoplus_{s \in \mathbb{N}} M_s$ be any finite R -module. Assume M is generated over $(R_i)_{i=0}^n$ by a finite set of homogeneous elements g_{ij} of degrees $d_{ij} \in \mathbb{N}$. Let $d(M) = \max\{d_{ij}\}$. Let $s_{ij'} \in M$ be any set of homogeneous elements of degree d_{ij} which generates M_s , where $s \leq d(M) + 1$, over $(R_i)_{i=0}^n$. Then the elements $s_{ij} \in M$ generate M over $(R_i)_{i=0}^n$.*

Moreover, if g_{ij} is a basis of M over $(R_i)_{i=0}^n$, and it majorizes s_{ij} up to degree $d(M) + 1$, then s_{ij} is a basis of M over $(R_i)_{i=0}^n$.

Proof. We can label the set of generators as $\{g_{ij} \mid j \in J_i, i = 0, \dots, n\}$, where $\{g_{ij} \mid j \in J_i\}$ are multiplied by elements in the rings $R_i = K[x_n, \dots, x_{n-i+1}]$ for $i \geq 1$, and $R_0 = K$.

We use double induction: on the number of variables n and on the degree. If $n = 0$ then M has only finitely many grades as it is generated over K by finitely many elements of degree $\leq d$. For $n \geq 1$ the elements g_{ij} define a generating set for the $K[x_1, \dots, x_{n-1}]$ -module $\overline{M} := M/(x_n \cdot M)$ over the rings

$$\overline{R_{i-1}} = R_i/(x_n) = K[x_{n-1}, \dots, x_{n-i+1}]$$

for $i \geq 2$ and $\overline{R_0} = K$. More precisely, we need to relabel the classes of elements g_{ij} accordingly. We set $\overline{J_0} := J_0 \cup J_1$, and $\overline{g_{0j}} = g_{0j}$ for $j \in J_0$ and $\overline{g_{0j}} = g_{1j}$ for $j \in J_1$. For $i \geq 1$ we set $\overline{g_{ij}} = g_{i-1,j}$ for $j \in \overline{J_i} = J_{i+1}$. It follows that the elements g_{ij} generate \overline{M} over $\overline{R_i}$. By the inductive assumption on the number of variables we conclude that the elements s_{ij} generate \overline{M} over $\overline{R_i}$. Then we show by induction on the degree $s \in \mathbb{N}$ that the elements $s_{ij} \in M$ generate M_s . By the assumption it is true for $s \leq d(M) + 1$. Suppose it is true for all $s' < s$, where $s > d(M) + 1$. We can write any element $a \in M_s$ as

$$a = \sum b_{ij} s_{ij} + x_n a',$$

where $b_{ij} \in (R_i)_{s-d'_{ij}}$, $i = 1, \dots, n$. But $a' \in M_{s-1}$, so by the inductive assumption we can write $a' = \sum b'_{ij} s_{ij}$, where again $b'_{ij} \in (R_i)_{s-1-d'_{ij}}$, and since $s-1-d'_{ij} \geq s-1-d(M) > 0$ the elements b'_{ij} are indexed by subscripts with $i > 0$. (The ring $R_0 = K$ has only gradation 0.) Consequently, $x_n a' = \sum x_n b'_{ij} s_{ij}$, where $x_n b'_{ij} \in (R_i)_{s-d'_{ij}}$ with $i > 0$, which yields

$$a = \sum (b_{ij} - x_n b'_{ij}) s_{ij}.$$

Now if g_{ij} is a basis which majorizes a generating set s_{ij} then by Corollary 4.1.5 each M_s is a free K -module and

$$H_M(s) = \sum_{i,j} \phi(s - d_{ij}, i) = \sum_{i,j} \phi(s - d'_{ij}, i).$$

But M is the image of the direct sum of free R_i -modules $\bigoplus N_i$ generated by elements of degree $d'_{ij} = d_{ij}$.

Since both graded modules are free over K and have the same K -ranks in each gradation then the epimorphism defines an isomorphism. (The relevant square matrix is invertible over K .)

Consequently, s_{ij} is a free basis of M over $(R_i)_{i=0}^n$. \square

Lemma 4.1.13. *Let M be a finitely generated graded R -module (with the standard gradation) and let $\overline{T} : \mathbb{N}^n \rightarrow \mathbb{R}^k$ be a normalized grading. If $A := \{a_{11}, \dots, a_{n,j_n}\}$ is a basis of M over $(R_i)_{i=0}^n$, and $B := \{b_{11}, \dots, b_{n,j_n}\}$ be a set in bijective correspondence with A , such that*

$$\deg(a_{ij}) = \deg(b_{ij}), \quad \deg_{\overline{T}}(a_{ij} - b_{ij}) > \deg_{\overline{T}}(a_{ij})$$

for all i, j then A majorizes B over $(R_i)_{i=0}^n$.

Proof. Consider the grade preserving K -algebra homomorphism

$$\Psi : \sum c_{ij} a_{ij} \rightarrow \sum c_{ij} b_{ij},$$

where $c_{ij} \in R_i$. Then for any x in the s -th gradation we have $\deg_{\overline{T}}(\Psi - I)(x) > \deg_{\overline{T}}(x)$, and $(\Psi - I)^n(x) = 0$ for $n \geq \dim(R)_s$. Then Ψ is a K -linear isomorphism with inverse $I - (\Psi - I) + (\Psi - I)^2 + \dots$ and defines the majorization. \square

Lemma 4.1.14. *Let X be an affine variety over a field K (respectively an analytic space or an open subset in \mathbb{R}^m), and let $\mathcal{O}(X)$ be the ring of regular functions on X . Consider the map evaluation at $x \in X$,*

$$\pi_x : \mathcal{O}_X[x_1, \dots, x_k] \rightarrow K[x_1, \dots, x_n].$$

Let $b_{ij} \in \mathcal{O}_X[x_1, \dots, x_n]$ be homogeneous elements whose evaluations at $x \in X$, $\pi_x(b_{ij}) = \overline{b_{ij}}$, form a basis of $K[x_1, \dots, x_n]$ over $(K[x_i, \dots, x_n])_{i=1}^n$. Then there is an open subset $U \subset X$ such that $b_{ij|U}$ is a basis of $\mathcal{O}(U)[x_1, \dots, x_n]$ over $(\mathcal{O}_U[x_i, \dots, x_n])_{i=1}^n$.

Proof. By Lemma 4.1.7, the set $(\overline{b_{ij}})$ can be thought of as a (Stanley's) basis of $\mathcal{O}_X[x_1, \dots, x_n]$. Then for any $\beta \in \mathbb{N}^i$, and the function

$x^\beta \cdot b_{ij}$, where $\deg(b_{ij}) + |\beta| = s \leq d + 1$, one can write

$$x^\beta \cdot b_{ij} = \sum_{\beta' \in \mathbb{N}^i, \deg(b_{i'j'}) + |\beta'| = s} A_{\beta, ij; \beta' i' j'} \cdot x^{\beta'} \cdot \overline{b_{i'j'}}.$$

Consider the matrix

$$A_s = [A_{\beta, ij; \beta' i' j'}]_{\deg(b_{ij}) + |\beta| = s}.$$

The evaluation $A_s(x)$ at x is the identity. Thus the subset

$$U := \{y \in X \mid \det(A_s)(y) \neq 0 \text{ for } s \leq d + 1\}$$

is a nonempty open subset such that b_{ij} majorizes $b_{ij|U}$ up to degree $d + 1$. Consequently, $b_{ij|U}$ is a basis of $\mathcal{O}(U)[x_1, \dots, x_n]$. \square

4.2. Neighborhood versions of Stanley decomposition.

Lemma 4.2.1. *Let \mathcal{F} be a coherent module of finite type on a regular scheme X over a field K of dimension n , with a coordinate system u_1, \dots, u_n at a K -rational point \bar{x} . Let $\mathcal{O}_{\bar{x}}^h \simeq K\langle u_1, \dots, u_n \rangle$ denote the Henselianization of the local ring $\mathcal{O}_{\bar{x}}$. Suppose the induced $\mathcal{O}_{\bar{x}}^h$ -module*

$$\mathcal{F}_x^h := \mathcal{F}_x \otimes_{\mathcal{O}_{\bar{x}}} \mathcal{O}_{\bar{x}}^h \simeq (\mathcal{O}_{\bar{x}}^h)^d$$

is free and finite. Then there exists an étale neighborhood $X' \rightarrow X$ of \bar{x} preserving the residue field K such that the induced coherent sheaf \mathcal{F} on X is free and finite over $\mathcal{O}_{\bar{x}, X'}$.

Proof. The module \mathcal{F}_x^h is generated by finitely many sections. Passing to an étale neighborhood $X' \rightarrow X$ we can assume that the generators are in $\mathcal{F}(X')$. Then there is a surjective morphism $\phi : \mathcal{O}_{X'}^d \rightarrow \mathcal{F}_x$ of sheaves and the induced epimorphism $\phi_x : \mathcal{O}_{x, X'}^d \rightarrow \mathcal{F}_x$ of stalks. Since the homomorphism $\mathcal{O}_{x, X'}^d \rightarrow (\mathcal{O}_{x, X'}^h)^d$ is injective and $(\mathcal{O}_{x, X'}^h)^d \rightarrow \mathcal{F}_x^h$ is an isomorphism it follows that their composition $\mathcal{O}_{x, X'}^d \rightarrow \mathcal{F}_x^h$ is injective. Thus ϕ_x is also injective and is an isomorphism of stalks. The latter implies that ϕ is an isomorphism of sheaves in a Zariski neighborhood of \bar{x} . \square

Theorem 4.2.2. *Let \mathcal{F} be a coherent module on a smooth scheme X over a field K of dimension n , with a coordinate system u_1, \dots, u_n at a K -rational point \bar{x} defining subschemes $X_i := V(u_1, \dots, u_{n-i+1}) \ni \bar{x}$, $X_0 = \{\bar{x}\} \subset X_1 \subset \dots \subset X_n = X$. Denote by $\mathcal{O}_{\bar{x}, X}$ the local sheaf at \bar{x} and by $\mathcal{O}_{\bar{x}, X}^h = K\langle u_1, \dots, u_n \rangle$ its Henselianization. Suppose there exists a Stanley basis $b_{ij} \in \mathcal{F}_x$ of $\mathcal{F}_x^h := \mathcal{F}_x \otimes_{\mathcal{O}_{\bar{x}, X}} \mathcal{O}_{\bar{x}, X}^h$ over $\mathcal{O}_{\bar{x}, X}^h$.*

Then there exist an étale neighborhood $X' \rightarrow X$ of \bar{x} preserving the residue field K with induced subschemes $X'_0 \subset X'_1 \subset \dots \subset X'_n = X'$ with $\mathcal{O}(X'_i) \subset K\langle u_{n-i+1}, \dots, u_n \rangle$ and (smooth) projections $\pi_i : X' \rightarrow X'_i$ such that

$$\mathcal{F}(X') = \bigoplus_{ij} \mathcal{O}(X'_i) b_{ij}.$$

Proof. The elements b_{nj} are defined in a certain affine étale neighborhood X' of X , and they generate a coherent free submodule \mathcal{G} for “sufficiently small” X' , that is

$$\mathcal{G}(X') = \bigoplus \mathcal{O}(X') b_{ni}.$$

Then $\mathcal{F}_{n-1} := \mathcal{F}/\mathcal{G}$ is a coherent $\mathcal{O}_{X'}$ -module. By Theorem 1.1.7, one can modify X' so that there exists the natural projection $\pi_{n-1} : X' \rightarrow X'_{n-1}$ on the subscheme X'_{n-1} (defined by $u_1 = 0$) induced by the inclusions $\mathcal{O}(X'_{n-1}) \subset K\langle u_2, \dots, u_n \rangle \subset K\langle u_1, \dots, u_n \rangle$. Then

$$\begin{aligned} (\mathcal{F}_{n-1}^h)_{\bar{x}} / ((u_2, \dots, u_n) \cdot (\mathcal{F}_{n-1})_{\bar{x}}^h) &= (\mathcal{F}_{n-1})_{\bar{x}} \otimes_{\mathcal{O}_{\bar{x}, X}} (\mathcal{O}_{\bar{x}, X}^h / ((u_2, \dots, u_n) \cdot \mathcal{O}_{\bar{x}, X}^h)) \\ &= (\mathcal{F}_{n-1})_{\bar{x}} \otimes_{\mathcal{O}_{\bar{x}, X}} (\mathcal{O}_{\bar{x}, X} / ((u_2, \dots, u_n) \cdot \mathcal{O}_{\bar{x}, X})) \\ &= (\mathcal{F}_{n-1})_{\bar{x}} / ((u_2, \dots, u_n) \cdot (\mathcal{F}_{n-1})_{\bar{x}}) = \mathcal{F}_{n-1} / \pi_{n-1}^*(m_{\bar{x}, X'_{n-1}}) \end{aligned}$$

is a finitely generated vector space over K . Thus, by Theorem 1.1.5, after possibly passing to an étale neighborhood of X' , the sheaf $\overline{\mathcal{F}}_{n-1} := \pi_{1*}(\mathcal{F}_{n-1})|_{X'_{n-1}}$ is coherent on X'_{n-1} with

$$\overline{\mathcal{F}}_{n-1}(X'_{n-1}) = \mathcal{F}_{n-1}(X') = \mathcal{F}(X')/G(X').$$

Then by the inductive assumption applied to $\overline{\mathcal{F}}_{n-1}$, after passing to an affine étale neighborhood $X''_{n-1} \rightarrow X'_{n-1}$, and the induced étale neighborhood $X'' \rightarrow X'$, we have the natural isomorphism of K -vector spaces

$$\phi' : \bigoplus_{i=0}^{n-1} \bigoplus_j \mathcal{O}(X''_i) b_{ij} \rightarrow \overline{\mathcal{F}}_{n-1}(X''_{n-1}) = \mathcal{F}_{n-1}(X'') = \mathcal{F}(X'')/G(X''),$$

which implies the existence of a natural isomorphism of K -vector spaces

$$\phi : \mathcal{G}(X'') \oplus \bigoplus_{i=0}^{n-1} \bigoplus_j \mathcal{O}(X'_i) b_{ij} \simeq \bigoplus_{i=0}^n \bigoplus_j \mathcal{O}(X''_i) b_{ij} \rightarrow \mathcal{F}(X'').$$

The fact that ϕ is surjective follows from surjectivity of ϕ' , while injectivity follows from injectivity of $\phi|_{\mathcal{G}(X'')} : \mathcal{G}(X'') \rightarrow \mathcal{F}(X'')$. \square

A similar result is valid in the holomorphic setting (with the same proof).

Theorem 4.2.3. *Let \mathcal{F} be a coherent module on a domain $U = U_1 \times \dots \times U_n$ in \mathbb{C}^n containing 0, where $U_i \subset \mathbb{C}$. Consider the natural projection $\overline{\pi}_i : U \rightarrow \overline{U}_i := U_i \times \dots \times U_n$. If the stalk \mathcal{F}_0 admits a Stanley basis b_{ij} then there exist neighborhoods $V_i \subset U_i$ containing 0 such that for $\overline{V}_i := V_i \times \dots \times V_n$, $i = 1, \dots, n$, $\overline{V}_0 = \{0\}$ there is an isomorphism*

$$\mathcal{F}(\overline{V}_n) = \bigoplus_{i=0}^n \bigoplus_j \mathcal{O}(\overline{V}_i) b_{ij}.$$

Proof. As before, we find a neighborhood $V_1 \times \overline{V}_{n-1}$ with the projection $\pi_{n-1} : V_1 \times \overline{V}_{n-1} \rightarrow \overline{V}_{n-1}$ such that the submodule generated by b_{nj} is free, and

$$\mathcal{G}(V_1 \times \overline{V}_{n-1}) = \bigoplus \mathcal{O}(V_1 \times \overline{V}_{n-1}) b_{ni}$$

and the induced sheaf $\mathcal{F}_{n-1} = \mathcal{F}/\mathcal{G}$ is coherent, with the finite-dimensional vector space $\mathcal{F}_{n-1, \overline{x}} / ((u_2, \dots, u_n) \cdot \mathcal{F}_{n-1, \overline{x}})$. Moreover by using Theorem 1.2.2, we can assume that the direct image $\overline{\mathcal{F}}_{n-1} := \pi_{n-1*}(\mathcal{F}_{n-1})$ of \mathcal{F}_{n-1} is coherent on \overline{V}_{n-1} , and then use the inductive assumption. Note that by the necessary modification of \overline{V}_{n-1} does not affect the previously constructed isomorphisms. \square

The following lemma is an immediate consequence of the definition.

Lemma 4.2.4. *Let \mathcal{F} be a module of finite type on a domain U containing x . If there exists a quasi-isomorphism $\mathcal{O}_x^d \rightarrow \mathcal{F}_x$ then there exists an open neighborhood $U' \subset U$ and a surjection of $\mathcal{O}^d(U)$ -modules $\mathcal{O}^d(U) \rightarrow \mathcal{F}(U)$ with kernel contained in $m_x^\infty \cdot \mathcal{O}^d(U)$.*

In the differential setting one can prove a weaker result.

Theorem 4.2.5. *Let \mathcal{F} be a module of finite type on a domain $U = U_1 \times \dots \times U_n$ in \mathbb{R}^n containing 0, where $U_i \subset \mathbb{R}$. Consider the natural projection $\overline{\pi}_i : U \rightarrow \overline{U}_i := U_i \times \dots \times U_n$. If \mathcal{F}_x admits a Stanley basis b_{ij} then there exist neighborhoods $V_i \subset U_i$ containing 0 such that for $\overline{V}_i := V_i \times \dots \times V_n$, $i = 1, \dots, n$, $\overline{V}_0 = \{0\}$ there is an epimorphism*

$$\bigoplus_{i=0}^n \bigoplus_j \mathcal{O}(\overline{V}_i) \cdot b_{ij} \rightarrow \mathcal{F}(\overline{V}_n)$$

with kernel contained in $\bigoplus_{i=0}^n m_{0, V_i}^\infty \cdot \mathcal{O}(\overline{V}_i) b_{ij}$.

Proof. The only difference here is that we consider the sheaf \mathcal{F} of finite type over the sheaf of differentiable functions. We use the previous lemma to get an epimorphism

$$\bigoplus \mathcal{O}(V_1 \times \overline{V}_{n-1}) \cdot b_{ni} \rightarrow \mathcal{G}(V_1 \times \overline{V}_{n-1})$$

with kernel in $\bigoplus m_{0, V}^\infty \cdot \mathcal{O}(V_1 \times \overline{V}_{n-1}) \cdot b_{ni}$. By Lemma 2.3.2, the sheaf $\mathcal{F}_{n-1} = \mathcal{F}/\mathcal{G}$ is of finite type. Moreover it has the finite-dimensional vector space $\mathcal{F}_{n-1, \overline{x}} / (u_2, \dots, u_n)$. Thus, by using Corollary 2.4.3, we

can assume that the direct image $\overline{\mathcal{F}}_{n-1} := \pi_{n-1}^*(\mathcal{F}_{n-1})$ is of finite type on \overline{V}_{n-1} , and then use the inductive assumption as in the algebraic or analytic cases. \square

4.3. Diagrams associated with functions. Let \overline{T} be any normalized linear grading. Let $f_{ij}(u, v) \in \mathcal{E}_{n+m}$ be a finite set of functions such that the initial exponents

$$\alpha_1 = \exp_{\overline{T}}(f_1(u, 0)) <_r \alpha_{2,1} = \exp(f_{2,1}(u, 0)) < \dots$$

define a diagram of initial exponents $\Delta \subset \mathbb{N}^n$ of finite type and its decomposition into disjoint subsets Δ_{ij} . For simplicity we assume that the coefficients $c_{\alpha_{ij}}(f_{ij})$ of f_{ij} are all equal to one.

By the above any $\beta \in \overline{B}_i = \bigcup \overline{B}_{i,j}$ can be written as $\beta = \alpha_{ij} + \gamma$, where $\gamma \in \overline{\Gamma}_{i+1,j}$ so we set

$$f_\beta := u^\gamma f_{i,j}, \quad F_\beta := \text{in}(f_\beta).$$

By Corollary 3.4.5 the set

$$S_1 := \left\{ x^\beta \mid \beta \in \bigcup_{i=1}^{n+1} (A_i \cup \overline{B}_{i-1}) \right\}$$

is a basis over $(R_i)_{i=0}^n$ of the graded R_n -module R_n .

Since $\exp(F_\beta) = \beta$, by Lemma 4.1.13, the set S_1 majorizes the set

$$S_2 := \left\{ x^\beta \mid \beta \in \bigcup_{i=1}^{n+1} A_i \right\} \cup \left\{ F_\beta \mid \beta \in \bigcup_{i=0}^n \overline{B}_i \right\},$$

which is thus a basis of R_n . Consequently, by Theorem 3.4.5,

$$\left\{ x^\beta \mid \beta \in \bigcup A_i \right\} \cup \left\{ f_\beta \mid \beta \in \bigcup_{i=1}^n \overline{B}_i \right\}$$

is a (Stanley's) basis of \mathcal{E}_n

over $(\mathcal{E}_i)_{i=0}^n$.

This leads to the following

Theorem 4.3.1. *There is a quasi-isomorphism over $(\mathcal{E}_n, \dots, \mathcal{E}_1)$ (preserving filtration by the powers (m_n^k))*

$$\phi : \bigoplus_{j=1}^n \bigoplus_{\beta \in \overline{B}_j} \mathcal{E}_{n-j+1} \cdot f_\beta \oplus \bigoplus_{j=1}^n \bigoplus_{\alpha \in A_j} \mathcal{E}_{n-j} \cdot x^\alpha \rightarrow \mathcal{E}_n.$$

Set

$$\mathcal{E}_n(\Gamma) := \{f \in \mathcal{E}_n \mid \text{supd}(f) \subset \Gamma\}.$$

The following theorem extends the original *Hironaka Henselian division theorem* [38] for algebraic functions to any smooth category.

Theorem 4.3.2 (Hironaka Henselian division theorem). *Let $\mathcal{I} \subset \mathcal{E}_n$ be an ideal generated by the functions f_{ij} . There exists an epimorphism*

$$\mathcal{E}_n(\Gamma) = \bigoplus_{j=1}^n \bigoplus_{\alpha \in A_j} \mathcal{E}_{n-j} \cdot x^\alpha \rightarrow \mathcal{E}_n/\mathcal{I}.$$

Proof. By the above there are epimorphisms

$$\mathcal{E}_n(\Gamma) = \bigoplus_{j=1}^n \bigoplus_{\alpha \in A_j} \mathcal{E}_{n-j} \cdot x^\alpha \rightarrow \mathcal{E}_n/M(\mathcal{I}) \rightarrow \mathcal{E}_n/\mathcal{I},$$

where $M(\mathcal{I}) := \bigoplus_{j=1}^n \bigoplus_{\beta \in \overline{B}_j} \mathcal{E}_{n-j+1} \cdot f_\beta \subset \mathcal{I}$. \square

Remark. The subspace $M(\mathcal{I}) \subsetneq \mathcal{I}$ is usually much smaller than \mathcal{I} .

Lemma 4.3.3. *Let $I \subset K[x_1, \dots, x_n]$ be a homogeneous ideal, where K is a field, and such that the diagram $\Delta := \exp_{\overline{T}}(\mathcal{I})$ is of finite type. Consider any set of homogeneous elements F_i in I such that $\exp(F_i) = \alpha_i$ are vertices of Δ . Then there is a decomposition*

$$\bigoplus_{j=1}^n \bigoplus_{\beta \in \overline{B}_j} R_{n-j+1} \cdot \overline{F}_{n,\beta} \oplus \bigoplus_{j=1}^n \bigoplus_{\alpha \in A_j} R_{n-j} \cdot x^\alpha = R_n,$$

where

$$I = \bigoplus_{j=1}^n \bigoplus_{\beta \in \overline{B}_j} R_{n-j+1} \cdot \overline{F}_{n,\beta}.$$

Theorem 4.3.4 (Existence of a filtered Stanley decomposition). *If K is an infinite field that any finite graded $K[x_1, \dots, x_n]$ -module M has a filtered Stanley decomposition over $(K[x_i, \dots, x_n])_{i=0}^{n-1}$ (possibly after a generic linear change of coordinates). That is, M can be written as (a vector space)*

$$M = \bigoplus_{i=0}^n N_i,$$

where $N_i \subset M$ are free finite graded R_i -submodules. If K is a finite field, such a decomposition exists over a certain finite extension of K .

Proof. Write M as the quotient $M = (R_n)^k / N$, where $N = N_k \subset (R_n)^k$ is a submodule. We prove the theorem by induction on k . Consider the projection $\pi_k : (R_n)^k \rightarrow R_n$ to the last coordinate, let $\pi_k(N_k) = \mathcal{I}_k \subset R_n$, and let Δ_k be a monotone diagram defined for \mathcal{I}_k and generic coordinates. Then there exists a standard basis $\pi_k(F_{k,\beta}) = \overline{F}_{k,\beta} \in \mathcal{I}_k$ of \mathcal{I}_k , as in Lemma 4.3.3, such that

$$\phi_k : \bigoplus_{j=1}^n \bigoplus_{\beta \in \overline{B}_j} R_{n-j+1} \cdot \overline{F}_{k,\beta} \oplus \bigoplus_{j=1}^n \bigoplus_{\alpha \in A_j} R_{n-j+1} \cdot x^\alpha \rightarrow R_n$$

is an isomorphism.

Set $\overline{B}_{k,j} := \overline{B}_j$, and define inductively an isomorphism

$$\text{id}_{k-1} \oplus \phi_k : R_n^{k-1} \oplus \bigoplus_{j=1}^n \bigoplus_{\beta \in \overline{B}_{k,j}} R_{k-j+1} \cdot F_{k,\beta} \oplus \bigoplus_{j=1}^n \bigoplus_{\alpha \in A_{k,j}} R_{k-j} \cdot (0, \dots, 0, x^\alpha) \rightarrow R_n^k.$$

Now consider the module

$N_{k-1} = (R_n)^{k-1} \cap N_k$ and repeat the procedure by induction. We eventually get an isomorphism

$$\phi := \phi_1 \oplus \dots \oplus \phi_n : \bigoplus_{s=1}^k \bigoplus_{j=1}^n \bigoplus_{\beta \in \overline{B}_{s,j}} R_{n-j+1} \cdot F_{s,\beta} \oplus \bigoplus_{j=1}^n \bigoplus_{\alpha \in A_{s,j}} R_{n-j} \cdot (0, \dots, x^\alpha(s), 0, \dots, 0) \rightarrow R_n^k$$

for the relevant $\overline{B}_{s,j}$ occurring in the process. The latter induces an isomorphism

$$\overline{\phi} : \bigoplus_{s=1}^n \bigoplus_{j=1}^n \bigoplus_{\alpha \in A_{s,j}} R_{n-j} \cdot (0, \dots, x^\alpha(s), 0, \dots, 0) \rightarrow R_n^k / \phi \left(\bigoplus_{s=1}^n \bigoplus_j \bigoplus_{\beta \in \overline{B}_{s,j}} R_{n-j} \cdot F_{s,\beta} \right) \rightarrow R_n^k / N = M.$$

Setting $N^j := \bigoplus_{s=1}^n \bigoplus_{\alpha \in A_{s,j}} R_{n-j} \cdot (0, \dots, x^\alpha(s), 0, \dots, 0)$ we get an isomorphism $\overline{\phi} : \bigoplus N^j \rightarrow M$. \square

Theorem 4.3.5 (Existence of a filtered Stanley decomposition 2). *Let \mathcal{E}_n be a smooth category over an infinite field K . For any finite filtered \mathcal{E}_n -module M , with filtration (M_i) satisfying $m_n^i \cdot M_j \subset M_{i+j}$, there exists (after a generic linear change of coordinates in \mathcal{E}_n) a filtered Stanley decomposition, that is, there exist free finite \mathcal{E}_i -modules $N^i = \mathcal{E}_i^{k_i}$ for $0 \leq i \leq n$, and a quasi-isomorphism over $(\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$*

$$\overline{\phi} : \bigoplus_{j=0}^n N^j \rightarrow M.$$

In particular, if the category (\mathcal{E}_n) is reduced then any finite \mathcal{E}_n -module M admits a decomposition

$$M = \bigoplus N^i$$

into a sum of free finite \mathcal{E}_i -modules $N^i = \mathcal{E}_i^{k_i}$ for $0 \leq i \leq n$. If K is finite then such a decomposition exists after passing to a finite extension K' of K .

Proof. As before, one can write M as the quotient module $M = \mathcal{E}_n^k/M_0$. By the proof of Theorem 4.3.5 there exists a basis $(\bar{b}_{ij})_{(i,j) \in S}$ of the graded module $R_n^k = \text{gr}(\mathcal{E}_n^k)$, whose part (for $S_1 \subset S$) is a basis $(\bar{b}_{ij})_{(i,j) \in S_1}$ of the submodule $\text{gr}(M_0)$, and $(\bar{b}_{ij})_{(i,j) \in S \setminus S_1}$ defines a basis of $\text{gr}(M) = \text{gr}(\mathcal{E}_n^k)/\text{gr}(M_0)$. Then, by Lemma 4.1.10, there is a basis $(b_{ij})_{(i,j) \in S}$ of \mathcal{E}_n^k such that $(b_{ij})_{(i,j) \in S_1}$ is a basis of M_0 and $(b_{ij})_{(i,j) \in S \setminus S_1}$ defines a basis of $M = \mathcal{E}_n^k/M_0$. We proceed by induction. \square

5. IMPLICIT FUNCTION AND WEIERSTRASS-HIRONAKA DIVISION

5.1. Generalized Weierstrass-Hironaka division and preparation. Consider any normalized grading \bar{T} on \mathbb{N}^n . Let $f_{ij}(u, v) \in \mathcal{E}_{n+m}$ be a finite set of functions such that the initial exponents (with respect to \bar{T}) are defined and generate a diagram Δ of finite type. Without loss of generality we assume that $\alpha_{i,j} = \exp(f_{i,j}(u, 0)) \in \mathbb{N}^i \setminus \mathbb{N}^{i-1}$ are ordered with respect to the reverse lexicographic order

$$\alpha_1 <_r \alpha_{2,1} <_r \dots <_r \alpha_{2,k_2} <_r \alpha_{3,1} <_r \dots$$

and define a decomposition of $\Delta \subset \mathbb{N}^n$ into disjoint subsets $\Delta_{ij} = \alpha_{ij} + \Gamma_{ij}$. By Corollary 3.4.4:

- (1) Each Δ_{ij} can be written as $\Delta_{ij} = \bar{B}_{ij} + \mathbb{N}^{*n-i,n}$, where \bar{B}_{ij} is a finite subset of $\mathbb{N}^i \setminus \mathbb{N}^{i-1}$ and $\mathbb{N}^{*n-i,n} = \{(0, \dots, 0, x_{i+1}, \dots, x_n \mid x_j \in \mathbb{N}\}$.
- (2) (Hironaka) The set $\Gamma = \mathbb{N}^n \setminus \Delta$ decomposes uniquely as the union of the sets $A_i \times \mathbb{N}^{n-i}$, where all subsets $A_i \subset \mathbb{N}^i \setminus \mathbb{N}^{i-1}$ are finite.

Theorem 5.1.1 (Generalized Weierstrass-Hironaka division theorem). *For any $g \in \mathcal{E}_{n+m}$, there exist $h_{ij} \in \mathcal{E}_{n+m}$ and $r(g) = h_{00} \in \mathcal{E}_{n+m}$ such that*

$$g = \sum h_{ij} f_{ij} + r(g)$$

where $\text{supd}(h_{ij}) \subset \Gamma_{ij} \times \mathbb{N}^m$, $\text{supd}(r(g)) \subset \Gamma \times \mathbb{N}^m$.

Moreover:

- (1) If (\mathcal{E}_n) is reduced the decomposition is unique.
- (2) If $\text{ord}(f_{ij}) = |\alpha_{ij}|$ then $\text{ord}(h_{ij}) \geq \text{ord}(g) - |\alpha_{ij}|$.
- (3) If \bar{T} is total then $\exp(h_{ij}(u, 0)) + \alpha_{ij} \geq \exp(g(u, 0))$.
- (4) $\text{ord}(h_{ij}(u, 0)) \geq \text{ord}(g(u, 0)) - |\alpha_{ij}|$.

Proof. This follows from Theorem 4.3.1. Uniqueness in (1) follows from uniqueness of formal division (Theorem 3.1.9), and the fact that there is a monomorphism $\mathcal{E}_n \rightarrow K[[u, v]]$. \square

Remark. More precisely, one can describe the coefficients in the division theorems as

$$h_{ij} = \sum_{\beta \in \bar{B}_{ij}} c_\beta \cdot u^\beta, \quad r(g) = \sum_{j=1}^n \sum_{\alpha \in A_j} c_\alpha \cdot u^\alpha,$$

and where $c_\beta(u_i, \dots, u_n, v) \in \mathcal{E}_{n-i}$ and $c_\alpha \in \mathcal{E}_{n-j}(u_{j+1}, \dots, u_n, v)$,

Theorem 5.1.2 (Generalized Weierstrass-Hironaka division theorem 2). *Let X be a smooth scheme of finite type over a field K (or a \mathbb{C} -analytic/differentiable manifold) of dimension $n + m$ with a given coordinate system at a K -rational point $x \in X$. Let $f_{ij}(u, v) \in \mathcal{O}(X)$ be a finite set of functions such that the initial exponents (with respect to a certain normalized grading \bar{T}) form a diagram Δ of finite type at $x \in V$. Then there exists an étale neighborhood $X' \rightarrow X$ preserving the residue field K (respectively an open neighborhood) such that for any $g \in \mathcal{O}(X')$, there exist $h_{ij}, r(g) \in \mathcal{O}(X')$ such that $\text{supd}(h_{i,j}) \subset \Gamma_{i,j} \times \mathbb{N}^m$, $\text{supd}(r(g)) \subset \Gamma \times \mathbb{N}^m$, and*

$$g = \sum h_{ij} f_{ij} + r(g).$$

Moreover this presentation is unique in the algebraic and the analytic situations, and conditions (1) through (4) of the previous theorem are satisfied at x .

Proof. For existence we apply Theorem 4.3.1 together with Theorems 4.2.2, 4.2.3, 4.2.5 respectively. Uniqueness follows from uniqueness of extensions in the algebraic and analytic situations, and uniqueness of division in the local ring. \square

Theorem 5.1.3 (Generalized preparation theorem). *Let \bar{T} be any normalized order on \mathbb{N}^n . Let $\{f_1(u, v), \dots, f_k(u, v)\}$ be a finite set of functions in \mathcal{E}_{n+m} for which the initial exponents with respect to \bar{T} exist and $\exp(f_i(u, 0)) = \alpha^i$ are the vertices of a diagram Δ of finite type in \mathbb{N}^n . Then there is a set of generators of the form $\bar{f}_i := u^{\alpha^i} + r_i$ of the ideal (f_1, \dots, f_k) such that $\exp(\bar{f}_i(u, 0)) = \alpha^i$, and $\text{supd}(r_i(u, 0))$ is contained in $\Gamma \times \mathbb{N}^n$. Moreover each \bar{f}_i can be written as a finite sum*

$$(5.1) \quad \bar{f}_i = u^{\alpha^i} + \sum_{j=1}^n \sum_{\alpha \in A_j} c_\alpha \cdot u^\alpha,$$

where $c_\alpha \in \mathcal{E}_{n+m-j}(u_{j+1}, \dots, u_n, v)$ for $j = 1, \dots, n$. Furthermore:

- (1) If (\mathcal{E}_n) is reduced the decomposition is unique.
- (2) If $\text{ord}(f_i) = |\alpha_i|$ then $\text{ord}(r_i) \geq |\alpha_i|$.
- (3) $\exp(\bar{f}_i(u, 0)) = \alpha_{ij}$.

Proof. We construct \bar{f}_i and show by induction that all the sets

$$\bar{f}_1, \dots, \bar{f}_{i-1}, \bar{f}_i, f_{i+1}, \dots, f_k$$

generate the same ideal. This is true for $i = 0$. Suppose it is valid for $i - 1$. Consider the division with remainder of u^{α^i} by $\bar{f}_1, \dots, \bar{f}_{i-1}, f_i, f_{i+1}, \dots, f_k$:

$$u^{\alpha^i} = \sum_{j=1}^{i-1} h_j \bar{f}_j + \sum_{j=i}^k h_j f_j + r(u^{\alpha^i}),$$

where $\text{supd}(r(u^{\alpha^i}(u, 0))) \subset \Gamma$. Then set

$$\bar{f}_i := u^{\alpha^i} - r(u^{\alpha^i}) = \sum_{j=1}^{i-1} h_j \bar{f}_j + \sum_{j=i}^k h_j f_j.$$

Note that $\exp(h_j \bar{f}_j(u, 0))$, and $\exp(h_j f_j(u, 0))$ are in Δ_j , and thus all are distinct for distinct j . We have

$$\alpha_i = \exp(\bar{f}_i(u, 0)) = \min_j \exp(h_j f_j(u, 0)) = \exp(h_i f_i(u, 0)) = \exp(h_i(u, 0)) + \exp(f_i(u, 0)).$$

Consequently, $\exp(h_i(u, 0)) = 0$, and the functions $h_i(u, 0)$ and $h_i(u, v)$ are invertible, and thus the ideals generated by $\bar{f}_1, \dots, \bar{f}_{i-1}, \bar{f}_i, f_{i+1}, \dots, f_k$ and $\bar{f}_1, \dots, \bar{f}_{i-1}, f_i, \dots, f_k$ are the same. The other properties follow from the previous theorem. \square

Theorem 5.1.3 generalizes the Malgrange- Weierstrass preparation theorem for a single variable.

We can consider another particularly simple situation of the above theorem.

Corollary 5.1.4. *Let \bar{T} be any monomial order on \mathbb{N}^n . Let $f_1, \dots, f_n \in \mathcal{E}_{n+m}$ be a set of functions for which $\exp(f_i(u, 0)) = k_i \cdot e_i$, where $i = 1, \dots, n$, $k_i \in \mathbb{N}$, $\{e_1, \dots, e_n\}$ is the standard basis of \mathbb{N}^n . Then:*

- (1) *There exists a set of generators \bar{f}_i of the ideal (f_1, \dots, f_k) of the form*

$$(5.2) \quad \bar{f}_i := u_i^{k_i} + \sum_{\alpha_i < k_i} c_\alpha(v) \cdot u_1^{\alpha_1} \cdot \dots \cdot u_k^{\alpha_k},$$

where $c_\alpha \in \mathcal{E}_m(v)$.

- (2) *For any $g \in \mathcal{E}_n$ there exist $h_i \in \mathcal{E}_n$ and $r(g) \in \mathcal{E}_n$ such that:*
 - (a) $g = \sum h_{ij} f_{ij} + r(g)$.
 - (b) $r(g) = \sum_{\alpha_i < k_i} c_\alpha(v) \cdot u_1^{\alpha_1} \cdot \dots \cdot u_k^{\alpha_k}$.
 - (c) $h_i = \sum_{\alpha_i < k_i} c_{i\alpha}(u_{i+1}, \dots, u_k, v) \cdot u_1^{\alpha_1} \cdot \dots \cdot u_i^{\alpha_i}$.

Proof. In that case $\Gamma = A_n = [0, k_1 - 1] \times \dots \times [0, k_n - 1]$ is finite and $c_\alpha \in \mathcal{E}_m(v)$, and we apply the generalized preparation and division theorems. \square

5.2. Hironaka standard basis for algebraic, analytic and smooth functions. The following theorem extends existence of the Hironaka standard basis theorem for formal analytic functions [9]. We note that Henselian Hironaka-Weierstrass division in [38] gives, in general, no control on the multiplicities of the remainders. On the other hand the Hironaka standard basis in the analytic case is convergent (see [38], [8]).

Theorem 5.2.1 (Existence of a standard basis). *Assume (\mathcal{E}_n) is a smooth category over an infinite field K , and let $\mathcal{I} \subset \mathcal{E}_n$ be any ideal. Consider a monotone grading \bar{T} on \mathbb{N}^n , and let $\Delta = \Delta(\mathcal{I}) = \{\exp_{\bar{T}}(f) \mid f \in \mathcal{I}\}$ be the monotone diagram of initial exponents defined for a generic coordinate system. Let $\alpha_1, \dots, \alpha_k$ be the set of vertices of Δ ordered by using the reverse lexicographic order. Then there exists a standard basis of \mathcal{I} with respect to \bar{T} , that is, a set of functions $f_i := u^{\alpha^i} + r_i \in \mathcal{I}$, where $\text{supd}(r_i)$ is contained in Γ for $i = 1, \dots, k$, with $\exp(f_i) = \alpha_i$ such that:*

- (1) $\text{mon}(f_i) := u^{\alpha_i}$ and $\text{ord}(f_i) = |\alpha_i|$.
- (2) The elements $\text{in}(f_i)$ generate $\text{in}(\mathcal{I})$.
- (3) Any function $f \in \mathcal{I}$ can be represented as $f = \sum h_i f_i = r(f)$, where $r(f) \in m_n^\infty$, $\text{supd}(r(f)) \subset \Gamma$, $\text{supd}(h_i) \subset \Gamma_i$, and the functions $h_i \in \mathcal{E}_n$ satisfy the conditions as in Theorem 5.1.1, and are uniquely defined modulo m_n^∞ .
- (4) In the reduced category (of algebraic and analytic functions) the ideal m_n^∞ is 0 and condition (3) can be stated as the equality $f = \sum h_i f_i$. In particular \mathcal{I} is generated by f_i . Moreover the standard basis is uniquely determined by f_i .

Proof. Let \bar{f}_i be any functions with $\exp(\bar{f}_i) = \alpha_i$. Write $u^{\alpha^i} = \sum h_i \bar{f}_i + r_i$. Then $f_i := u^{\alpha^i} + r_i$ satisfies $\exp(f_i) = \alpha_i$. This shows that the elements $\exp(f_i)$ generate $\Delta(\mathcal{I})$, and the $\text{in}(f_i)$ generate $\text{in}(\mathcal{I})$. If $f \in \mathcal{I}$ then by the division theorem we can write $f = \sum h_i f_i + r(f)$, where $f \in \mathcal{I}$, $\text{supd}(r(f)) \subset \Gamma$. Then we get $\exp(r(f)) \subset \Gamma \cap \Delta = \emptyset$ and $r(f) \notin m^\infty$. This argument also implies uniqueness of the standard basis in the reduced category. \square

In a nonreduced category the standard basis need not generate the ideal \mathcal{I} . One can remedy this under stronger assumptions by using the preparation theorem.

Theorem 5.2.2. *Let Δ be a monotone diagram of the initial exponents with respect to a certain total normalized monomial grading \bar{T} . Consider the ideal \mathcal{I} generated by functions f_i such that the elements $\alpha_i := \exp(f_i)$ are the vertices of Δ . Then there exists a standard basis f_i satisfying conditions (1)–(3) and generating \mathcal{I} .*

Proof. The proof and construction of f_i are the same. To show that the elements f_i generate \mathcal{I} we apply the preparation theorem. \square

Theorem 5.2.3 (Existence of a standard basis 2). *Let X be a regular scheme of finite type over a field K (or a complex analytic/differentiable manifold) with a given coordinate system, and $x \in X$ be a K -rational point. Let \mathcal{I} be a sheaf of ideals on X of finite type. Consider a monotone grading \bar{T} on \mathbb{N}^n , and let $\Delta = \Delta(\mathcal{I}) = \{\exp_{\bar{T}}(f) \mid f \in \mathcal{I}\}$ be the diagram of initial exponents (which is monotone and thus of finite type for a generic coordinate system). Let $\alpha_1, \dots, \alpha_k$ be the set of vertices of Δ ordered according to the reverse lexicographic order. Then there is an étale neighborhood $X' \rightarrow X$ of x preserving the residue field at $x \in X$ (respectively an open neighborhood) and a standard basis of \mathcal{I} on X' , which is a uniquely determined set of functions $f_1, \dots, f_k \in \mathcal{I}(X')$, of the form*

$$f_i := u^{\alpha^i} + r_i \in \mathcal{I},$$

where $\text{supd}(r_i) \subset \Gamma$ for $i = 1, \dots, k$ and $\text{ord}_x(f_i) = |\alpha_i|$. Moreover (in the algebraic and analytic setting) any function $f \in \mathcal{I}(X')$ can be uniquely represented as $f = \sum h_i f_i$, where $h_i \in \mathcal{O}(X')$ with $\text{supd}(h_i) \subset \Gamma_i$. (In the differential category the function $f \in \mathcal{I}$ can be uniquely represented as $f \equiv \sum h_i f_i \pmod{m_n^\infty}$.)

Proof. Follows from Theorems 4.2.2, 4.2.3, 4.2.5 and 5.2.1. \square

5.3. Implicit function theorems. Let $\Delta \subset \mathbb{N}^n$ be a diagram of finite type with vertices α_i ordered reverse-lexicographically as in Section 3.4.

Consider the finite decomposition 3.2 after Corollary 3.4.4, and write

$$\Gamma = \bigcup_{i=1}^n A_i \times \mathbb{N}^{n-i}, \quad \Delta = \bigcup_{i=1}^n \bar{B}_i + \mathbb{N}^{*n-i+1, n},$$

Set

$$d(\Delta) := \max \left(\bigcup_i (A_i \cup \overline{B}_i) \right), \quad \Delta(s) := \{\alpha \in \Delta \mid |\alpha| = s\}.$$

Let $\{f_1(u, v), \dots, f_k(u, v)\}$ be a finite set of functions in $\mathcal{E}_{n+m} = \mathcal{E}_{n+m}(u_1, \dots, u_n, v_1, \dots, v_m) = \mathcal{E}_{n+m}(u, v)$ which is in set-theoretic bijective correspondence with vertices

$$f_i \mapsto \alpha_i, \quad \text{ord}(f_i(u, 0)) = |\alpha_i|.$$

Any $\beta \in \Delta_i$ can be written as $\beta = \alpha_i + \gamma_i$, where $\gamma_i \in \Gamma_i$. Set

$$f_\beta := u^\gamma f_i, \quad i(\beta) := i.$$

We define *generalized Jacobians* as

$$J^s(f_1, \dots, f_k : u^{\alpha_1}, \dots, u^{\alpha_k}) := \det [D_{u^\alpha}(f_\beta)]_{\alpha, \beta \in \Delta(s)} = \det [D_{u^{\alpha - \beta + \alpha_{i(\beta)}}}(f_{i(\beta)})]_{\alpha, \beta \in \Delta(s)}.$$

Recall that $D_{u^\alpha} = \frac{1}{\alpha!} \frac{\partial^{|\alpha|}}{\partial u^\alpha}$, and note that there is an obvious equality

$$D_{u^\alpha}(f_\beta) = D_{u^\alpha}(u^{\beta - \alpha_{i(\beta)}} f_{i(\beta)}) = D_{u^{\alpha - \beta + \alpha_{i(\beta)}}}(f_{i(\beta)}),$$

which allows us to compute Jacobians in two different ways.

Remark. Here $D_{u^{\alpha - \beta + \alpha_{i(\beta)}}}$ is assumed to be 0 if $\alpha - \beta + \alpha_{i(\beta)}$ contains a negative component.

In the case of functions of multiplicity one this notion coincides with the standard Jacobian

$$J^1(f_1, \dots, f_k : u_1, \dots, u_k) := \det [D_{u_j}(f_i)].$$

Theorem 5.3.1 (Generalized division theorem 3). *If for all $s \leq d(\Delta) + 1$, the Jacobians*

$$J^s(f_1, \dots, f_k : u^{\alpha_1}, \dots, u^{\alpha_k})$$

are invertible then for any $g \in \mathcal{E}_n$ there exist $h_i \in \mathcal{E}_n$ and $r(g) \in \mathcal{E}_n$ such that $\text{supd}(h_i) \subset \Gamma_i$, $\text{supd}(r(g)) \subset \Gamma$, and

$$g = \sum h_i f_i + r(g).$$

Also

$$r(g) = \sum_{j=1}^n \sum_{\alpha \in A_j} c_\alpha \cdot u^\alpha,$$

where $c_\alpha \in \mathcal{E}_{n-j}(u_{j+1}, \dots, u_n, v)$

Moreover, if $\text{ord}(f_i) = \text{ord}(f_i(u, 0)) = |\alpha_i|$ then $\text{ord}(h_i) \geq \text{ord}(g) - |\alpha_i|$.

Proof. The theorem is a consequence of Theorem 4.1.6 and Lemma 4.1.9. Consider the rings of polynomials R_i over K . By Corollary 3.4.5, the set of monomials

$$\left\{ u^\alpha \mid \alpha \in \bigcup (A_{i-1} \cup \overline{B}_i) \right\}$$

is a basis of the module R_n over $(R_i)_{i=0}^n$. The condition of invertibility of $J^s(f)(0)$ means exactly that

$$\{\text{in}(f_\beta(u, 0)) \mid \beta \in \Delta(s)\} \cup \{u^\alpha \mid \alpha \in \Gamma(s)\}$$

is a basis of the s gradation $(R_n)_s$ of R_n (over K).

Consequently, by the stabilization theorem (Theorem 4.1.10), the set

$$\{\text{in}(f_\beta(u, 0)) \mid \beta \in \bigcup \overline{B}_i\} \cup \{u^\alpha \mid \alpha \in \bigcup A_{i-1}\}$$

is a basis of R_n because it generates the module R_n over $(R_i)_{i=0}^n$ up to degree $d(\Delta) + 1$.

Thus, by Lemma 4.1.8, the set

$$\{\text{in}(f_\beta(u, v)) \mid \beta \in \bigcup \overline{B}_i\} \cup \{u^\alpha \mid \alpha \in \bigcup A_i\}$$

is a basis of R_{n+m} because it generates the module R_{n+m} over $(R_i)_{i=0}^n$.

Finally, by Theorem 4.1.6, the set

$$\{f_\beta(u, v) \mid \beta \in \bigcup \overline{B}_i\} \cup \{u^\alpha \mid \alpha \in \bigcup A_i\}$$

is a basis of \mathcal{E}_{n+m} over $(\mathcal{E}_{i+m})_{i=0}^n$. In the “moreover” part we can use the standard filtration on \mathcal{E}_{n+m} and R_{n+m} and Lemma 4.1.8. \square

Corollary 5.3.2. *Under the previous assumption the set*

$$\left\{ \text{in}(f_\beta(u, v)) \mid \beta \in \bigcup \overline{B}_i \right\} \cup \left\{ u^\alpha \mid \alpha \in \bigcup A_i \right\} = \{ \text{in}(f_\beta(u, v)) \mid \beta \in \Delta \} \cup \{ u^\alpha \mid \alpha \in \Gamma \}$$

the set generates R_{n+m} over $(R_i)_{i=0}^n$.

Theorem 5.3.3 (Generalized division theorem 4). *Let X be a smooth scheme of finite type over a field K (or a \mathbb{C} -analytic/differentiable manifold) of dimension n with a given coordinate system at a K -rational point $x \in X$. Let $\Delta \subset \mathbb{N}^r \subset \mathbb{N}^n$ be a diagram of finite type, for some $r \leq n$, generated by $\alpha_i \in \mathbb{N}^r$. Let $f_1, \dots, f_k \in \mathcal{O}(X)$ be functions in a bijective correspondence with $\alpha_1, \dots, \alpha_k$, and such that $\text{ord}_x(f_i) = |\alpha_i|$. Assume moreover that for all $s \leq d(\Delta) + 1$, the Jacobians*

$$J^s(f_1, \dots, f_k : u^{\alpha_1}, \dots, u^{\alpha_k})$$

are invertible at x . Then there exists an étale neighborhood $X' \rightarrow X$ preserving the residue field K (respectively an open neighborhood) such that for any $g \in \mathcal{O}(X')$ there exist $h_i \in \mathcal{O}(X')$ and $r(g) \in \mathcal{O}(X')$ with $\text{supd}(h_i) \subset \Gamma_i$, $\text{supd}(r(g)) \subset \Gamma$, and

$$g = \sum h_i f_i + r(g).$$

Moreover, if $\text{ord}_x(f_{ij}) = \text{ord}_x(f_i(u, 0)) = |\alpha_i|$ then $\text{ord}_x(h_i) \geq \text{ord}_x(g) - |\alpha_i|$.

Proof. This follows directly from the previous theorem and Theorems 4.2.2, 4.2.3, 4.2.5. □

Denote by $\text{Vert}(\Delta)$ the set of vertices of Δ , and set $\text{Vert}(\Delta(s)) := \text{Vert}(\Delta) \cap \Delta(s)$. We define

$$\Delta^0(s) := \Delta(s) \setminus \text{Vert}(\Delta(s))$$

if $\text{Vert}(\Delta(s)) \neq \emptyset$, otherwise $\Delta^0(s) := \emptyset$. Set

$$J_s^0(f_1, \dots, f_k : u^{\alpha_1}, \dots, u^{\alpha_k}) := \det [D_{u^\alpha}(f_\beta)]_{\alpha, \beta \in \Delta^0(s)} = \det [D_{u^{\alpha-\beta+i(\beta)}}(f_{i(\beta)})]_{\alpha, \beta \in \Delta^0(s)}.$$

The Jacobians J_s^0 occur only for functions of distinct multiplicities.

Theorem 5.3.4 (Generalized implicit function theorem). *With the notation of Theorem 5.3.1, assume that for all $s \leq d(\Delta) + 1$, the Jacobians*

$$J^s(f_1, \dots, f_k : u^{\alpha_1}, \dots, u^{\alpha_k}) \quad \text{and} \quad J_s^0(f_1, \dots, f_k : u^{\alpha_1}, \dots, u^{\alpha_k})$$

are invertible. Then there is a set of generators $\overline{f}_1, \dots, \overline{f}_k$ of the ideal (f_1, \dots, f_k) of the form

$$(5.3) \quad \overline{f}_i := u^{\alpha_i} + r_i = u^{\alpha_i} + \sum_{j=1}^n \sum_{\alpha \in A_j} c_\alpha \cdot u^\alpha,$$

where $c_\alpha \in \mathcal{E}_{n+m-j}(u_{j+1}, \dots, u_n, v)$ for $j = 1, \dots, n$, and $\text{supd}(r_i(u)) \subset \Gamma \times \mathbb{N}^k$.

Moreover, $\text{ord}(\overline{f}_i(u, 0)) = \text{ord}(f_i(u, 0)) = |\alpha_i|$, and if $\text{ord}(f_i) = |\alpha_i|$ then $\text{ord}(\overline{f}_i) = |\alpha_i|$.

Proof. The proof is similar to the proof of the preparation theorem. We construct \overline{f}_i with multiplicities $|\alpha_i|$ and show by induction on the multiplicity s that all the ideals

$$(\overline{f}_i \mid |\alpha_i| \leq s) + (f_i \mid |\alpha_i| > s) \quad \text{and} \quad (\overline{f}_i \mid |\alpha_i| \leq s') + (f_i \mid |\alpha_i| > s')$$

are the same.

This is true for $s = 0$. Suppose it is valid for s . Let s' be the smallest integer $> s$ for which there exists a subscript $1 \leq j \leq k$ such that $s' = |\alpha_j|$. For all j such that $\alpha_j = s'$ consider the division with remainder of u^{α_j} by f_1, \dots, f_k :

$$u^{\alpha_j} = \sum_{j=1}^k h_{ij} f_j + r(u^{\alpha_j}),$$

where $\text{supd}(r(u^{\alpha_j})) \subset \Gamma \times \mathbb{N}^m$ and $\text{supd}(h_{ij}) \subset \Gamma_j \times \mathbb{N}^m$. Then set

$$(5.4) \quad \overline{f}_i := u^{\alpha_i} - r(u^{\alpha_i}) = \sum_{j=1}^{i-1} h_{ij} f_j = \sum_{|\alpha_i| < s'} h_{ij} f_j + \sum_{|\alpha_i| = s'} h_{ij} f_j + \sum_{|\alpha_i| > s'} h_{ij} f_j.$$

Passing to the initial forms gives

$$\text{in}(\overline{f}_i(u, 0)) = \sum_{|\alpha_i| < s'} \text{in}(h_{ij}(u, 0)) \text{in}(f_j(u, 0)) + \sum_{|\alpha_i| = s'} \text{in}(h_{ij}(u, 0)) \text{in}(f_j(u, 0)).$$

We see that if $|\alpha_i| < s'$ then $\text{ord}(h_{ij}) > 0$. (The summation in the above formula is taken only over terms of degree s' .)

The condition of J_s^0 being invertible can be written as invertibility of

$$\det [D_{u^\alpha}(f_\beta)]_{\alpha, \beta \in \Delta^0(s)},$$

which implies that

$$\{\overline{\text{in}(f_i(u, 0))} \mid |\alpha_i| = s'\} \cup \{\text{in}(f_\beta(u, 0)) \mid \beta \in \Delta_0(s')\}$$

forms a basis of $\text{in}_{s'}(\mathcal{I}(u, 0))$.

The invertibility of J_s means that

$$\det [D_{u^\alpha}(f_\beta)]_{\alpha, \beta \in \Delta(s)}$$

is invertible and that

$$\{\text{in}(f_i(u, 0)) \mid |\alpha_i| = s'\} \cup \{\text{in}(f_\beta(u, 0)) \mid \beta \in \Delta_0(s')\}$$

is another basis of $\text{in}_{s'}(\mathcal{I}(u, 0))$. Consequently, the determinant

$$\det [\overline{h_{ij}(0)}]_{|\alpha_i|=|\alpha_j|=s'}$$

is invertible.

By the inductive assumption, for any i' with $|\alpha_{i'}| < s'$ we can write

$$f_{i'} = \sum_{|\alpha_j| < s'} g_{i'j} \overline{f_j} + \sum_{|\alpha_j| \geq s'} g_{i'j} f_j.$$

Substituting this formula for $f_{i'}$ in (5.4) gives

$$\begin{aligned} \overline{f_i} &= \sum_{|\alpha_i| < s'} h_{ij'} \left(\sum_{|\alpha_{j'}| < s'} g_{i'j'} \overline{f_{j'}} + \sum_{|\alpha_{j'}| \geq s'} g_{i'j'} f_{j'} \right) + \sum_{|\alpha_i|=s'} h_{ij} f_j + \sum_{|\alpha_i| > s'} h_{ij} f_j \\ &= \sum_{|\alpha_i| < s'} \overline{h_{ij}} \cdot \overline{f_j} + \sum_{|\alpha_i|=s'} \overline{h_{ij}} f_j + \sum_{|\alpha_i| > s'} \overline{h_{ij}} f_j, \end{aligned}$$

where the matrix

$$[\overline{h_{ij}(0)}]_{|\alpha_i|=|\alpha_j|=s'} = [h_{ij}(0)]_{|\alpha_i|=|\alpha_j|=s'}$$

is invertible. The latter implies that

$$(\overline{f_i} \mid |\alpha| \leq s) + (f_i \mid |\alpha| > s) = (\overline{f_i} \mid |\alpha| \leq s') + (f_i \mid |\alpha| > s'),$$

of which completes the inductive step. \square

Theorems 5.3.1 and 5.3.3 generalize the preparation and division theorems. One can easily see that the Jacobian condition generalizes the condition for the initial exponents with respect to a monomial order.

Lemma 5.3.5. *Let K be any commutative ring. Let $F_1, \dots, F_k \in K[x_1, \dots, x_n]$ be forms. Let \overline{T} be a normalized order on \mathbb{N}^n and Δ be a diagram of finite type with vertices $\alpha_i = \exp_{\overline{T}}(f_i)$. Assume all the highest coefficients are invertible. Then all the Jacobians*

$$J^s(F_1, \dots, F_k : u^{\alpha_1}, \dots, u^{\alpha_k}) \quad \text{and} \quad J_0^s(F_1, \dots, F_k : u^{\alpha_1}, \dots, u^{\alpha_k})$$

are invertible.

Proof. Order the monomials according to \overline{T} . Then each generalized Jacobian matrix is lower triangular with invertible entries on the main diagonal. \square

Example 5.3.6. In the original implicit function theorem the relevant diagram Δ is generated by the standard basis $e_1, \dots, e_n \in \mathbb{N}^n$, and $A_0 = \{0\}$, $A_i = \emptyset$, $i > 0$, $\overline{B}_i = \{e_i\}$, so that we get $d(\Delta) = 1$. The condition $J^s(f)(x) \neq 0$ in Theorem 5.3.4 is assumed only for $s = 1 = d(\Delta)$. In the theorems above we need to verify the Jacobian conditions for $s \leq d(\Delta) + 1 = 2$.

The generalized Jacobians depend only on the initial forms of the function, so we use only the initial forms in the computations below.

Let $F_1 = a_{11}x_1 + a_{12}x_2$, $F_2 = a_{21}x_1 + a_{22}x_2$. Then

$$J^1 = J^1(F_1, F_2 : x_1, x_2) = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is the ordinary Jacobian and

$$J^2 = \det \begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix} = a_{11} \cdot J^1$$

We observe the occurrence of the extra condition of invertibility of a_{11} in the generalized implicit and division theorems when comparing to the condition $J^1 \neq 0$ in the classical implicit function theorem. This condition is due to lack of symmetry of a Stanley basis (and division), and it can be fulfilled by merely swapping coordinates. In that sense the condition is not essential from the point of view of the formulation of the classical implicit function theorem and can be dropped.

The generalized implicit function theorem allows us to choose the exponents and corresponding monotone diagrams Δ in many different ways as long as the multiplicities of generators and degrees of the corresponding monomials are equal and the Jacobian conditions hold. Among all those diagrams, particularly useful is the simplest one defined by the k vectors which are multiples of the standard basis or its part. Again such a diagram can be applied to a generic set of k functions.

The invertibility of Jacobians can be regarded as a generalization of the transversality condition in the smooth case (see Theorem 5.3.7(1)).

Let $f_1, \dots, f_k \in \mathcal{E}_n(u_1, \dots, u_k, v_1, \dots, v_{n-k})$, where $k \leq n$, be a set of functions such that $\text{ord}(f_i(u, 0)) = d_i$, where $d_1 \leq \dots \leq d_n$. Let Δ be the diagram generated by $\alpha_i = d_i \cdot e_i$, where $i = 1, \dots, k$. Then

$$\Delta = \{\alpha \in \mathbb{N} \mid \exists i \alpha_i \geq d_i\}, \quad \text{and} \quad i(\alpha) := \min\{i \mid \alpha_i \geq d_i\} \quad \text{for any } \alpha \in \Delta.$$

Corollary 5.3.7 (Generalized implicit function theorem 2). *Let $f_1, \dots, f_k \in \mathcal{E}_n(u_1, \dots, u_k, v_1, \dots, v_{n-k})$, where $k \leq n$, be a set of functions such that $\text{ord}(f_i(u, 0)) = d_i$, where $d_1 \leq \dots \leq d_n$. Denote by $\mathcal{I} \subset \mathcal{E}_n$ the ideal generated by f_1, \dots, f_k . If for all $s \leq \sum d_i - k + 2$, the Jacobians*

$$J^s(f_1, \dots, f_k : u_1^{d_1}, \dots, u_k^{d_k})$$

are invertible then the following conditions are satisfied:

- (1) *The module $\mathcal{E}_n/\mathcal{I}$ is free (or quasifree for the nonreduced category) and finite over $\mathcal{E}_{n-k}(v)$ with a basis $u_1^{\alpha_1} \cdot \dots \cdot u_k^{\alpha_k}$, where $\alpha_i < d_i$, and*

$$\text{rank}_{\mathcal{E}_{n-k}(v)}(\mathcal{E}_n/\mathcal{I}) = d_1 \cdot \dots \cdot d_k.$$

- (2) *Each $f \in \mathcal{E}_n$ can be written uniquely (up to flat functions) as*

$$f = \sum c_i f_i + r(f),$$

where $\text{supd}(r(f)) \subset \Gamma \times \mathbb{N}^{n-k}$ and $\text{supd}(c_i) \subset \Gamma_i \times \mathbb{N}^{n-k}$, that is,

$$r(f) = \sum_{\alpha_j < d_j} c_\alpha(v) \cdot u_1^{\alpha_1} \cdot \dots \cdot u_i^{\alpha_i}, \quad c_i = \sum_{\alpha_j < d_j} c_\alpha(u_{j+1}, \dots, u_n, v) \cdot u_1^{\alpha_1} \cdot \dots \cdot u_i^{\alpha_i}.$$

- (3) *If additionally $f \in \mathcal{I}$ then $r(f) \in m_{n+m}^\infty$.*
(4) *If $\text{ord}(f_i) = d_i$ (or $m = 0$) then the Hilbert function of \mathcal{I} is equal to*

$$H_{\mathcal{I}} = H(\Delta) = \sum_{0 \leq \alpha_i \leq d_i - 1} \phi(n + |a|, d) = d_1 \cdot \dots \cdot d_n \cdot t^{n-k} + a_{n-k-1} t^{n-k-1} + \dots + a_0.$$

- (5) *If additionally*

$$J_0^s(f_1, \dots, f_k : u_1^{d_1}, \dots, u_k^{d_k})$$

are invertible for $s \leq \sum d_i - n + 2$, then there exists a set of generators \overline{f}_i of the ideal \mathcal{I} of the form

$$\overline{f}_i := u_i^{d_i} + \sum_{\alpha_i < d_i} c_\alpha(v) \cdot u_1^{\alpha_1} \cdot \dots \cdot u_k^{\alpha_k},$$

where $c_\alpha \in \mathcal{E}_m(v)$.

- (6) $\text{ord}(\overline{f}_i(u, 0)) = \text{ord}(f_i(u, 0)) = |\alpha_i|$, and if $\text{ord}(f_i) = |\alpha_i|$ then $\text{ord}(\overline{f}_i) = |\alpha_i|$.
(7) *The generators \overline{f}_i satisfy condition (2) above.*

Proof. In this situation $\Gamma = A_n = [0, d_1 - 1] \times \dots \times [0, d_n - 1]$ is finite, $B_i = d_i e_i + \{(\alpha_1, \dots, \alpha_{i-1}, 0, \dots, 0) \mid \alpha_i < d_i\}$ and $d(\Delta) = (\sum d_i) - n + 1$. Conditions (2) and (5) follow from the previous theorem. The conditions on the Jacobians imply, by Theorem 5.4.2, that the functions f_i form a Cohen-Macaulay regular sequence. Condition (1) is a consequence of the more general Theorem 5.4.5 below. The other conditions follow from Theorem 4.3.1. \square

Remark. The above theorems are valid with unchanged proofs for homogeneous (and nonhomogeneous) polynomials F_1, \dots, F_k in $K[x_1, \dots, x_n]$, where K is a commutative ring with 1. Under the Jacobian conditions, $K[x_1, \dots, x_n]/(F_1, \dots, F_k)$ is a free module over $K[x_{k+1}, \dots, x_n]$ of rank $d_1 \cdot \dots \cdot d_n$.

When $k = n$ in the theorem above, and K is a field, the integer $\dim_K(K[x_1, \dots, x_n]/\mathcal{I}) = d_1 \cdot \dots \cdot d_n$ can be understood as the number of zeroes with multiplicities of the polynomials f_1, \dots, f_n in the affine space \mathbb{A}_K^n , as in Bézout's theorem.

5.4. Cohen-Macaulay Weierstrass isomorphism. Let us analyze the conditions imposed in Theorem 5.3.7, and consider the following example.

Example 5.4.1. Let $F_1 = a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_2^2$ and $F_2 = a_{21}x_1^2 + a_{22}x_1x_2 + a_{23}x_2^2$. Then $d(\Delta) = 2$, and the computation of Jacobians will be carried out up to order $s \leq 4$ (starting from $s = 2$). We have

$$J^2 = J^2(F_1, F_2 : x_1^2, x_2^2) = \det \begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix}.$$

The condition $J^2 \neq 0$ implies the linear independence of the pure quadratic parts of the (initial) forms. Next,

$$J^3 = \det \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ 0 & a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{21} & a_{22} & a_{23} \end{pmatrix}.$$

Thus J^3 is nothing but the resultant of the forms. Its invertibility implies that the forms F_1, F_2 have no common linear factor. A simple computation shows that $J^4 = a_{11}J^3$, and $J^5 = a_{11}^2J^3$ as before. The additional conditions of invertibility of a_{11} and J_2 are due to lack of symmetry of the construction and imply, together with other conditions, certain asymmetry properties of filtered Stanley decompositions like the division theorem. Once the forms are in general position, that is, J^3 is invertible, the conditions of invertibility of a_{11} and J^2 can be ensured by a generic change of coordinates, and as such are not essential for many properties.

Observe that the example above easily generalizes to the case of any two forms $F_1, F_2 \in K[x, y]$ of degrees d_1, d_2 . We get $J^d(F_1, F_2) = \text{Res}(F_1, F_2)$, where $d = d(\Delta) = d_1 + d_2 - k + 1$.

In general, the determinants of the matrices $J^s(F_1, \dots, F_k : x_1^{d_1}, \dots, x_k^{d_k})$ were first considered by Macaulay, and used in his definition of (projective) resultant for homogeneous polynomials. He proved in particular the following results:

- Theorem 5.4.2** (Macaulay [57], [44]). (1) $J^s(F_1, \dots, F_k) = \text{Res}(F_1, \dots, F_k) \cdot \Delta(F_1, \dots, F_k, s)$ for all $s \geq d := \sum d_i - k + 1$, where $\Delta(F_1, \dots, F_k, s)$ is, so called, an extraneous factor.
- (2) The square-free part of $\Delta(F_1, \dots, F_k, s)$ is equal (up to sign) to the minor of $M(F_1, \dots, F_n; s)$ obtained by deleting all rows and columns that are indexed by any power product, that is, divisible by exactly one x_{d_i} for $1 \leq i \leq n$.

Moreover it follows from his theorem and his considerations that $\Delta(F_1, \dots, F_k, s)$ can be made invertible by some coordinate change.

Let K denote a field and $F_1, \dots, F_k \in R = K[x_1, \dots, x_k]$ be forms of degree d_1, \dots, d_k respectively. Set $I = (F_1, \dots, F_k)$, and denote by I_s and R_s the forms in I and R of degree s . Let t be an indeterminate over \mathbb{Z} and write the Hilbert function as the formal power series

$$H(R/I) = \sum_{s=0}^{\infty} (R_s/I_s)t^s.$$

The following result is essentially due to Macaulay.

Theorem 5.4.3 (Macaulay). *The following conditions are equivalent:*

- (1) The forms F_1, \dots, F_k form a regular sequence.

- (2) The resultant $\text{Res}(F_1, \dots, F_k)$ does not vanish.
- (3) The vector space $K[x_1, \dots, x_k]/(F_1, \dots, F_k)$ has finite dimension.
- (4) The ideal $I := (F_1, \dots, F_k)$ is (x_1, \dots, x_k) -adic.
- (5) The forms F_1, \dots, F_k have no nontrivial solutions over the algebraic closure of K .

Moreover, if these conditions are satisfied then:

- (1) $\dim_K(K[x_1, \dots, x_n]/(F_1, \dots, F_k)) = d_1 \cdot \dots \cdot d_k$.
- (2) $H_{(F_1, \dots, F_i)}(t) = (1 - t^{d_1}) \cdot \dots \cdot (1 - t^{d_i})(1 - t)^n = H_{(x_1^{d_1}, \dots, x_i^{d_i})}(t)$ for any $i = 1, \dots, k$.
- (3) If K is an infinite field then after some generic change of coordinates,

$$K[x_1, \dots, x_k]/(F_1, \dots, F_i) = (K[x_1, \dots, x_k]/(F_1, \dots, F_i, x_{i+1}, \dots, x_k))[x_{i+1}, \dots, x_k]$$

is finite over $K[x_{i+1}, \dots, x_k]$ of degree $d_1 \cdot \dots \cdot d_i$.

It follows from the Macaulay definition of resultant that invertibility of the generalized Jacobians implies invertibility of the resultant. It is natural to ask whether the converse is true.

Conjecture 5.4.4. *Let K be an infinite field, and F_1, \dots, F_k be homogeneous polynomials in $K[x_1, \dots, x_k]$. Then the following conditions are equivalent:*

- (1) $\text{Res}(F_1, \dots, F_k)$ is invertible (or F_1, \dots, F_k is a regular sequence).
- (2) There exists a generic coordinate change such that

$$J^s(F_1, \dots, F_k : x_1^{d_1}, \dots, x_k^{d_k})$$

is invertible for all s .

- (3) There exists a generic linear coordinate change such that $K[x_1, \dots, x_k]/(F_1, \dots, F_i, x_{i+1}, \dots, x_k)$ is generated by x^α , with $0 \leq \alpha_i < d_i$.

It is quite easy to show the conjecture for $k \leq 2$. The problem is related to a famous Eisenbud-Green-Harris conjecture.

The following results can be considered as a generalization of Theorem 2.3.7 on the Weierstrass isomorphism for Cohen-Macaulay singularities.

Theorem 5.4.5 (Cohen-Macaulay-Weierstrass isomorphism). *Let $\mathcal{E}_{n+k}(x, y)/\mathcal{I}$ be a Cohen-Macaulay local ring (of dimension k) with a regular sequence defined by coordinates y_1, \dots, y_k . Let $\Delta = \exp_T(\mathcal{I}(x, 0)) \subset \mathbb{N}^n$ (for a normalized total order T) be the induced diagram. Then there is an isomorphism (quasi-isomorphism in the nonreduced category) of free $\mathcal{E}_k(y)$ -modules*

$$\phi : \bigoplus_{\alpha \in \Gamma} \mathcal{E}_k(y)x^\alpha \rightarrow \mathcal{E}_{n+k}/\mathcal{I}.$$

Proof. Consider the ideal $\mathcal{J} = (y_1, \dots, y_k) \subset \mathcal{E}_k$, and set $R = \mathcal{E}_n/\mathcal{I}$. Let $\mathcal{J}_R \subset R$ denote the ideal generated by y_1, \dots, y_k . Then $\mathcal{J}_R^i = \mathcal{J}^i \cdot R$. The homomorphism ϕ is surjective by Weierstrass-Hironaka division, and likewise its restriction

$$\phi : \bigoplus_{\alpha \in \Gamma} \mathcal{J}^i \cdot x^\alpha \rightarrow \mathcal{J}_R^i.$$

By a theorem of Matsumura [52],

$$\text{gr}_{\mathcal{J}_R}(R) = \bigoplus \mathcal{J}_R^i/\mathcal{J}_R^{i+1} \simeq (R/\mathcal{J}_R)[y_1, \dots, y_k].$$

The epimorphism ϕ defines a gradation preserving epimorphism

$$\bar{\phi} : \bigoplus_{\alpha \in \Gamma} \mathcal{J}^i/\mathcal{J}^{i+1} \cdot x^\alpha = \bigoplus_{\alpha \in \Gamma} K[y_1, \dots, y_k] \cdot x^\alpha \rightarrow \mathcal{J}_R^i = (R/\mathcal{J}_R)[y_1, \dots, y_k],$$

which is an isomorphism. If a function $f \in \bigoplus_{\alpha \in \Gamma} \mathcal{E}_k(y)x^\alpha$ is in the kernel of ϕ , and

$$f \in \mathcal{J}^i \cdot \left(\bigoplus_{\alpha \in \Gamma} \mathcal{E}_k(y)x^\alpha \right) \setminus \mathcal{J}^{i+1} \cdot \left(\bigoplus_{\alpha \in \Gamma} \mathcal{E}_k(y)x^\alpha \right),$$

then it defines a nonzero element in the kernel of $\bar{\phi}$, which is impossible. Thus $f \in \mathcal{J}^\infty \cdot \left(\bigoplus_{\alpha \in \Gamma} \mathcal{E}_k(y)x^\alpha \right)$. \square

Theorem 5.4.6 (Weierstrass isomorphism 2). *Let $\pi : X \rightarrow Y$ be a smooth of dimension k of smooth schemes of dimension $n+k$ and n over K (respectively X, Y are open subsets of \mathbb{C}^{n+k} or \mathbb{R}^{n+k} and of \mathbb{C}^n or \mathbb{R}^n , and $\pi : X \rightarrow Y$ is the restriction of the natural projection $\pi_0 : \mathbb{C}^{n+k} \rightarrow \mathbb{C}^n$ or $\pi_0 : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^n$).*

Let $z = (0, 0)$ be a (K -rational) point of X , and $(x, y) = (x_1, \dots, x_k, y_1, \dots, y_n)$ be a local coordinate system at $z = (0, 0)$ with projection π defined by (x_1, \dots, x_k) . Let $\mathcal{I} \subset \mathcal{O}_X$ be an ideal sheaf of finite type, and suppose the local ring $\mathcal{O}_{X,z}/\mathcal{I}_z$ is Cohen-Macaulay. Assume that $\mathcal{O}_X(x, 0)/\mathcal{I}(x, 0)$ is a finite K -space, with $\mathcal{I}(x, 0)$ defining a finite diagram Δ in \mathbb{N}^n .

Then there exist étale neighborhoods X' of X and Y' of Y preserving the residue field K (respectively open neighborhoods) with the induced projection $X' \rightarrow Y'$ and the Weierstrass isomorphism (surjection in the differential setting with kernel contained in m_z^∞) of free $\mathcal{O}_{Y'}$ -modules

$$\bigoplus_{\alpha \in \Gamma} \mathcal{O}_{Y'} \cdot x^\alpha \rightarrow \pi_*(\mathcal{O}_{X'}/\mathcal{I}),$$

and $\mathcal{O}(Y')$ -modules

$$\bigoplus_{\alpha \in \Gamma} \mathcal{O}(Y') \cdot x^\alpha \rightarrow \mathcal{O}(X')/\mathcal{I},$$

Proof. Put $\bar{y} := \pi(z) \in Y$. The sheaf $\mathcal{F} := \mathcal{O}_X/\mathcal{I}$ is of finite type. Moreover the vector space $\mathcal{F}_z/(y_1, \dots, y_k) = \mathcal{F}_z/m_{\bar{y}, Y}$ is of finite dimension. By Theorems 1.1.7 (in the algebraic setting) 1.2.3 (in the analytic situation), and 2.4.2 (in the differential setting) there exist neighborhoods $X' \times Y'$ of $X \times Y$ such that $\pi_*(\mathcal{O}_{X'}/\mathcal{I})$ is coherent on Y (or of finite type in the differential situation).

Then in the analytic case, by Theorem 5.4.5, $\pi_*(\mathcal{O}_{X'}/\mathcal{I})_z \simeq (\mathcal{O}_{X'}/\mathcal{I})_z$ is a free $\mathcal{O}_{\bar{y}}$ -module with basis x^α , $\alpha \in \Gamma$. This implies that after possibly shrinking Y' , $\pi_*(\mathcal{O}_{X'}/\mathcal{I})$ is a free $\mathcal{O}_{Y'}$ -module with the same basis.

In the algebraic setting, by Theorem 5.4.5, after passing to the Henselianizations of the stalks we get an isomorphism of free $\mathcal{O}_{Y', \bar{y}}^h$ -modules

$$(5.5) \quad \bigoplus_{\alpha \in \Gamma} \mathcal{O}_{Y', \bar{y}}^h \cdot x^\alpha \simeq \mathcal{O}_{X', z}^h/(\mathcal{I} \cdot \mathcal{O}_{X', z}^h).$$

In a certain étale neighborhood X' of X the generators of the coherent modules can be expressed in terms of x^α . In other words, by modifying X' and Y' we may assume additionally that x^α , $\alpha \in \Gamma$, generate the coherent module $\pi_*(\mathcal{O}_{X'}/\mathcal{I})$. This defines an epimorphism of sheaves

$$\phi : \bigoplus_{\alpha \in \Gamma} \mathcal{O}_{Y'} \cdot x^\alpha \rightarrow \pi_*(\mathcal{O}_{X'}/\mathcal{I}).$$

The corresponding homomorphism of stalks

$$\phi_z : \bigoplus_{\alpha \in \Gamma} \mathcal{O}_{Y', \bar{y}} \cdot x^\alpha \rightarrow \pi_*(\mathcal{O}_{X'}/\mathcal{I})_z \simeq (\mathcal{O}_{X'}/\mathcal{I})_z = \mathcal{O}_{X', z}/\mathcal{I}_z$$

is also injective as $\mathcal{O}_{Y', \bar{y}} \rightarrow \mathcal{O}_{Y', \bar{y}}^h$ is injective, and we get an injective homomorphism, in fact an isomorphism, of the Henselianizations as in (5.5).

Thus ϕ_z is an isomorphism of stalks which defines an isomorphism of coherent free \mathcal{O}_Y -modules in an open Zariski neighborhood.

In the differential setting, by shrinking X and Y we can assume that ϕ is a morphism of sheaves of modules of finite type. By Theorem 5.4.5, it defines an epimorphism ϕ_z of stalks with local generators x^α , $\alpha \in \Gamma$. Since $\pi_*(\mathcal{O}_{X'}/\mathcal{I})$ is of finite type we can assume by further shrinking that they generate $\pi_*(\mathcal{O}_{X'}/\mathcal{I})$, and ϕ is epimorphism. Since ϕ_z is a quasi-isomorphism of stalks, and its kernel is contained in m_z^∞ . \square

6. MARKED IDEALS AND STANDARD BASIS ALONG SAMUEL STRATUM

In the next chapters we give some applications of the previous results to desingularization and description of the Samuel stratum. The main goal of Chapter 6 is to study the notion (introduced here) of standard basis along Samuel stratum (Definition 6.3.3). It allows one to describe and modify singularities controlled by the Hilbert-Samuel function.

6.1. Resolution of marked ideals. Recall the definition of Hilbert-Samuel function of an ideal sheaf \mathcal{I} at a closed point $x \in X$, where X is a manifold or a smooth scheme is defined as

$$H_{x,\mathcal{I}}(k) = \dim \mathcal{O}_{X,x}/(m_{x,X}^{k+1} + \mathcal{I}).$$

As before the order of \mathcal{I} at $x \in X$ is denoted by $\text{ord}_x(\mathcal{I}) := \max\{k \in \mathbb{N} \mid \mathcal{I} \subset m_{x,X}^k\}$

Definition 6.1.1. (Hironaka (see [36]), Bierstone-Milman (see [7]), Villamayor (see [67])) A *marked ideal* (respectively an H-marked ideal) is a collection (X, \mathcal{I}, E, μ) , (respectively (X, \mathcal{I}, E, H)) where X is a smooth scheme of finite type over a field K (or an analytic/differentiable manifold), \mathcal{I} is a sheaf of ideals on X of finite type, E is a totally ordered collection of divisors with SNC, whose irreducible components are smooth pairwise disjoint and all have multiplicity one, and μ is a nonnegative integer (respectively H is a function $H : \mathbb{N} \rightarrow \mathbb{N}$ with integral values). Moreover the irreducible components of divisors in E have simultaneously simple normal crossings.

A collection of marked ideals $\{(X, \mathcal{I}_i, E, \mu_i)\}$ will be called a *multiple marked ideal*. Marked functions (f, μ) are pairs of regular functions on X and $\mu \in \mathbb{N}$.

The functions H can be identify with infinite sequence of nonnegative integers ordered lexicographically.

Definition 6.1.2. (Hironaka ([36]), Bierstone-Milman (see [7]), Villamayor (see [67])) By the *cosupport* (originally *singular locus*) of (X, \mathcal{I}, E, μ) we mean

$$\text{cosupp}(X, \mathcal{I}, E, \mu) := \{x \in X \mid \text{ord}_x(\mathcal{I}) \geq \mu\}.$$

Similarly

$$\text{cosupp}\{(X, \mathcal{I}_i, E, \mu_i)\} := \{x \in X \mid \text{ord}_x(\mathcal{I}_i) \geq \mu_i\} = \bigcap_i \text{cosupp}(X, \mathcal{I}_i, E, \mu_i).$$

By the *cosupport* of (X, \mathcal{I}, E, H) we mean

$$\text{cosupp}(X, \mathcal{I}, E, H) := \{x \in X \mid \mathcal{H}_x(\mathcal{I}) \geq H\}.$$

Remarks. (1) In most of the applications $\text{cosupp}(\mathcal{I}, H)$ coincides with so called Samuel stratum.

(2) For any sheaf of ideals \mathcal{I} on X we have $\text{cosupp}(\mathcal{I}, 1) = \text{cosupp}(\mathcal{I})$.

(3) For any marked ideals (\mathcal{I}, μ) on X , $\text{cosupp}(\mathcal{I}, \mu)$ is a closed subset of X (Lemma 6.2.2).

Let u_1, \dots, u_n be a local coordinate system of a smooth variety (or an analytic or differentiable manifold) X , and $C \subset X$ be a closed smooth subspace (submanifold) of X defined by $u_1 = \dots = u_r = 0$, with $r \leq n$. Denote by \mathbb{P}^{r-1} the projective space with homogenous coordinates y_1, \dots, y_r . Recall that *the blow-up of C* is defined (locally on X) as the map

$$X' := \{(x, y) \in X \times \mathbb{P}^{r-1} \mid u_i y_j = u_j y_i\} \rightarrow X$$

Then there are open neighborhoods U_i of X' where $y_i \neq 0$ with coordinate system $u'_j = y_j/y_i = u_j/u_i$ for $j \neq i$, $j \leq r$ and $u'_i = u_i$ otherwise.

Definition 6.1.3. The blow-ups with the smooth centers $C \subset \text{cosupp}(X, \mathcal{I}, E, \mu)$ (respectively $C \subset \text{cosupp}(X, \mathcal{I}, E, H)$ or $C \subset \text{cosupp}\{(X, \mathcal{I}_i, E, \mu_i)\}$) will be called *admissible for (X, \mathcal{I}, E, μ)* , (respectively for (X, \mathcal{I}, E, H) or for $\{(X, \mathcal{I}_i, E, \mu_i)\}$) if the centers are contained in the cosupport of marked ideals and have SNC with E . Likewise we called the centers *admissible* for the marked ideals.

Lemma 6.1.4. *Let $C \subset \text{cosupp}(\mathcal{I}, \mu)$ be a smooth center of the blow-up $\sigma : X \leftarrow X'$ and let D denote the exceptional divisor. Let \mathcal{I}_C denote the sheaf of ideals defined by C . Then*

- (1) $\mathcal{I} \subset \mathcal{I}_C^\mu$.
- (2) $\sigma^*(\mathcal{I}) \subset (\mathcal{I}_D)^\mu$.

Proof. (1) We can assume that the ambient variety X is affine. Let u_1, \dots, u_k be parameters generating \mathcal{I}_C . Suppose $f \in \mathcal{I} \setminus \mathcal{I}_C^\mu$. Then we can write $f = \sum_\alpha c_\alpha u^\alpha$, where either $|\alpha| \geq \mu$ or $|\alpha| < \mu$ and $c_\alpha \notin \mathcal{I}_C$. By the assumption there is α with $|\alpha| < \mu$ such that $c_\alpha \notin \mathcal{I}_C$. Take α with the smallest $|\alpha|$. There is a point $x \in C$ for which $c_\alpha(x) \neq 0$ and in the Taylor expansion of f at x there is a term $c_\alpha(x)u^\alpha$. Thus $\text{ord}_x(\mathcal{I}) < \mu$. This contradicts to the assumption $C \subset \text{cosupp}(\mathcal{I}, \mu)$.

$$(2) \sigma^*(\mathcal{I}) \subset \sigma^*(\mathcal{I}_C)^\mu = (\mathcal{I}_D)^\mu. \quad \square$$

Definition 6.1.5. Let $\sigma : X' \rightarrow X$ be an admissible blow-up for (X, \mathcal{I}, E, μ) with the exceptional divisor D then a marked ideal $(X', \mathcal{I}', E', \mu) = \sigma^c(X, \mathcal{I}, E, \mu)$ is called *the controlled transform* of (X, \mathcal{I}, E, μ) if

- (1) $\mathcal{I}' = \mathcal{I}(D)^{-\mu} \sigma^*(\mathcal{I})$.
- (2) $E' = \sigma^c(E) \cup \{D\}$, where $\sigma^c(E)$ is the set of strict transforms of divisors in E_{i-1} .
- (3) The order on $\sigma^c(E)$ is defined by the order on E while D is the maximal element of E .

Similarly the *controlled transform* of $\{(X, \mathcal{I}_i, E, \mu_i)\}$ is given as the collection of the controlled transforms of $(X, \mathcal{I}_i, E, \mu)$.

Definition 6.1.6. Let \mathcal{I} be any ideal sheaf of finite type on a smooth variety X over a field K (or an analytic or differentiable manifold), and let $C \subset X$ be a smooth subvariety. Consider the blow-up $\sigma : X' \rightarrow X$ at a smooth closed center $C \subset X$ contained in the Samuel stratum. By the *strict transform* of \mathcal{I} we mean here the ideal generated locally by $(1/y^{c(f)})\sigma^*(f)$, where y is a local equation of the exceptional divisor, and $c(f)$ is the maximal exponent for which $y^{c(f)}$ divides $\sigma^*(f)$.

By the *the strict transform* of (X, \mathcal{I}, E, H) under an admissible blow-up $\sigma : X' \rightarrow X$ we mean H -marked ideal (X, \mathcal{I}, E, H) , where \mathcal{I}' is the strict transform and E' satisfies (2) and (3).

By [9, Proposition 3.13] the condition of the strict transform is equivalent in the analytic and algebraic setting to the following:

The strict transform is generated by all the functions $f \in \mathcal{O}(U')$ for which $y^k f$, for some k , is in the ideal generated by $\sigma^(g)$, where $g \in \mathcal{O}(U)$.*

Definition 6.1.7. (Hironaka (see [36]), Bierstone-Milman (see [9]), Villamayor (see [67])) By a *admissible sequence of blow-ups* of (X, \mathcal{I}, E, μ) (respectively (X, \mathcal{I}, E, H)) we mean a sequence of blow-ups $\sigma_i : X_i \rightarrow X_{i-1}$ of smooth centers $C_{i-1} \subset X_{i-1}$,

$$X_0 = X \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \xleftarrow{\sigma_3} \dots X_i \xleftarrow{\dots} \dots \xleftarrow{\sigma_r} X_r,$$

which defines a sequence of marked ideals $(X_i, \mathcal{I}_i, E_i, \mu)$ (respectively $(X_i, \mathcal{I}_i, E_i, H)$), such that the centers C_{i-1} are admissible for $(X_{i-1}, \mathcal{I}_{i-1}, E_{i-1}, \mu)$ (respectively for $(X_{i-1}, \mathcal{I}_{i-1}, E_{i-1}, H)$), and $(X_i, \mathcal{I}_i, E_i, H)$ are controlled transforms of $(X_{i-1}, \mathcal{I}_{i-1}, E_{i-1}, \mu)$ (respectively $(X_i, \mathcal{I}_i, E_i, H)$ are the strict transforms of $(X_{i-1}, \mathcal{I}_{i-1}, E_{i-1}, H)$). If additionally

$$\text{cosupp}(X_r, \mathcal{I}_r, E_r, \mu) = \emptyset$$

(resp. $\text{cosupp}(X_r, \mathcal{I}_r, E_r, H) = \emptyset$) then we call the sequence a *resolution* of (X, \mathcal{I}, E, μ) .

The definition of admissible sequence and a resolution sequence applies also to multiple marked ideals.

6.2. Ideals of derivatives. Ideals of derivatives were first introduced and studied in the resolution context by Giraud.

Definition 6.2.1. (Giraud, Villamayor) Let \mathcal{I} be a sheaf of ideals of finite type on a smooth variety X (or an analytic/differentiable manifold). For any $i \in \mathbb{N}$, by the *i -th derivative* $\mathcal{D}^i(\mathcal{I})$ of \mathcal{I} we mean the sheaf of ideals generated by all functions $f \in \mathcal{I}$ with their (Hasse) derivatives of $D_{u^\alpha} = \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f_j}{\partial u^\alpha}$ for all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$, where $|\alpha| := \alpha_1 + \dots + \alpha_n \leq i$.

If (\mathcal{I}, μ) is a marked ideal and $i \leq \mu$ then we define

$$\mathcal{D}^i(\mathcal{I}, \mu) := (\mathcal{D}^i(\mathcal{I}), \mu - i).$$

Lemma 6.2.2. (Giraud, Villamayor) For any $i \leq \mu - 1$,

$$\text{cosupp}(\mathcal{I}, \mu) \subset \text{cosupp}(\mathcal{D}^i(\mathcal{I}), \mu - i).$$

(with equality in characteristic zero). In particular case

$$\text{cosupp}(\mathcal{I}, \mu) = \text{cosupp}(\mathcal{D}^{\mu-1}(\mathcal{I}), 1) = V(\mathcal{D}^{\mu-1}(\mathcal{I}))$$

is a closed subspace of X . □

Proof. If $\text{ord}_x(\mathcal{I}) \geq \mu$ then $\mathcal{D}^i(\mathcal{I}) \geq \mu - i$. If $\text{ord}_x(\mathcal{I}) < \mu$ then $\mathcal{D}^{\mu-1}(\mathcal{I})$ is invertible. □

We write $(\mathcal{I}, \mu) \subset (\mathcal{J}, \mu)$ if $\mathcal{I} \subset \mathcal{J}$.

Lemma 6.2.3. (*Giraud, Villamayor*) Let (\mathcal{I}, μ) be a marked ideal and $C \subset \text{cosupp}(\mathcal{I}, \mu)$ be a smooth center and $r \leq \mu$. Let $\sigma : X \leftarrow X'$ be a blow-up at C . Then

$$\sigma^c(\mathcal{D}^r(\mathcal{I}, \mu)) \subseteq \mathcal{D}^r(\sigma^c(\mathcal{I}, \mu)).$$

Proof. See simple computations using chain rule in [69], [72].

Lemma 6.2.4. Let ϕ be any étale morphism. Then $\phi^*(\mathcal{D}^a(\mathcal{I})) = \mathcal{D}^a(\phi^*(\mathcal{I}))$ for any $a \in \mathbb{N}$.

6.3. Standard basis along Samuel stratum. The implicit function theorem proven in the previous sections allows relaxing the condition of the Hironaka standard basis. This idea was first used in the papers of Bierstone-Milman [9], [10] and applied to functions in formal coordinate charts. Our construction is expressed in a different language which is closed to Hironaka's Henselian Theorem.

Recall that the *Samuel stratum* S through a closed point $x \in X$ on a scheme (or an analytic or differentiable manifold) X is a locally closed subset $S \subset X$ consisting of all the closed points $y \in X$ with the same Hilbert-Samuel function $H_{x,X} = H_{y,Y}$,

If \mathcal{I} is an ideal sheaf of finite type on a smooth scheme (or an analytic/differentiable manifold) X then Samuel stratum of \mathcal{I} on X is a locally closed subset S of X , such that $H_{x,\mathcal{I}} = H_{y,\mathcal{I}}$ for any two closed points $x, y \in S$.

By using the singular implicit function theorem we are going to construct a standard basis of any ideal sheaf of finite type on X along Samuel stratum.

Consider a monotone diagram Δ with vertices $\alpha_1, \dots, \alpha_k$.

Lemma 6.3.1. *If Δ is monotone in \mathbb{N}^n and $\mathbb{N}^s \subset \mathbb{N}^n$ is the smallest “sublattice” containing all the vertices $\alpha_1, \dots, \alpha_k$ then the vertices span \mathbb{N}^s . Moreover, if there is a vertex α_i with s -coordinate nonzero, then for any $s' \leq s$ there is a vertex α_j with s' -coordinate not zero and $|\alpha_j| \leq |\alpha_i|$.*

Proof. Let $\alpha_i = (\alpha_{i1}, \dots, \alpha_{is})$ be a vertex with $\alpha_{is} \neq 0$. Such a vertex exists by the minimality of \mathbb{N}^s . Then we need to show that for any $s' < s$ there exists a vertex with nonzero s' -coordinate. It follows by monotonicity that $\beta_i := (\alpha_{i1}, \dots, \alpha_{i,s'-1}, \alpha_{is'} + \dots + \alpha_{is}, 0, \dots, 0) \in \Delta$. Then $\beta_i = \alpha_j + \gamma$ for some vertex α_j with $|\alpha_j| < |\alpha_i|$ and $\gamma \in \mathbb{N}^n$. If $\alpha_{js'} = 0$ it follows that α_j has all coordinates not greater than the vertex α_i . This implies that $\alpha_i = \alpha_j + \gamma'$, where $\gamma' \in \mathbb{N}^n$, which contradicts the definition of vertex. \square

Let u_1, \dots, u_n be a coordinate system on a smooth scheme (or a manifold) X . Consider now a monotone diagram $\Delta \subset \mathbb{N}^s$, for $s \leq n$ with vertices $\alpha_1, \dots, \alpha_k$ corresponding to the functions f_1, \dots, f_k at a point $x \in X$, and suppose each f_i has the form $f_i = u^{\alpha_i} + r_i$ (in other words, the monomial u^{α_i} occurs in the expansion of f_i at x).

Assume now that $\alpha_1, \dots, \alpha_k$ span \mathbb{N}^s . For any coordinate u_i consider a vertex $\alpha_{j(i)}$ with i -th coordinate $\alpha_{j(i)i}$ nonzero, and suppose $\alpha_{j(i)i} = p^{k_i} b_i$, where p is the characteristic of K , and p does not divide b_i . Set

$$a_i := \begin{cases} p^{k_i} & \text{if } \text{char}(K) = p, \\ 1 & \text{if } \text{char}(K) = 0, \end{cases} \quad \beta_i := \alpha_{j(i)} - a_i e_i.$$

Then $D_{u^{\beta_i}} f_{j(i)}$ has the form $D_{u^{\beta_i}} f_{j(i)} = u_i^{a_i} + \text{other terms}$.

For a given coordinate system u_1, \dots, u_n on a smooth scheme over a field K or on a manifold and a sequence of natural numbers $\bar{a} = (a_1, \dots, a_s)$ one can introduce the resultant Jacobian differential operator which plays the role of the “main Jacobian” in the Cohen-Macaulay case:

$$JR^{\bar{a}}(f_1, \dots, f_s)(x) := \text{Res} \left(\sum_{\alpha \in \mathbb{N}^s, |\alpha|=a_1} D_{u^\alpha}(f_1)(x) X^\alpha, \dots, \sum_{\alpha \in \mathbb{N}^s, |\alpha|=a_s} D_{u^\alpha}(f_s)(x) \cdot X^\alpha \right).$$

(Here $X := (X_1, \dots, X_s)$ are formal unknowns, and Res denotes the resultant as in the previous section. We shall often skip the index \bar{a} in $JR^{\bar{a}} = JR$)

Example 6.3.2. If $\bar{a} = (1, \dots, 1)$ then $JR^{\bar{a}}(f_1, \dots, f_s)$ is the usual Jacobian determinant.

Definition 6.3.3. Let $x \in X$ be a point on a smooth scheme of dimension n over a field K (respectively a complex or real analytic or differentiable manifold) with a coordinate system u_1, \dots, u_k . Let \mathcal{I} be an ideal of finite type on X . Let $\Delta \subset \mathbb{N}^s \subset \mathbb{N}^n$ be a monotone diagram with vertices $\alpha_1, \dots, \alpha_k$ ordered reverse-lexicographically, which span \mathbb{N}^s .

A set of functions $f_1, \dots, f_k \in \mathcal{I}(U)$ on an étale neighborhood U of x will be called a *standard basis of \mathcal{I} at x with respect to Δ* if:

- (1) $H_x(\mathcal{I}) = H(\Delta \times \mathbb{N}^{n-s})$.
- (2) $\text{ord}_x(f_i) = |\alpha_i|$.
- (3) $\text{supd}(f_i) \subset \{\alpha_i\} \cup (\Gamma \times \mathbb{N}^{n-s})$ and $D_{u^\alpha}(f) \equiv 1$ in a neighborhood of x .
- (4) $J^s(f_1, \dots, f_k : u^{\alpha_1}, \dots, u^{\alpha_k})(x) \neq 0$ for all $s \leq d(\Delta) + 1$.
- (5) $JR^{\bar{\alpha}}(D_{u^{\beta_1}} f_{j_1}, \dots, D_{u^{\beta_k}} f_{j_k})(x) \neq 0$.

We shall call $f_1, \dots, f_k \in \mathcal{I}(U)$ a *standard basis of \mathcal{I} on U* along Samuel stratum S if it is a standard basis at any (closed) point $y \in S$ with Hilbert-Samuel function. $H_y(\mathcal{I}) = H(\Delta \times \mathbb{N}^{n-s})$.

We shall call a coordinate system for which the above conditions hold *compatible* with the standard basis of \mathcal{I} .

In the algebraic situation we call functions f'_1, \dots, f'_k on an open Zariski neighborhood $V \subset X$ of $x \in X$ a *standard pre-basis of \mathcal{I} at x* if there is an étale neighborhood $U \rightarrow V$ of x and invertible functions c_1, \dots, c_k on U such that

$$f_1 := c_1 \cdot \sigma^*(f'_1), \dots, f_k := c_k \cdot \sigma^*(f'_k)$$

is a standard basis of \mathcal{I} at x .

Remark. The notion of standard pre-basis satisfies all the conditions of a standard basis except for (3) which is replaced with a weaker form.

It can be easily shown that the standard basis in the above sense determines at any $x \in S$ a (formal analytic) standard basis relative to a diagram of $\widehat{\mathcal{I}}_x \subset \widehat{\mathcal{O}}_{X,x} = K[[u_1, \dots, u_n]]$ in the sense of Bierstone-Milman [9],[10]. On the other hand the construction is conceived in the language which is related to Hironaka's Henselian approach ([38]).

Example 6.3.4. If C is a smooth center on X of codimension s , then we consider the diagram $\Delta \subset \mathbb{N}^s$ generated by the standard basis e_1, \dots, e_s . Then C coincides with the Samuel stratum for \mathcal{I}_C . The standard basis of \mathcal{I}_C along $S = C$ with respect Δ is a set of generators of the form $f_1, \dots, f_s \in \mathcal{I}_C$, where

$$f_i = u_i + h_i(u_{s+1}, \dots, u_n)$$

by condition (3). In this case conditions (4) and (5) and are (essentially) equivalent to $\det[\frac{\partial f_i}{\partial u_j}(x)]_{i,j=1,\dots,s} \neq 0$. (See also Example 5.3.6)

Condition (1) of Definition 6.3.3 together with other conditions implies the following

Lemma 6.3.5. *There is a natural isomorphism of vector spaces over a field K*

$$\bar{r} : K[x_1, \dots, x_n]^{\Gamma \times \mathbb{N}^{n-s}} = \{f \in K[x_1, \dots, x_n] \mid \text{supp}(f) \subseteq \Gamma \times \mathbb{N}^{n-s}\} \rightarrow K[x_1, \dots, x_n] / \text{in}_x \mathcal{I}_x,$$

induced by inclusion $K[x_1, \dots, x_n]^{\Gamma \times \mathbb{N}^{n-s}} \subset K[x_1, \dots, x_n]$.

Proof. By Lemma 5.3.2, there is a basis of $K[x_1, \dots, x_n]$ of the form

$$\{\text{in}_x(f_\beta) \mid \beta \in \Delta \times \mathbb{N}^{n-s}\} \cup \{u^\alpha \mid \alpha \in \Gamma \times \mathbb{N}^{n-s}\}$$

Since $\{\text{in}_x(f_\beta) \mid \beta \in \Delta \times \mathbb{N}^{n-s}\}$ is contained in $\text{in}_x(\mathcal{I})$ we get that \bar{r} is an epimorphism of vector spaces preserving the degrees of polynomials. But since $H_x(\mathcal{I}) = H(\Delta \times \mathbb{N}^{n-s})$ we see that \bar{r} defines an isomorphism in each gradation and thus it is an isomorphism. □

Corollary 6.3.6. *The subset*

$$\mathcal{I}_x^{\Gamma \times \mathbb{N}^{n-s}} := \{f \in \mathcal{I}_x \mid \text{supd}(f) \subseteq \Gamma \times \mathbb{N}^{n-s}\} \subset \mathcal{O}_{X,x}$$

is zero in algebraic and analytic setting and is contained in $m_{x,X}^\infty$ in the differential setting.

Proof. Assume that $f \in \mathcal{I}_x^{\Gamma \times \mathbb{N}^{n-s}}$ then $\text{in}_x(f) \in \text{in}_x(\mathcal{I}_x)$ and $\text{supd}(\text{in}_x(f)) \subset \Gamma \times \mathbb{N}^{n-s}$. By the previous Lemma, $\text{in}_x(f) = 0$. Then the germ $f_x = 0$ in algebraic and analytic setting and it is flat in the differential setting. □

Corollary 6.3.7. *Given a coordinate system and a monotone diagram Δ . The standard basis (with respect to Δ and the coordinate system) of ideal sheaf is unique if exists in algebraic and analytic setting, and it is unique up to flat functions $m_{x,X}^\infty$ at each point x in the Samuel stratum.*

Proof. If (f_i) and (f'_i) are two standard bases then by condition (3), $f_i - f'_i \in \mathcal{I}^{\Gamma \times \mathbb{N}^{n-s}}$ and we can use the previous corollary. □

It follows from Lemma 6.2.2 that Condition (2) of Definition 6.3.3 is equivalent to

$$(2') \quad D_{u^\alpha}(f_i(x)) = 0, \quad i = 1, \dots, k, \quad |\alpha| < |\alpha_i|.$$

Condition (3) of Definition 6.3.3 is coherent in the sense that it is satisfied in a neighborhood.

Lemma 6.3.8. *Condition (3) is equivalent to each of the following:*

$$(3A) \quad \text{supd}(f_i - u^{\alpha_i}) \subset \Gamma \times \mathbb{N}^{n-s}.$$

$$(3B) \quad D_{u^{\alpha_i}}(f_i) \equiv 1 \text{ and } D_{u^\alpha}(f_i) \equiv 0 \text{ for } \alpha \in \Delta \setminus \{\alpha_i\}.$$

$$(3C) \quad f_i = u^{\alpha_i} + r(f_i), \text{ where } \text{supp}(r(f_i)) \in \Gamma \times \mathbb{N}^{n-s} \text{ (with respect to a coordinate system vanishing at } x).$$

Proof. This follows from the relation between the support and differential support in Lemma 3.1.2. Observe that the set $\mathbb{N}^n \setminus (\{\alpha\} \cup \Gamma)$ is \mathbb{N}^n -invariant and condition (3) of Lemma 3.1.2 is satisfied at x . □

Condition (4) of Definition 6.3.3 is coherent and can be stated in the form

$$J^s(f_1, \dots, f_k : u^{\alpha_1}, \dots, u^{\alpha_k})(x) = \det [D_{u^{\alpha - \beta + \alpha_i(\beta)}}(f_{i(\beta)})]_{\alpha, \beta \in \Delta(s)} \neq 0$$

for all $s \leq d(\Delta) + 1$. It follows from Lemma 5.3.5 that this condition is satisfied for any standard basis at a point x .

Condition (5) of Definition 6.3.3 is coherent. It implies that the variables u_1, \dots, u_s are *essential* for the presentation of the initial forms $F_i = \text{in}_x(f_i(u_1, \dots, u_s, 0, \dots, 0))$ in the sense that there are no translations $u_i + a_i t$, where $a_i \in K$, preserving F_i . In other words, there is no linear coordinate change u'_1, \dots, u'_s such that all $F_i(u'_1, \dots, u'_s) = F_i(u'_1, \dots, u'_{s-1}, 0)$ depend only on $s - 1$ coordinates. Suppose such a translation exists for F_i . Then the forms $D_{u^{\beta_i}} F_{j(i)}$ also depend on u'_1, \dots, u'_{s-1} . In particular they have common nontrivial zeroes, and thus their resultant $\text{Res}(D_{u^{\beta_j(i)}} F_i) = JR^{\bar{\alpha}}(D_{u^{\beta_i}} f_i)(x)$ is zero.

On the other hand, this condition is automatically satisfied for the standard basis with $\exp(f_i) = \alpha_i$. In this case $\exp(D_{u^{\beta_i} F_i}) = a_i e_i$, which implies that $K[u_1, \dots, u_s]/(D_{u^{\beta_i} F_i})$ is finite as it corresponds to a finite diagram contained in $\Gamma = \prod [0, a_i - 1]$. Consequently, the forms have no nontrivial zeroes and their resultant $\text{Res}(D_{u^{\beta_j(i)}} F_i) = JR(D_{u^{\beta_i}} f_i)(x)$ does not vanish.

This also implies

Lemma 6.3.9. *The standard basis of $\mathcal{I} \subset \mathcal{E}_n$ with respect to a monotone order is a standard basis with respect to the induced diagram.* □

Remark. Note, however that the monotone order defines different diagrams along Samuel stratum.

6.4. Description of the Samuel stratum. Let \mathcal{I} be a sheaf of ideals of finite type on a smooth scheme X over K , or a complex analytic or differentiable manifold. Let \mathcal{I}^Γ denote the subsheaf generated by the functions $f \in \mathcal{I}$ with $\text{supd}(f) \subset \Gamma$. Then, by section 6.2, the set $\text{cosupp}(\mathcal{I}^\Gamma, \infty)$ is a closed subspace described as the vanishing locus $V(\mathcal{D}^\infty(\mathcal{I}^\Gamma))$.

The standard basis (and the standard pre-basis) along Samuel stratum is a coherent notion, and gives a local description of the Samuel stratum.

Theorem 6.4.1 (Existence of a weak standard basis along the Samuel stratum). *Let X be a smooth scheme of finite type over a field K (or an analytic/differentiable manifold) with a given coordinate system, and let $x \in X$ be a closed point. Let \mathcal{I} be a sheaf of ideals of finite type on X . There is an étale (respectively open) neighborhood $U \subset X$ of x and regular functions $f_1, \dots, f_k \in \mathcal{I}(U)$ which form a standard basis of \mathcal{I} along the Samuel stratum through x with respect to a monotone diagram Δ and a certain coordinate system u_1, \dots, u_n on U . Moreover:*

- (1) *The Samuel stratum through $x \in U$ can be described as*

$$S = S_x = \{y \in U \mid H_{y, \mathcal{I}} = H_{x, \mathcal{I}}\} = \{y \in U \mid \text{ord}_y(f_i) = |\alpha_i|\}.$$

In the differential setting,

$$S = S_x = \{y \in U \mid \text{ord}_y(f_i) = |\alpha_i|\} \cap \text{cosupp}(\mathcal{I}^\Gamma, \infty).$$

- (2) *$H(\Delta \times \mathbb{N}^{n-s}) = \max\{H_y(\mathcal{I}) \mid y \in U\}$ is the maximum value of the Hilbert-Samuel function on U .*

- (3) In the algebraic and complex analytic setting any function $f \in \mathcal{I}(U)$ can be uniquely written as $f = \sum h_i f_i$ where $h_i \in \mathcal{O}(U)^{\Gamma_i} = \{f \in \mathcal{O}(U) \mid \text{supd}(f) \subset \Gamma_i\}$.
- In the differential setting $f = \sum h_i f_i + r(f)$, where $r(f) \in m_S^\infty \cap \mathcal{I}^\Gamma$.
 - In the real analytic case there is a presentation $f = \sum h_i f_i$ in a neighborhood U_f of S (depending on f).

Proof. Consider a monotone order \bar{T} on \mathbb{N}^n and an étale neighborhood, possibly extending the residue field, (respectively an open neighborhood) U of x for which there exists a generic coordinate system u_1, \dots, u_n defining a monotone diagram $\Delta = \exp_x(\mathcal{I})$ at x and a standard basis f_1, \dots, f_k of \mathcal{I}_x with respect to \bar{T} (see Theorems 3.3.5, and 5.2.3). By Theorem 5.2.3, and 5.2.2, and Lemma 6.3.9 the functions f_1, \dots, f_k generate \mathcal{I}_x satisfy the conditions of Definition 6.3.3 at x . Since \mathcal{I} is of finite type, by shrinking U if necessary we can assume that:

- \mathcal{I} is generated by $f_1, \dots, f_k \in \mathcal{I}(U)$ on U .
- f_1, \dots, f_k satisfy the differential conditions (3) through (5) on U of Definition 6.3.3.
- In the algebraic, complex analytic and differential situation there exists Weierstrass-Hironaka division on U by $f_1, \dots, f_k \in \mathcal{I}(U)$.

Let

$$S_0 := \{y \in U \mid \text{ord}_y(f_i) = |\alpha_i|\}.$$

In the real analytic case we also assume that U contains a single connected component of S_0 . Let y be a closed point in X . If $y \notin S_0$ then we can find a largest integer d such that $\text{ord}_y(f_j) = |\alpha_j|$ for all f_j with $\text{ord}_y(f_j) < d$.

Then $d = \text{ord}_y(f_i) < |\alpha_i|$ for a certain f_i . Consider a coordinate system $\bar{u}_1, \dots, \bar{u}_n$, where $\bar{u}_i := u_i - u_i(y)$, vanishing at y induced by u_1, \dots, u_n (possibly after passing to an étale neighborhood, and extending the base field). It follows from Theorem 5.3.2 that a basis of $\text{gr}(\mathcal{O}_{X_y})/m_y^d \simeq \mathcal{O}_{X_y}/m_y^d$ is given by

$$\Psi_y^d := \{\bar{u}^\alpha \mid \alpha \in \Gamma, |\alpha| < d\} \cup \{\bar{u}^\alpha \text{ in}_y(f_i) \mid \alpha \in \Gamma_i, |\alpha| + |\alpha_i| < d\}$$

with the subsets $\Psi_{y,0}^d := \{\bar{u}^\alpha \mid \alpha \in \Gamma, |\alpha| < d\}$ and $\Psi_{y,1}^d := \{\bar{u}^\alpha \text{ in}_y(f_i) \mid \alpha \in \Gamma_i, |\alpha| + |\alpha_i| < d\}$ being in bijective correspondence with $\Gamma \times \mathbb{N}^{n-s}$ and $\Delta \times \mathbb{N}^{n-s}$. Since $\text{supd}(f_i) \subset \{\alpha_i\} \cup (\Gamma \times \mathbb{N}^{n-s})$ and $\text{ord}_y(f_i) < d$, we conclude that $\text{supd}(\text{in}_y(f_i))$ is contained in $\Gamma \times \mathbb{N}^{n-s}$ and thus it is linearly independent of $\Psi_{y,1}^d$, which implies that $H_y(\mathcal{I}) < H(\Delta \times \mathbb{N}^{n-s})$ with respect to the lexicographic order.

If $y \in S_0$ but $y \notin \text{cosupp}(\mathcal{I}^\Gamma, \infty)$ (in the differential setting), then there is a nonzero $g \in \mathcal{I}^\Gamma$, of a certain order e , with $\text{supd}(\text{in}_y(g)) \subset \Gamma \times \mathbb{N}^{n-s}$. Then $\text{in}_y(g)$ is not in the vector space Ψ_y^{e+1} . This implies that $H_{y, \text{in}_y(\mathcal{I})} = H_{y, \mathcal{I}} < H(\Delta)$ (proving condition (2)).

If $y \in S_0$ (or $y \in S_0 \cap \text{cosupp}(\mathcal{I}^\Gamma, \infty)$ in the differential setting) then in the algebraic and the complex analytic setting \mathcal{I} is coherent, and for any $g \in \mathcal{I}$ (also in the differential setting) there is Weierstrass-Hironaka division by f_1, \dots, f_k on U , yielding $g = \sum h_i f_i + h_0$, where $\text{supd}(h_0) \subset \Gamma$ (by Theorem 5.3.1), and thus, by Corollary 6.3.7, $h_0 \in \mathcal{I}^\Gamma$ is flat at y or $h_0 \equiv 0$ on U in the algebraic/analytic case (as $(h_0)_x \equiv 0$). We conclude that $\text{in}_y(g) = \sum H_i \text{in}_y(f_i)$.

In the real analytic situation the division exists at any point of $y \in S_0$. Using uniqueness of the extension, we can define the functions h_i in the neighborhood of S_0 . Since h_0 is zero in a neighborhood of $x \in S_0$, it is zero in a neighborhood of the connected component of S_0 through x , and again $g = \sum h_i f_i$ with $\text{in}_y(g) = \sum H_i \text{in}_y(f_i)$.

This shows that the functions in Ψ_y^∞ form a basis of $\text{in}_y(\mathcal{I})$. And since they are in bijective correspondence with the elements of $\Delta \times \mathbb{N}^{n-s}$, we conclude that $H_{y, \mathcal{I}} = H(\Delta \times \mathbb{N}^{n-s})$. \square

Remark. The theorem implies existence of a standard pre-basis of \mathcal{I} on a Zariski open neighborhood V of x , as the functions f_i on U define principal divisors whose images determine a pre-basis on V .

Lemma 6.4.2. *Let \mathcal{I} be a sheaf of ideals on X , and $C \subset S$ be a smooth center contained in the Samuel stratum S of dimension c . Consider a standard basis f_1, \dots, f_k in a neighborhood of $x \in S$ defined for a monotone diagram $\Delta \subset \mathbb{N}^s \subset \mathbb{N}^n$, with vertices spanning \mathbb{N}^s . Then there exists a coordinate system v_1, \dots, v_n which is compatible with f_1, \dots, f_k , and such that v_1, \dots, v_c with $s \leq c \leq n$ describe the ideal \mathcal{I}_C in a neighborhood of $x \in X$.*

Proof. Denote by u_1, \dots, u_n a given coordinate system compatible with the standard basis f_1, \dots, f_k . Consider any monotone order T on \mathbb{N}^n . Let v_1, \dots, v_c be the standard basis of the ideal \mathcal{I}_C at x defined for

T . Then v_1, \dots, v_c is a set of parameters describing \mathcal{I}_C , and $\exp(\mathcal{I})$ is a diagram spanned by some vectors $e_{j_1} = \exp(v_1), \dots, e_{j_c} = \exp(v_c)$ which is part of the basis of \mathbb{N}^n with $j_1 < \dots < j_c$. That is, we can express v_i as

$$v_i = u_{j_1} + r_i(u_{j_1+1}, \dots, u_n).$$

Denote by $\mu_i = \text{ord}_x(f_i)$ the multiplicity of f_i . Then $f_i \in \mathcal{I}_C^{\mu_i}$, and the initial form

$$F = \text{in}_x(f_i) = \sum c_\alpha v^\alpha$$

at x is a function of v_1, \dots, v_c , where $v_i = v_i(u_1, \dots, u_n)$. Since the coordinates u_1, \dots, u_s are essential,

$$F(u_1, \dots, u_s, 0, \dots, 0) = \text{in}_x(f_i(u_1, \dots, u_s, 0, \dots, 0)) = \sum c_\alpha v^\alpha(u_1, \dots, u_s, 0, \dots, 0)$$

is a function depending upon the s variables u_1, \dots, u_s with no translations. This implies that $(j_1, \dots, j_s) = (1, \dots, s)$, and since v_1, \dots, v_c is the standard basis of \mathcal{I}_C , it can be written in the form $v_i = u_i + f(u_{s+1}, \dots, u_n)$ for $i = 1, \dots, s$, $v_i = v_i(u_{s+1}, \dots, u_n)$ for $i = s+1, \dots, c$. This set can be extended to a coordinate system v_1, \dots, v_n with $v_i = v_i(u_{s+1}, \dots, u_n)$ for $i = s+1, \dots, n$. The coordinate change $u_i \mapsto v_i$ does not change the derivatives $D_{u_i} = D_{v_i}$ for $i = 1, \dots, s$, defining the differential support and used in conditions (4) and (5) of a standard basis. Consequently, the conditions for a standard basis are satisfied for the coordinate system v_1, \dots, v_n . \square

We shall refer to such a coordinate system v_1, \dots, v_n as *compatible* with the center C and the weak standard basis f_1, \dots, f_k .

Let $C \subset X$ be any smooth center on a smooth scheme (or an analytic or differentiable manifold X). Suppose u_1, \dots, u_c are the coordinates describing C on a neighborhood U of X . Consider a graded locally free \mathcal{O}_C -sheaf

$$\text{gr}_C(\mathcal{O}_X) = \bigoplus \mathcal{I}_C^s / \mathcal{I}_C^{s+1}$$

over C which can be described over $U \cap C$ as

$$\text{gr}_C(\mathcal{O}_U) \simeq \mathcal{O}_{C \cap U}[u_1, \dots, u_c].$$

Then any ideal \mathcal{I} defines the ideal of the initial forms

$$\text{in}_C(\mathcal{I}) := \bigoplus (\mathcal{I} \cap \mathcal{I}_C^s) / (\mathcal{I} \cap \mathcal{I}_C^{s+1}) = \bigoplus ((\mathcal{I} \cap \mathcal{I}_C^s) + \mathcal{I}_C^{s+1}) / \mathcal{I}_C^{s+1} \subset \text{gr}_C(\mathcal{O}).$$

Theorem 6.4.3 (Hironaka's normal flatness theorem). *Let X be a smooth scheme of finite type over a field K (or an analytic or differentiable manifold). The following conditions are equivalent for a smooth subvariety (submanifold) $C \subset X$:*

- (1) $H_{x, \mathcal{I}}$ is locally constant along C (defined for closed points).
- (2) $\text{gr}_C(\mathcal{O}_X) / \text{in}_C(\mathcal{I})$ is a locally free \mathcal{O}_C -module.

Moreover, if $\{f_1, \dots, f_k\}$ is a standard basis of \mathcal{I} in a neighborhood of $x \in C$ and u_1, \dots, u_n is a compatible coordinate system with the standard basis and the center so that $\mathcal{I}_C = (u_1, \dots, u_c)$ for $c \geq s$, then the set

$$\{u^\alpha \mid \alpha \in \Gamma \times \mathbb{N}^{c-s}\}$$

is a basis of the free \mathcal{O}_C -module $\text{gr}_C(\mathcal{O}_X / \mathcal{I}_C)$ on $U \cap C$ for some étale (respectively open) neighborhood U of $x \in X$. On the other hand, the set

$$\{u^\alpha \text{in}_C(f_i) \mid \alpha \in \Gamma_i \times \mathbb{N}^{c-s}\}$$

is a basis of the free \mathcal{O}_C -module $\text{in}_C(\mathcal{I})$ on $U \cap C$.

Proof. (1) \Rightarrow (2). Let $C \subset S$ be any smooth center contained in the Samuel stratum on X , and let $x \in C$. Let f_1, \dots, f_k be a standard basis of \mathcal{I} in a neighborhood of x corresponding to a certain monotone diagram $\Delta \in \mathbb{N}^s$ (it exists by Theorem 6.4.1).

By Lemma 6.4.2, we can find a coordinate system $u := (u_1, \dots, u_n)$ compatible with f_1, \dots, f_k and such that (u_1, \dots, u_c) define (locally) the ideal \mathcal{I}_C of C . Denote by $v := (u_{c+1}, \dots, u_n)$ the remaining coordinates. By Theorem 6.4.1, each f_i has a constant multiplicity μ_i along S and thus along C . Consequently, by Lemma 6.1.4, $f_i \in \mathcal{I}_C^{\mu_i}$. Write $\text{in}_C(f_i) = F_i(u_1, \dots, u_c) \in \mathcal{O}_C(v)[u_1, \dots, u_c]$ as the form of degree μ_i with coefficients in $\mathcal{O}_C(v)$. Thus $\text{in}_x(f_i) = \text{in}_C(f_i)(x)$, where $\text{in}_C(f_i)(x) = F_i(v(x))(u_1, \dots, u_c)$ denotes the evaluation of the \mathcal{O}_C -form $\text{in}_C(f_i) = F_i(u_1, \dots, u_c)$ at $x \in C$.

In particular, $\text{in}_x(f_i) \in K[u_1, \dots, u_c]$, and $\{u^\alpha \mid \alpha \in \Gamma \times \mathbb{N}^{c-s}\} \cup \{u^\alpha F_i(v(x)) \mid \alpha \in \Gamma_i \times \mathbb{N}^{c-s}\}$ is a basis of the evaluation $K[u_1, \dots, u_c]$ of $\text{gr}_C(\mathcal{O}_X) = \mathcal{O}_C[u_1, \dots, u_c]$ at $x \in C$.

By Lemma 4.1.14, the set $\{u^\alpha \mid \alpha \in \Gamma \times \mathbb{N}^{c-s}\} \cup \{u^\alpha F_i \mid \alpha \in \Gamma_i \times \mathbb{N}^{c-s}\}$ is a basis over $\mathcal{O}_{C \cap U}$ of $\text{gr}_C(\mathcal{O}_U) = \mathcal{O}_{C \cap U}[u_1, \dots, u_c]$ in an open neighborhood U of $x \in X$. Then for any $F \in \text{in}_C(\mathcal{I}|_U)$ we can write uniquely

$$F = \sum H_i \text{in}_C(f_i) + H_0,$$

where the functions H_i are homogeneous in $\text{gr}_C(\mathcal{O}_U)$, and $\text{supd}(H_i) \subset \Gamma_i \times \mathbb{N}^{c-s}$. Since $\text{in}_C(F_i) \in \text{in}_C(\mathcal{I})$, the set $\{u^\alpha \mid \alpha \in \Gamma \times \mathbb{N}^{c-s}\}$ generates $\mathcal{O}_{C \cap U}[u_1, \dots, u_c]/\text{in}_C(\mathcal{I})$, and there is an epimorphism

$$\phi: \bigoplus_{\alpha \in \Gamma \times \mathbb{N}^{c-s}} \mathcal{O}_{C \cap U} \cdot u^\alpha \rightarrow \mathcal{O}_{C \cap U}[u_1, \dots, u_c]/\text{in}_C(\mathcal{I}).$$

The kernel of ϕ consists of all \mathcal{O}_C -forms F with $\text{supd}(F) \subset \Gamma \times \mathbb{N}^{c-s}$. If there is a nonzero \mathcal{O}_C -form $F = \text{in}_C(f) \in \text{in}_C(\mathcal{I})$ with $\text{supd}(F) \subset \Gamma \times \mathbb{N}^{c-s}$ then its evaluation $F(y)$ at some closed point $y \in C \cap U$ is not zero. But this implies that $\text{supd}(\text{in}_y(f)) \subset \Gamma$ and is linearly independent of

$$\{u^\alpha \cdot \text{in}_y(f_i)(u) \mid \alpha \in \Gamma_i \times \mathbb{N}^{n-k}\},$$

and $H_y(\mathcal{I}) < H(\Delta \times \mathbb{N}^{n-k}) = H_x(\mathcal{I})$, which contradicts the assumption.

Consequently, the kernel of ϕ is trivial, and ϕ is an isomorphism. This also implies that any $F \in \text{in}_C(\mathcal{I})$ can be uniquely written as $F = \sum H_i \text{in}_C(f_i)$, where $\text{supd}(H_i) \subset \Gamma_i \times \mathbb{N}^{c-s}$. This proves the implication (1) \Rightarrow (2) and the “moreover” part of the theorem.

(2) \Rightarrow (1). Observe that $\text{gr}_C(\mathcal{O}_X) = \mathcal{O}_c[u_1, \dots, u_c]$ is a graded \mathcal{O}_c -module (with the standard grading), and its evaluation at $x \in C$ is a graded K -module $K[u_1, \dots, u_c]$ for a base field K . Let $\text{gr}_C(\mathcal{I})_x \subset K[u_1, \dots, u_c]$ be the evaluation of the ideal $\text{gr}_C(\mathcal{I})$ at $x \in C$. Consider a monotone monomial order \overline{T} such that after a generic change of coordinates the diagram $\Delta := \exp_{\overline{T}}(\text{gr}_C(\mathcal{I})(x))$ is monotone.

Let $f_1, \dots, f_k \in \text{in}_C(\mathcal{I})$ be the homogeneous polynomials over \mathcal{O}_C such that for their evaluations $\overline{f}_1, \dots, \overline{f}_k$ at x , the initial exponents $\exp_{\overline{T}}(\overline{f}_i) = \alpha_i$ are the vertices of Δ . Then the corresponding set

$$\{u^\alpha \mid \alpha \in \Gamma \times \mathbb{N}^{c-s}\} \cup \{u^\alpha \overline{f}_i(x) \mid \alpha \in \Gamma_i \times \mathbb{N}^{c-s}\}$$

is a basis of $K[u_1, \dots, u_c]$. Thus, by Lemma 4.1.14,

$$\{u^\alpha \mid \alpha \in \Gamma \times \mathbb{N}^{c-s}\} \cup \{u^\alpha f_i(x) \mid \alpha \in \Gamma_i \times \mathbb{N}^{c-s}\}$$

is a basis of the free $\mathcal{O}_{C \cap U}$ -module $\mathcal{O}_{C \cap U}[u_1, \dots, u_c]$. On the other hand, the module $\mathcal{O}_C[u_1, \dots, u_c]/\text{in}_C(\mathcal{I})$ is locally free (so we may assume it is free on U). Moreover, as before there is an epimorphism

$$\phi: \bigoplus_{\alpha \in \Gamma \times \mathbb{N}^{c-s}} \mathcal{O}_{C \cap U} \cdot u^\alpha \rightarrow \mathcal{O}_{C \cap U}[u_1, \dots, u_c]/\text{in}_C(\mathcal{I}),$$

which is an isomorphism after evaluating at x , which means that both the free \mathcal{O}_C -modules have the same rank in each grading. Thus the epimorphism ϕ is an isomorphism. Then as before each element in $\text{gr}_C(\mathcal{I})$ can be written as $f = \sum h_i f_i$, where $\text{supd}(h_i) \subset \Gamma_i$. This implies that each function in \mathcal{I} can be written as $f \simeq \sum h_i f_i \pmod{\mathcal{I}_c^s}$ up to a power \mathcal{I}_c^s for any $s \gg 0$. Consequently, the initial form $\text{in}_y(f) \in \text{in}_y(\mathcal{I})$ can be written as $\text{in}_y(f) = \sum \text{in}_y(h_i) \text{in}(f_i)$, where $\text{supd}(\text{in}_y(h_i)) \subset \Gamma_i \times \mathbb{N}^{n-c}$.

Thus the Hilbert-Samuel function $H_y(\mathcal{I}) = H(\Delta \times \mathbb{N}^{n-c})$ at any point $y \in C$ is determined by the diagram $\Delta \times \mathbb{N}^{n-c}$ and is the same for all closed points $y \in U \cap C$. \square

6.5. Hilbert-Samuel function. To deduce the important Bennett theorem, we shall need a useful extension of Bierstone-Milman result [8] Corollary 5.2.2. In the Bierstone-Milman paper [8] a (stronger) pointwise order is considered for the set of Hilbert-Samuel functions. They show that the set of Hilbert-Samuel functions with pointwise order has d.c.c. property (see below). Here we need a similar result for a (weaker) lexicographic order. The main idea of the proof to use diagrams of initial exponents, remains the same.

Theorem 6.5.1 (Descending chain condition of the Hilbert-Samuel function). *The set of values of the Hilbert-Samuel function (ordered lexicographically)*

$$\mathcal{H}(n) := \{H_{\mathcal{I}}(k) = \dim(K[x_1, \dots, x_n]/(\mathcal{I} + \mathfrak{m}^{k+1})) \mid \mathcal{I} \subset K[x_1, \dots, x_n]\}$$

is d.c.c. (satisfies the descending chain condition). In other words, any decreasing sequence of functions

$$(6.1) \quad H_1 \geq \dots \geq H_n \geq \dots$$

stabilizes: $H_s = H_{s+1} = \dots$ for sufficiently large s .

Proof. Consider any normalized and total order \bar{T} on \mathbb{N}^n . For any finite subset $\alpha := \{a^1, \dots, a^s\}$ in \mathbb{N}^n denote by $\Delta(\alpha) := \Delta(a^1, \dots, a^s) := \bigcup a^i + \mathbb{N}^n$ the diagram of the initial exponents defined by a^i .

Let $H(\alpha)(k) := H(\Delta(a^1, \dots, a^s))$ denote the corresponding Hilbert-Samuel function. Then let

$$\mathcal{H}(n) := \{H(\alpha)(k) \mid \alpha := (a^1, \dots, a^s) \in \mathbb{N}^n\}$$

be the set of all possible Hilbert-Samuel functions obtained that way. We shall assume that the elements of α are ordered: $a^1 < \dots < a^s$ with respect to \bar{T} , and that $a^i \notin \bigcup_{j=1}^{i-1} a^j + \mathbb{N}^n$. Denote by S the set of all finite subsets of \mathbb{N}^n of that form, and by S_k those subsets in S for which all elements have multiplicity $|a^i| \leq k$.

Write (H_i) from the sequence (6.1) as $(H(\alpha_i))$, where (α_i) is the corresponding sequence of finite subsets of \mathbb{N}^n . For any $\alpha = (a^1, \dots, a^s)$ set $\alpha + \mathbb{N}^n := \bigcup a^i + \mathbb{N}^n$.

Immediately from the definition we see that if $\alpha \subset \beta$ then $\alpha + \mathbb{N}^n \subset \beta + \mathbb{N}^n$ and $H(\alpha) \geq H(\beta)$. Also if $\alpha \subseteq \beta$ then for $b \in \beta \setminus \alpha$, we have $b \notin \alpha + \mathbb{N}^n$.

For any finite subset α of \mathbb{N} we define its “restriction” to be the subset

$$\text{res}_k(\alpha) = \{a \in \alpha \mid |a| \leq k\} \in S_k.$$

By the *truncated Hilbert-Samuel function* $H^{\leq k}$ we mean the truncation of $H : \mathbb{N} \rightarrow \mathbb{N}$ to the set $\{1, \dots, k\}$.

We see that

$$H(\text{res}_k(\alpha)) \geq H(\alpha) \quad \text{and} \quad H^{\leq k}(\alpha) = H^{\leq k}(\text{res}_k(\alpha)).$$

Since each set S_k is finite, we can construct by induction a sequence $\beta_k \in S_k$ such that:

- (1) $\text{res}_{k-1}(\beta_k) = \beta_{k-1}$.
- (2) For any β_k there exist infinitely many α_i from the sequence such that $\text{res}(\alpha_i) = \beta_k$.

Thus we get a sequence $\beta_1 \subset \beta_2 \subset \dots$. But the relevant sequence

$$\beta_1 + \mathbb{N}^n \subset \beta_2 + \mathbb{N}^n \subset \beta_m + \mathbb{N}^n = \beta_{m+1} + \mathbb{N}^n = \dots$$

stabilizes since it corresponds to an increasing chain of monomial ideals in the Noetherian ring $K[x_1, \dots, x_n]$. This implies that the sequence of the sets of vertices $\beta_m = \beta_{m+1} = \dots$ stabilizes. In other words, there is an infinite sequence α_{i_j} such that $\text{res}_j(\alpha_{i_j}) = \beta_m$ for $j \geq m$. We show that $\alpha_{i_j} = \beta_m$ for $j \geq m$. Observe that if $\beta_m \subsetneq \alpha_{i_{j_0}}$ then $\beta_m \subsetneq \text{res}_j(\alpha_{i_j})$ for sufficiently large $j > j_0$ and

$$H^{\leq j}(\alpha_{i_j}) > H^{\leq j}(\beta_m).$$

Since $\beta_m = \text{res}_j(\alpha_{i_j})$ we have the contradiction:

$$H^{\leq j}(\beta_m) = H^{\leq j}(\alpha_{i_j}) \geq H^{\leq j}(\alpha_{i_{j_0}}) > H^{\leq j}(\beta_m).$$

This shows that $\alpha_{i_{j_1}} = \alpha_{i_{j_2}} = \dots$ stabilizes, and consequently $H_i = H(\alpha_i)$ stabilizes for $i \geq i_{j_m}$. □

Corollary 6.5.2. *There exist only finitely many diagrams of initial exponents Δ in \mathbb{N}^k having the same Hilbert-Samuel function.*

Proof. We use the same notation as in the previous proofs. Suppose there exist infinitely many diagrams Δ_i with the same Hilbert-Samuel function. For each s one can find the subsets α_s of $\mathbb{N}_{\leq s}^k := \{x \in \mathbb{N}^k \mid |x| \leq s\}$, such that α_s is the restriction of infinitely many of Δ_i . Each of the subsets generates the diagram β_s . Then as before it leads to an infinite sequence of the subsets $\beta_1 \subset \beta_2 \subset \dots \subset \beta_i \subset \dots$, such that $\text{res}_{i-1}\beta_i = \beta_{i-1}$ for all $i = 1, \dots$. But then the set $\beta := \bigcup \beta_i$ corresponds to a finitely generated ideal $I_\beta := \{x^a \mid a \in \beta\}$ in $K[x_1, \dots, x_n]$. Thus the sequences (I_{β_i}) (and (β_i)) stabilize $\beta_i = \beta_{i+1} = \dots$ and define the same diagram. □

The following theorem extends Bennett’s result to the differential (and analytic) setting.

Corollary 6.5.3 (Bennett [15]). *Let X be a smooth scheme of finite type over a field K (or a compact analytic or differentiable manifold). Let \mathcal{I} be a sheaf of ideals of finite type on X . Then the Hilbert-Samuel function $H_{x, \mathcal{I}}(k)$ of \mathcal{I} on X is upper semicontinuous and attains only finitely many values. Consequently, there is a finite Samuel decomposition into locally closed strata such that two closed points are in the same stratum if they have the same Hilbert-Samuel function $H_{\mathcal{I}, x}(s) = \dim_{K_x} \mathcal{O}_X / (m_x^{s+1} + \mathcal{I}_x)$.*

Proof. Let us consider the case of a perfect field in the algebraic setting. It follows from Theorem 6.4.1 that the set

$$\{x \in X \mid H_{x,\mathcal{I}} \leq H\}$$

is open. Since X is quasi-compact in Zariski topology (or in the case of compact manifold), we conclude that the Hilbert-Samuel function attains its maximum value on X , and consequently, by d.c.c., it has finitely many values. This proves the theorem over a perfect field.

In general, we can pass to an algebraic closure \overline{K} , where we find the finite Samuel decomposition of $\overline{X} := X \times_{\text{Spec } K} \text{Spec } \overline{K}$ which is stable under the action of the Galois group $\text{Gal}(\overline{K}/K)$. This implies that it descends to the Samuel decomposition of X . \square

Corollary 6.5.4 (Bennett [15]). *Let X be any scheme of finite type over a field K (or a compact analytic space or a differentiable space). Then there exists a finite Samuel decomposition of X into a locally closed Samuel strata such that two closed points are in the same stratum if they have the same Hilbert-Samuel function $H_{X,x}(s) = \dim_{K_x} \mathcal{O}_X/m_x^{s+1}$.*

Proof. We can locally embed X into a smooth scheme over \mathbf{A}_K^n or into K^n , where $K = \mathbb{R}, \mathbb{C}$, and apply the previous theorem. \square

The following theorem shows that the problem of resolution of singularities controlled by the Hilbert-Samuel function can be reduced to the desingularization of marked ideals (f_i, μ_i) defined for the standard basis. Thus the standard basis is a counterpart of Hironaka's distinguished data or Bierstone-Milman's semicoherent presentation of ideals (see [38], [9], [10]).

Theorem 6.5.5 (Stability of standard basis under blow-ups). *Let \mathcal{I} be any ideal sheaf of finite type on a smooth variety (or an analytic or differentiable manifold) X , and let $C \subset X$ be a smooth subvariety. Consider the blow-up $\sigma : X' \rightarrow X$ at a smooth closed center $C \subset X$ contained in the Samuel stratum, and let \mathcal{I}' be its strict transform. Let $U' \subset \sigma^{-1}(U)$ be an open subset where a coordinate y on U describes the exceptional divisor of σ . Then:*

- (1) *If f_1, \dots, f_k is a standard basis of \mathcal{I} on an étale (respectively open) neighborhood U of X with respect to a monotone diagram Δ then*

$$f'_1 := \sigma^*(f_1)/y^{\mu_1}, \dots, f'_k := \sigma^*(f_k)/y^{\mu_k}$$

is a standard basis of \mathcal{I}' on U' with respect to Δ and the induced coordinate systems.

- (2) (Bennett) $H_{x,\mathcal{I}} \geq H_{x',(\mathcal{I}')}.$

Proof. Let u_1, \dots, u_n be local parameters compatible with the standard basis and the center C (Lemma 6.4.2). Here the coordinates u_1, \dots, u_c define C locally.

Let μ_i denote the multiplicity of the function f_i . It follows from Lemma 6.1.4 that $f_i \in \mathcal{I}^{\mu_i} \setminus \mathcal{I}_C^{\mu_i+1}$ and correspondingly $\sigma^*(f) \in \mathcal{I}_D^{\mu_i} \setminus \mathcal{I}_D^{\mu_i+1}$, where $D = \sigma^{-1}(C)$ is the exceptional divisor of σ . Then $f'_i := (1/y^{\mu_i})\sigma^*(f_i)$. Consider the effect of the blow-up of C near a point $x \in X$.

The points x' in $\sigma^{-1}(x)$ are defined by the lines in the normal space N_C , where $N_C^* = \mathcal{I}_C/\mathcal{I}_C^2$. For the line $t[a_1, \dots, a_k]$, where $a_i \in K$, we consider the dual hyperplane in N_C^* and the corresponding linear system. More precisely, let $r = \max\{i : a_i \neq 0\}$ and consider the change of coordinates

$$(6.2) \quad \bar{u}_1 = u_1 - (a_1/a_r)u_r, \dots, \bar{u}_{r-1} = u_{r-1} - (a_{r-1}/a_r)u_r, \bar{u}_r = u_r, \bar{u}_{r+1} = u_{r+1}, \dots, \bar{u}_n = u_n.$$

The effect of the blow-up at x' (in new coordinates) is described by

$$u'_1 = \bar{u}_1/y, \dots, u'_r = \bar{u}_r/y, u'_r = \bar{u}_r = y, u'_{r+1} = \bar{u}_{r+1}/y, \dots, u'_n = \bar{u}_n/y$$

with the exceptional divisor $y = \bar{u}_r$. Since $f_i \in \mathcal{I}_C^{\mu_i} \setminus \mathcal{I}_C^{\mu_i+1}$, we can write $f_i = \sum_{|\alpha| \geq \mu_i} c_{i\alpha}(v)\bar{u}^\alpha$, where $\bar{u} := (\bar{u}_1, \dots, \bar{u}_c)$ and $v := (\bar{u}_{k+1}, \dots, \bar{u}_n)$. Then the initial form with respect to the \mathcal{I}_C -grading is

$$F_i(v, \bar{u}) = \text{in}_C(f_i) = \sum_{|\alpha| = \mu_i} c_{i\alpha}(v)\bar{u}^\alpha,$$

and $f_i = F_i + G_i$, where $G_i \in \mathcal{I}_C^{\mu_i+1}$ and $\sigma^*(G_i) = y^{\mu_i+1}G'$. For any $\alpha = (\alpha_1, \dots, \alpha_c) \in \mathbb{N}^c$ define

$$\alpha' := (\alpha_1, \dots, 0_r, \dots, \alpha_c)$$

with 0 as the r -th coordinate. The transformed function will have the form

$$f'_i := (1/y^{\mu_i})\sigma^*(f_i) = (1/y^{\mu_i}) \sum \sigma^*(c_i)(u')^{\beta'} \cdot y^{|\beta|-\mu_i} = F_i(v)(u'_1, \dots, 1_r, \dots, u'_c) + yG'_i(v, u'_1, \dots, u'_c).$$

Let

$$\bar{F}_i(\bar{u}_1, \dots, \bar{u}_k) := \text{in}_x(f_i) = F_i(v(x)) = \sum_{|\alpha|=\mu_i} c_{i\alpha}(v)(x)\bar{u}^\alpha$$

be the initial form at x . Then $\bar{F}_i(\bar{u}_1, \dots, \bar{u}_c)$ is a form of degree μ_i . Consider the transform of \bar{F}_i under the blow-up:

$$F'_i(u'_1, \dots, u'_c) = (1/y^{\mu_i})\sigma^*(F_i(\bar{u}_1, \dots, \bar{u}_c)) = \bar{F}_i(u'_1, \dots, 1_r, \dots, u'_c).$$

This implies that $\text{ord}_{x'}(f'_i) = \text{ord}_{x'}((1/y^{\mu_i})\sigma^*(f_i)) \leq \text{ord}_x(\bar{F}_i(u'_1, \dots, 1_r, \dots, u'_c)) \leq \mu_i$.

Suppose that $\text{ord}_{x'}(f'_i) = \mu_i$ for all i . Then $\deg(F'_i) = \deg(F_i)$ and we conclude that \bar{F}_i does not contain \bar{u}_r , that is,

$$\bar{F}_i(u'_1, \dots, 0_r, \dots, u'_c) = \bar{F}_i(\bar{u}_1, \dots, \bar{u}_r, \dots, \bar{u}_c) = F_i(\bar{u}_1, \dots, 0_r, \dots, \bar{u}_c) = \bar{F}_i(u_1, \dots, 0_r, \dots, u_c).$$

Since the variables u_1, \dots, u_s are essential, this implies that $r \geq s + 1$. The set of coordinates $\bar{u}_1, \dots, \bar{u}_n$ is then compatible with the standard basis, and the derivatives $D_{u_i} = D_{\bar{u}_i}$ are the same for $i = 1, \dots, s$. Moreover by the chain rule $D_{u_i} = D_{\bar{u}_i} = (1/y)D_{u'_i}$ (see for instance [68]). The latter implies that the differential conditions (3) through (5) in the definition of a standard basis are preserved after the blow-up in the new coordinate system u'_1, \dots, u'_n . For instance for the second part of condition (3) we can write

$$D_{u'^{\alpha_i}}(f'_i) = y^{|\alpha_i|} D_{\bar{u}^{\alpha_i}} y^{-|\alpha_i|} \sigma^*(f_i) \equiv 1$$

with the natural identification of $\sigma^*(f_i)$ with f_i . The other differential conditions (4) and (5) follow in the same way. In other words, $\text{supd}(f'_i) \subset \Gamma \times \mathbb{N}^{n-s}$. To prove that f'_i is a standard basis of \mathcal{T}' at x' with respect to the diagram Δ , we need to show that $H(\Delta) = H_{x', X'}(\mathcal{T}')$. We show this in a series of lemmas below.

Now suppose $\text{ord}_{x'}(f'_i) < \mu_i$ for some i and let

$$d = \mu_j := \min\{\mu_i : \text{ord}_{x'}(f'_i) < \mu_i\}.$$

If such a d does not exist, that is, $\text{ord}_{x'}(f'_i) = \mu_i$ for all i , then set $d = \infty$.

Lemma 6.5.6. *If $r \leq s$ then all the vertices α_i with $|\alpha_i| < d$ are in \mathbb{N}^{r-1} .*

Proof. Let α_i be a vertex of Δ with $|\alpha_i| = \mu_i < d$. Suppose it has a coordinate $s \geq r$ which is not zero. Then by Lemma 6.3.1 there is another vertex α_{i_1} with r -th coordinate not zero and $|\alpha_{i_1}| \leq |\alpha_i| < d$. But since $D_{u^{\alpha_{i_1}}}(f_{i_1}) = 1$, the initial form \bar{F}_{i_1} depends on u_r , and $\text{ord}(\bar{F}_{i_1}(u'_1, \dots, 1_r, \dots, u'_c)) < \mu_{i_1} < d$, which contradicts the assumption on d . \square

Consider the monotone diagram Δ^{r-1} generated by the vertices with $|\alpha_i| < d$. Then Δ^{r-1} is contained in \mathbb{N}^{r-1} , and let $\Gamma^{r-1} = \mathbb{N}^{r-1} \setminus \Delta^{r-1}$. By the lemma, both sets $\Gamma \times \mathbb{N}^{n-s} \subset \Gamma^{r-1} \times \mathbb{N}^{n-r+1}$ coincide for the exponents $\alpha \in \mathbb{N}^s$ of degree $|\alpha| < d$.

Lemma 6.5.7. *The set*

$$\{(u')^\alpha \text{in}_{x'}(f'_i) \mid \alpha \in \Gamma_i^r \times \mathbb{N}^{n-r+1}, |\alpha| + |\alpha_i| < d\} \cup \{(u')^\alpha \mid \alpha \in \Gamma^r \times \mathbb{N}^{n-r+1}, |\alpha| < d\}$$

is a basis of the $K[u'_r, \dots, u'_n]$ -module $K[u'_1, \dots, u'_n]/(u'_1, \dots, u'_{r-1})^d$ if d is finite and of $K[u'_1, \dots, u'_n]$ if $d = \infty$.

Proof. If $r \leq s$ then for any i , the vertex α_i with $\mu_i < d$ is in \mathbb{N}^{r-1} . If $r > s$ then the vertices belong to \mathbb{N}^s and we shall simply replace $\Gamma_i^{r-1} \times \mathbb{N}^{n-r+1}$ with $\Gamma_i^s \times \mathbb{N}^{n-s}$ in the considerations below.

For $\mu_i < d$ we have $\text{in}_{x'}(f_i) = \bar{F}_i + yG_i$, and

$$\text{in}_{x'}(f_i)(u'_1, \dots, u'_{r-1}, 0, u'_{r+1}, \dots, u'_s) = \bar{F}_i(\bar{u}_1, \dots, \bar{u}_{r-1}, 0, \bar{u}_{r+1}, \dots, \bar{u}_c) = \bar{F}_i(\bar{u}_1, \dots, \bar{u}_c).$$

By the assumption,

$$\{u^\alpha \bar{F}_i \mid \alpha \in \Gamma_i^{r-1} \times \mathbb{N}^{n-r+1}, |\alpha| + |\alpha_i| < d\} \cup \{u^\alpha \mid \alpha \in \Gamma^{r-1} \times \mathbb{N}^{n-r+1}, |\alpha| < d\}$$

is a basis of the $K[u_r, \dots, u_n]$ -module $K[u_1, \dots, u_n]/(u_1, \dots, u_{r-1})^d$.

Since $\bar{u}_i - u_i$ is divisible by $y = u_r$ we conclude, by the proof of Lemma 4.1.8, that

$$\{\bar{u}^\alpha \bar{F}_i \mid \alpha \in \Gamma_i^{r-1} \times \mathbb{N}^{n-r+1}, |\alpha| + |\alpha_i| < d\} \cup \{\bar{u}^\alpha \mid \alpha \in \Gamma^r \times \mathbb{N}^{n-r+1}, |\alpha| < d\}$$

is also a basis of the $K[u_r, \dots, u_n]$ -module $K[u_1, \dots, u_n]/(u_1, \dots, u_{r-1})^d$.

Similarly the differences

$$(u')^\alpha \text{in}_{x'}(f_i)(u'_1, \dots, u'_c) - (u')^\alpha \bar{F}_i(u'_1, \dots, u'_c)$$

are divisible by $y = u'_r$, and thus again

$$\{(u')^\alpha \text{in}_{x'}(f'_i) \mid \alpha \in \Gamma_i^r \times \mathbb{N}^{n-r+1}, |\alpha| + |\alpha_i| < d\} \cup \{(u')^\alpha \mid \alpha \in \Gamma^r \times \mathbb{N}^{n-r+1}, |\alpha| < d\}$$

is a basis of the $K[u'_r, \dots, u'_n]$ -module $K[u'_1, \dots, u'_n]/(u'_1, \dots, u'_{r-1})^d$. \square

Lemma 6.5.8. *The set*

$$\{u^\alpha \text{in}_{x'}(f'_i) \mid |\alpha| + \mu_i < d, \alpha \in \Gamma_i \times \mathbb{N}^{n-s}\}$$

is a basis of the K -space $\{\text{in}_{x'}(f) \mid f \in \mathcal{I}'_{x'}, \text{ord}_{x'}(f) < d\}$ if d is finite and of $\text{in}_{x'} \mathcal{I}'$ if $d = \infty$.

Proof. Any function $f \in \mathcal{I}'_{x'}$ of order e can be written as a combination $f = \sum c_i \sigma^*(g_i)/y^{k_i}$ of $\sigma^*(g_i)/y^{k_i}$, where $g_i \in \mathcal{I}_x$, $y^{k_i} \mid \sigma^*(g_i)$ and $c_i \in \mathcal{O}_{X', x'}$. One can approximate c_i up to m_x^{e+1} by a polynomial $p_i(u'_1, \dots, u'_n)$ of the form

$$p_i(u'_1, \dots, u'_n) = \sigma^*(p_i(\bar{u}_1/\bar{u}_r, \dots, \bar{u}_r, \dots, \bar{u}_c/u_r, \dots, \bar{u}_n)) = (1/y^{\bar{k}_i}) \sigma^* \bar{p}_i(\bar{u}_1, \dots, \bar{u}_n)$$

for suitable polynomials \bar{p}_i .

Then f can be approximated by

$$\sum p_i \sigma^*(g_i)/y^{k_i} = \sum \sigma^*(\bar{p}_i) \sigma^*(g_i)/y^{k_i + \bar{k}_i} = (1/y^k) \sigma^* \left(\sum \bar{p}_i g_i \bar{u}_r^{k - (k_i + \bar{k}_i)} \right)$$

for sufficiently large k . This implies that $\text{in}_{x'}(f) = \text{in}_{x'}((1/y^k) \sigma^*(\bar{f}))$ with $\bar{f} := \sum \bar{p}_i g_i \bar{u}_r^{k - (k_i + \bar{k}_i)} \in \mathcal{I}_x$.

Write $\bar{f} = \sum h_i f_i$ with $\text{supd}(h_i) \subset \Gamma_i$. By Lemma 6.4.3, we get $\text{ord}_C(\bar{f}) = \min\{\text{ord}_C(h_i f_i)\} \geq k$. Consequently,

$$\sigma^*(\bar{f})/y^k = \sum \sigma^*(h_i)/y^{k - \mu_i} \sigma^*(f_i)/y_i^{\mu_i} = \sum (\sigma^*(h_i)/y^{k - \mu_i}) f'_i.$$

Note that since $\mu_i < d$ and $\text{supd}(h_i) \subset \Gamma_i \times \mathbb{N}^{n-s} = \Gamma_i^{r-1} \times \mathbb{N}^{n-r+1}$, we get $D_{u^\alpha}(h_i) = 0$ for $\alpha \in \Delta_i^{r-1}$, and consequently $D_{(u')^\alpha}(\sigma^*(h_i)/y^{k - \mu_i}) = 0$ for $\alpha \in \Delta_i^{r-1}$, which means that

$$\text{supd}(\sigma^*(h_i)/y^{k - \mu_i}) \subset \Gamma_i^{r-1} \times \mathbb{N}^{n-r+1}$$

as well. Since $\Gamma_i^{r-1} \times \mathbb{N}^{n-r+1}$ coincides with $\Gamma_i \times \mathbb{N}^{n-s}$ for the exponents $\alpha < d$, we see that for $d' := \text{ord}_{x'}(f) < d$ we have

$$\text{in}_{x'}(f) = \sum_{\text{ord}_{x'}(h) + \mu_i = d'} \text{in}_{x'}(\sigma^*(h_i)/y^{k - \mu_i}) \text{in}_x(f'_i) = \sum H_i \text{in}_x(f'_i)$$

for some H_i with $\text{supd}(H_i) \subset \Gamma_i \times \mathbb{N}^{n-s}$, which completes the proof of the lemma. \square

Lemma 6.5.8 implies that $H_{x', \mathcal{I}'} = H(\Delta \times \mathbb{N}^s)$ if $\text{ord}_{x'}(f'_i) = \mu_i$ for all i ($d = \infty$). Since all other properties were proven before, we conclude that f'_i is a standard basis at such points.

Now suppose that $d = \min\{\mu_i : \text{ord}_{x'}(f'_i) < \mu_i\}$ is finite. We will show that $H_{x', X'}(\mathcal{I}') < H(\Delta \times \mathbb{N}^{n-s})$. The argument splits into two quite similar cases:

Case 1. Suppose that $\text{ord}_{x'}(\bar{F}_j(u'_1, \dots, 1_r, \dots, u'_s, 0, \dots, 0)) < d = \mu_j$. In this case $r \leq s$ defines the essential unknown. By condition (3) of Definition 6.3.3,

$$\text{supd}(f_j) \subset \{\alpha_j\} \cup (\Gamma \times \mathbb{N}^{n-s}) \supset \Gamma^{r-1} \times \mathbb{N}^{n-r+1}.$$

This implies that $D_{u^\alpha}(f_j) = D_{\bar{u}^\alpha}(f_j) = 0$ for $\alpha \in \Delta^r \subset \Delta \setminus \{\alpha_j\}$. Then, by the chain rule, $D_{u'^\alpha}(\text{in}_{x'}(f'_j)) = 0$ for $\alpha \in \Delta^r$, $|\alpha| < d$, or equivalently $\alpha \in \Delta$, $|\alpha| < d$. Thus, by Lemmas 6.5.7 and 6.5.8 we see that $\text{supd}(\text{in}_{x'}(f'_j)) \subset \Gamma \times \mathbb{N}^{n-s}$ and $\text{in}_{x'}(f'_j)$ is independent of

$$\{u^\alpha \text{in}_{x'}(f'_i) \mid \alpha \in \Gamma_i \times \mathbb{N}^{n-s}\}$$

and $H_{x, X}(\mathcal{I}) = H(\Delta \times \mathbb{N}^{n-s}) > H_{x', X'}(\mathcal{I}')$.

Case 2. Suppose that $r \geq s + 1$ and $\text{ord}(\bar{F}_i(\bar{u}_1, \dots, \bar{u}_s, 0, \dots, 0)) = \mu_i$ for all i , and let $\text{ord}_{x'}(f'_j) = d < \mu_j$ for a certain j . In this situation $D_{u'^\alpha} f'_j \equiv 0$ for $\alpha \in \Delta \times \mathbb{N}^{n-s}$, $|\alpha| < \mu_j$ as in Case 1. Then again $\text{supd}(\text{in}_{x'}(f'_j)) \subset \Gamma \times \mathbb{N}^{n-s}$, and $\text{in}_{x'}(f'_j)$ is independent of $u^\alpha \text{in}_{x'}(f'_i)$, where $\alpha \in \Gamma_i \times \mathbb{N}^{n-s}$. Consequently, $H_{x, X}(\mathcal{I}) = H(\Delta \times \mathbb{N}^{n-s}) > H_{x', X'}(\mathcal{I}')$ as in Case 1.

Remark. The fact that one can find a basis of an ideal which preserves its optimal form under blow-ups (mainly condition (3) of Definition 6.3.3) is very important. It reduces a problem of the lowering Hilbert-Samuel function and thus strong desingularization to resolution of marked ideals. To ensure canonicity of the reduction one considers the relevant equivalence relation as in [9] (Section 4).

In characteristic zero, however the situation is even simpler. We show that the standard basis, though not unique, generates a unique canonical Rees Algebra giving a canonical reduction of Hironaka desingularization to resolution of marked ideals. Thus the reduction is automatic and requires no additional glueing relations. □

7. CANONICAL REES ALGEBRA AND STANDARD BASIS

7.1. First properties of Rees algebras.

Definition 7.1.1. Let X some smooth space over a field K (or analytic or differentiable manifold).

By the Rees Algebra

$$\mathcal{R} = \bigoplus_{\mu \in \mathbb{N}} (\mathcal{R}^\mu, \mu)$$

we mean a graded algebra satisfying the conditions

- (1) $\mathcal{R}^0 = \mathcal{O}(U)$
- (2) $\mathcal{R}^\mu \subset \mathcal{O}(U)$ is an ideal sheaf of \mathcal{O}_X .
- (3) $\mathcal{R}^\mu \subset \mathcal{R}^{\mu'}$ if $\mu' \geq \mu$
- (4) $\mathcal{R}^\mu \cdot \mathcal{R}^{\mu'} \subset \mathcal{R}^{\mu+\mu'}$

By the Rees Algebra generated by $\{\bar{\mathcal{I}}\} := (\mathcal{I}_i, \mu_i)$ we mean the smallest graded algebra $\mathcal{R} = \bigoplus_{\mu \in \mathbb{N}} (\mathcal{R}^\mu, \mu)$ such that $\mathcal{I}_i \subseteq \mathcal{R}^{\mu_i}$.

A Rees algebra will be called a *differential Rees Algebra* if it satisfies

- (5) $\mathcal{D}^a(\mathcal{R}^\mu) \subset \mathcal{R}^{\mu-a}$ if $a \in \mathbb{Z}_{>0}$, $\mu \geq a$.

Similarly the differential Rees Algebra $\mathcal{R} = \mathcal{R}(\{\bar{\mathcal{I}}\})$ is *diff-generated by $\bar{\mathcal{I}} := \{(\mathcal{I}_i, \mu_i)\}$* if it is the smallest differential Rees algebra for which $\mathcal{I}_i \in \mathcal{R}^{\mu_i}$.

Remark. Different notions of Rees algebras defined by marked ideals were studied in the context of resolution by Giraud, Hironaka, Oda, and more recently Kawanoue-Matsuki, and Villamayor. The above definition is essentially equivalent to the one used Villamayor's papers. (See [32],[37],[63],[70],[45].)

The differential Rees algebras are natural extensions of marked ideals. They possess important properties generalizing the notion of coefficient ideals and homognization used in the simple proofs of the (weaker) desingularization in characterist zero [73], [47]

In this paper the notion will be used mainly to study more subtle properties related to the Hilbert-Samuel function, and strong resolution (see Definition 7.2.10).

It follows from the definition that

$$\text{cosupp}(\mathcal{R}(\bar{\mathcal{I}})) = \bigcap_{\mu \in \mathbb{N}} \text{cosupp}(\mathcal{R}^\mu, \mu) = \text{cosupp}(\bar{\mathcal{I}}),$$

for any multiple marked ideal $\bar{\mathcal{I}}$.

Moreover, an immediate consequence of the definition is the folowing:

Proposition 7.1.2. *Let $\bar{\mathcal{I}} = \{(\mathcal{I}_i, \mu_i)\}$ be a finite collection of marked ideals on a smooth scheme of finite type X over a field K . Let u_1, \dots, u_n be a system of coordinates on X . Denote by f_{i1}, \dots, f_{ij_i} the finite sets of generators of \mathcal{I}_i . Then the differential Rees algebra $\mathcal{R}(\bar{\mathcal{I}})$ is (finitely) generated by marked functions $(D_{u^\alpha} f_{ij}, \mu_i - |\alpha|)$, where $\alpha \in \mathbb{N}^n$, $|\alpha| \leq \mu_i$.*

Also, by Lemma 6.2.4, for any étale morphism $\phi : X \rightarrow Y$ of smooth varieties over a field K we have that

$$\phi^*(\mathcal{R}(\bar{\mathcal{I}})) = \mathcal{R}(\phi^*(\bar{\mathcal{I}})).$$

7.2. Canonical Rees Algebra along Samuel stratum and essential variables. Let $F \in K[x_1, \dots, x_n]$ be a form of degree d . For any nonnegative integer $k \leq d$ denote by $\overline{D}^d(F)$ the vector space spanned by the derivatives of order d . This definition does not depend upon a linear change of coordinates. Using this operation one can define a homogenous counterpart of Rees algebra and Rees ideal.

Definition 7.2.1. By the *homogenous Rees Algebra* generated by the homogenous polynomials $F_i \in K[x_1, \dots, x_n]$ of degree d_i we mean the smallest graded subalgebra

$$R = R(F_1, \dots, F_r) = \bigoplus_{d \in \mathbb{N}} R^d$$

containing $F_i \in R^{d_i}$ and which is \overline{D} -stable, that is

$$\overline{D}^a(R^d) \subset R^{d-a}$$

if $a \in \mathbb{Z}_{>0}$, $d \geq a$.

The graded ideal

$$I = I(F_1, \dots, F_r) \subset R = R(F_1, \dots, F_r)$$

generated over R by (F_i) will be called the *Rees ideal* generated by (F_i) .

Definition 7.2.2. Let $\mathcal{I} = \bigoplus_{a \in \mathbb{N}} \mathcal{I}_a \subset K[x_1, \dots, x_n]$ be a homogenous ideal. By the *essential set of coordinates* we mean a set of lineally independent linear forms u_1, \dots, u_k such that

- (1) There exists a set of homogenous generators $F_1, \dots, F_r \in \mathcal{I}$, such that $F_i = F_i(u_1, \dots, u_k)$.
- (2) The vector space $V := \text{span}(u_1, \dots, u_k) \subset \text{span}(x_1, \dots, x_n)$ is minimal for all sets u_1, \dots, u'_k satisfying the condition (1). The vector space V will be called the *essential space* of \mathcal{I} .

The notion of essential coordinates makes sense for homogenous polynomials or their sets.

Lemma 7.2.3. ([8], Lemma 6.2a) *In characteristic 0 the vector space $V := \overline{D}^{d-1}(F)$ is essential for F .*

Proof. If u_1, \dots, u_k is a basis of V then after extending the set to a complete coordinate system u_1, \dots, u_n and F does not depend upon u_{k+1}, \dots, u_n thus $F = F(u_1, \dots, u_k)$. On the other hand if u'_1, \dots, u'_k are essential unknowns then $F = F(u'_1, \dots, u'_k)$ and $\text{span}(u'_1, \dots, u'_k) \supseteq V = \overline{D}^{d-1}(F)$. \square

In characteristic p we shall use homogenous Rees algebras to isolate essential variables. Let us first reformulate the above lemma first in characteristic zero

Proposition 7.2.4. *Let K be a field of the characteristic 0 and $F_1, \dots, F_r \in K[x_1, \dots, x_n]$ denote homogenous polynomials. The homogenous Rees algebra $R(F_1, \dots, F_r)$ is generated by the essential unknowns u_1, \dots, u_s for F_1, \dots, F_r . That is*

$$R := R(F_1, \dots, F_r) = K[u_1, \dots, u_k]$$

Proof. The result follows from the fact that $u_1, \dots, u_k \in R^1$ and any form $G \in R^d$ can be expressed as function in u_1, \dots, u_k . This is true since the generators have this form and the property is preserved by the derivations, sums, and products. \square

Proposition 7.2.5. *Let K be a perfect field of $\text{char}(K) = p$. Let $F_1, \dots, F_r \in K[x_1, \dots, x_n]$ denote homogenous polynomials. generating $R = R(F_1, \dots, F_r)$*

Then the essential space V for F_1, \dots, F_r is unique. Moreover there exists a unique filtration

$$V_0 \subset V_1 \subset \dots \subset V_\ell = V$$

such that $V_i^{(p^i)} = \{G \in K[x_1, \dots, x_n]_1 \mid G^{p^i} \in R^{p^i}\}$. Moreover consider a coordinate system

$$u_1, \dots, u_{k_1}, u_{k_1+1}^p, \dots, u_{k_2}^p, \dots, u_{k_{\ell-1}+1}^{p^\ell}, \dots, u_{k_\ell}^{p^\ell} = \bar{u}_1, \dots, \bar{u}_\ell^{p^\ell}$$

with $V_i = \text{span}(u_1, \dots, u_{k_i}) = \text{span}(\bar{u}_1, \dots, \bar{u}_i)$ Then

- (1) $R(F_1, \dots, F_r) = K[\bar{u}_0, \dots, \bar{u}_\ell^{p^\ell}]$.
- (2) $J(F_1, \dots, F_r) = K[\bar{u}_0, \dots, \bar{u}_\ell^{p^\ell}] \cap (F_1, \dots, F_r)$.

Proof. Consider $V_0 := R^1$. By rearranging coordinates we can assume that $V_0 = \text{span}(u_1, \dots, u_{k_1})$. This implies that $D_{u_j}(G) = 0$ for any $G \in R$ and $j > k_1$. Thus it can be expressed as $G(u_1, \dots, u_{k_1}, u^{p_{k_1+1}}, \dots, u^{p_n})$. Consider the canonical subalgebra $R_1 \subset R$ defined by the conditions $D_{u_j}(G) = 0$ for all j . Then R_1 consists of the homogenous polynomials of the forms $G(u_1^p, \dots, u_n^p) = G'(u_1, \dots, u_n)^p$. This defines the subalgebra

$$\sqrt[p]{R_1} := \{G \in K[u_1, \dots, u_n] \mid G^p \in R_1\}.$$

By the inductive assumption there exists a unique essential space V^1 for $\sqrt[p]{R_1}$, with the induced canonical filtration $\overline{V}_1 \subset \dots \subset \overline{V}_{\ell-1}$. Since $V_1 \subset \overline{V}_1$, It defines a unique filtration $V_1 \subset \dots \subset V_\ell$, where $V_i := \overline{V}_{i-1}$ for $i = 2, \dots, r$. □

The above proposition leads to a more subtle notion than essential space in characteristic p .

Definition 7.2.6. Let K be a perfect field of characteristic $p > 0$. The *essential flag*

$$V_0 \subset V_1 \subset \dots \subset V_\ell = V$$

of a homogenous ideal $\mathcal{I} \subset [x_1, \dots, x_n]$ is a filtration of vector subspaces of the essential space E such that for any partitioned basis

$$u_1, \dots, u_{k_0}, u_{k_0+1}, \dots, u_{k_1}, \dots, u_{k_\ell}$$

of E such that u_1, \dots, u_{k_i} is a basis of V_i there exist homogenous generators

$$F_i := F_i(u_1, \dots, u_{k_0}, u_{k_0+1}^p, \dots, u_{k_1}^p, \dots, u_{k_\ell}^p).$$

of I . Moreover the sequence of numbers $(k_0, \dots, k_\ell, 0, \dots)$ is minimal

Proposition 7.2.7. Let K be a perfect field and $I \subset K[x_1, \dots, x_n]$ be a homogenous ideal, and let $F_1, \dots, F_r \in I$ denote a standard basis of \mathcal{I} (at 0). Then

- (1) F_1, \dots, F_r are homogenous and moreover $F_i = F_i(u_1, \dots, u_k)$, where u_1, \dots, u_k is essential set of coordinates for I .
- (2) $R(I) := R(F_1, \dots, F_r)$ is independent of the choice of the standard basis of I
- (3) If $\text{char}(K) = 0$ then $R(I) = K[u_1, \dots, u_k]$ and $J(I) = I \cap K[u_1, \dots, u_k]$.
- (4) If $\text{char}(K) = p$ then $R(I) = K[\overline{u}_0, \dots, \overline{u}^{\ell}]$ and $J(I) = I \cap K[\overline{u}_0, \dots, \overline{u}_\ell^{\ell}]$
- (5) The essential space and the essential flag for I are unique.

Remark. The parts (1) and (2) are proven for a standard basis with respect to a certain monotone order in char. 0 in ([8]) (Lemma 6.7(2)).

Proof. We give here a proof in positive characteristic. The case of characteristic zero is the same but slightly simpler. Let F_1, \dots, F_r be any basis of I . The algebra $R(F_1, \dots, F_r) = K[\overline{u}_0, \dots, \overline{u}_r^{\ell}]$ is characterized uniquely by the properties $D_{\overline{u}_j^t}(G) = 0$ where $0 \leq j < t \leq \ell$, or $t > \ell$, and j is arbitrary. This property is valid thus also for generators F_i . This implies that for any $G = \sum H_i F_i \in I$, $D_{\overline{u}_j^t}(G) \in I$. Thus the property holds for a standard basis (G_i) of I associated with some coordinate system (v_i) and diagrams Δ and Γ .

But, by the property of the standard basis $G_i = v^{\alpha_i} + r_i$ with $\text{supd}(r_i) \subset \Gamma$. More precisely, by Condition (3) of Definition 6.3.3, $D_{v^\alpha}(G_i) = 0$ for $\alpha \in \Delta \setminus \{\alpha_i\}$, and $D_{v^{\alpha_i}}(G_i) = 1$. In both cases, that is, if $\alpha \in \Delta$ we have $D_{v^\alpha}(D_{\overline{u}_j^t})(G_i) = D_{\overline{u}_j^t}(D_{v^\alpha}(G_i)) = 0$ which means that $\text{supd}(D_{\overline{u}_j^t})(G_i) \subset \Gamma$.

Since additionally $(D_{\overline{u}_j^t})(G_i) \in I$ we conclude that $\text{supd}(D_{\overline{u}_j^t})(G_i) = 0$. The latter implies that $G_i \in R(F_1, \dots, F_r)$, and $R(G_1, \dots, G_s) \subseteq R(F_1, \dots, F_r)$. This implies that if (F_i) is another standard basis then by symmetry we get the equality $R(F_1, \dots, F_r) = R(G_1, \dots, G_s)$.

This also shows that any standard basis determines a unique essential flag.

Now write $G_j = \sum H_i F_i$, with $\text{cosupp}(H_i) \subset \Gamma_i$. Again we easily see by induction on i that $0 = D(G_i) = \sum D(H_i F_i) = \sum D(H_i) F_i$, with $\text{cosupp}(D(H_i)) \subset \Gamma_i$. Thus $G_i \in J(F_1, \dots, F_r)$ showing inclusion $J(G_1, \dots, G_m) \subseteq J(F_1, \dots, F_r)$, which implies (by symmetry) the equality of both algebras. One can choose a standard basis with respect to the coordinate system (u_i) . This implies that $J(I) = I \cap K[\overline{u}_0, \dots, \overline{u}_r^{\ell}]$. □

The result on homogenous Rees algebras can be extended to any ideal sheaf. We will show it here in characteristic zero (Theorem 7.2.9). The case of positive characteristic is more subtle and will be dealt in a separate paper.

Let $f \in (\mathcal{I}, \mu)$ be a marked function, and $x \in \text{cosupp}(f, \mu)$. Then we define the *initial form* $\text{in}_x(f)$ of (f, μ) at x to be the class of f in $(\mathcal{I} + m_{x,X}^{\mu+1})/m_{x,X}^{\mu+1}$. Thus $\text{in}_x(f)$ can be identified with the initial μ -form of f if $\text{ord}_x(\mathcal{I}) = \mu$ and is 0 otherwise (if $\text{ord}_x(\mathcal{I}) > \mu$).

We shall need the following result:

Lemma 7.2.8. *Let (f_i, d_i) be marked functions of maximal orders and consider the generated Rees algebra $\mathcal{R} = \mathcal{R}(f_1, \dots, f_r)$ and the homogenous Rees algebra $R(\text{in}_x(f_1), \dots, \text{in}_x(f_r))$. Then*

$$R(\text{in}_x(f_1), \dots, \text{in}_x(f_r)) = \text{in}_x(\mathcal{R}(f_1, \dots, f_r))$$

Proof. It follows by definition that $\text{in}_x(Df) = D(\text{in}_x(f))$, for any $D \in \overline{D}^a$, and $f \in \mathcal{R}$. Moreover in_x preserves products and sums of the marked functions in Rees algebra \mathcal{R} . □

Theorem 7.2.9. *Assume X is a smooth scheme of finite type over the ground field K of the characteristic 0 (respectively a complex manifold). Let \mathcal{I} be a coherent sheaf of ideals on X , and (f'_i, d_i) be a standard pre-basis of \mathcal{I} along a Samuel Stratum S . Then the Rees algebra $\mathcal{R}(\mathcal{I}) = \mathcal{R}(f'_1, \dots, f'_k)$ and the Rees ideal $\mathcal{J}(\mathcal{I}) := \mathcal{J}(f'_1, \dots, f'_k)$ are independent of choice of standard basis of \mathcal{I} in a neighborhood of the Samuel stratum S .*

Proof. Let f'_1, \dots, f'_r , and g'_1, \dots, g'_m be two standard pre-bases of \mathcal{I} along S on a Zariski open neighborhood of x corresponding to two standard bases f_1, \dots, f_r , and g_1, \dots, g_m of \mathcal{I} along S on a common étale neighborhood X' preserving the residue field of $x \in X$ (see Definition 6.5.5). Denote by u_1, \dots, u_n , and v_1, \dots, v_n the corresponding compatible coordinates. We can assume here that u_1, \dots, u_s are distinguished, and u_1, \dots, u_k are essential, with $s \leq k \leq n$. By symmetry it suffices to show that $f_i \in \mathcal{J}^{\mu_i}(g_1, \dots, g_s)$. We will show that f_i is in the completion of $\widehat{\mathcal{J}^{\mu_i}(g_1, \dots, g_s)}_x \subset \widehat{\mathcal{O}_{X',x}} = \widehat{\mathcal{O}_{X,x}}$ in a neighborhood of any $x \in S = \text{cosupp}(\mathcal{J})$.

First observe that the initial forms $\text{in}_x(f_1), \dots, \text{in}_x(f_r)$, and $\text{in}_x(g_1), \dots, \text{in}_x(g_s)$ form two different bases of the initial ideal $\text{in}_x(\mathcal{I})$. Then we can find the essential linear forms

$$\bar{u}_1 = \text{in}_x(u_1) = \text{in}_x(\tilde{u}_1), \dots, \bar{u}_s = \text{in}_x(\tilde{u}_s)$$

in the grading

$$R^1(\text{in}_x(\mathcal{I})) = R^1(\text{in}_x(g_1), \dots, \text{in}_x(g_s)),$$

for a certain coordinates $\tilde{u}_1, \dots, \tilde{u}_k \in \mathcal{R}^1(g_1, \dots, g_s)$, and $\tilde{f}_i \in \mathcal{R}^{d_i}(g_1, \dots, g_s)$, such that

$$(\text{in}_x(\tilde{f}_i))(\bar{u}_1, \dots, \bar{u}_s, 0, \dots, 0) = \text{in}_x(f_i)(u_1, \dots, u_s, 0, \dots, 0).$$

This implies that (\tilde{f}_i) satisfies (in particular) the condition (4) of the weak standard basis. Using it one can perform in the completion ring $\widehat{\mathcal{O}_{X,x}}$ the following Euclidean division algorithm. Consider the function \tilde{f}_j . Its initial form coincides with that of f_j . We shall modify \tilde{f}_j to get f_j with all intermediate steps performed in $\widehat{\mathcal{J}^{\mu_i}(g_1, \dots, g_s)}_x$. We just need to eliminate all the higher degree monomials in \tilde{f}_j which are not in $\Delta \times \mathbb{N}^{n-s}$. Set $h_0 := \tilde{f}_j$

For any natural s consider the vector space V_s spanned by the ordered set of forms

$$V_s := \{u^\alpha \mid \alpha \in \Delta = \bigcup \Delta_i, |\alpha| = s\},$$

and the natural projection

$$\pi_s : K[[u_1, \dots, u_n]] \rightarrow V_s.$$

Let

$$T^s := [t_{\alpha,\beta}] = [\dots, \pi_s(\tilde{f}_\alpha), \dots]$$

be the square matrix whose subscripts are labeled by the ordered set

$$\Delta^s =: \{\alpha \in \Delta, |\alpha| = s\}$$

and containing as $\alpha = \alpha' + \alpha_i$ - column the vector $\pi_s(\tilde{f}_\alpha)$, where $\tilde{f}_\alpha := \tilde{u}^{\alpha'} \tilde{f}_i$, for $\alpha \in \Delta^s, \alpha = \alpha_i + \alpha' \in \Delta_i^s \subset \Delta^s$, with α_i the vertex of Δ_i . Then it follows from condition (4) of Definition 6.5.5 and Stabilization Theorem that the matrices T^s are invertible and let $(T^s)^{-1} := [r_{\alpha, \beta}^s]$.

For any $h = \sum_{(\alpha, \gamma) \in \mathbb{N}^s \times \mathbb{N}^{n-s}} c_{\alpha, \gamma} u^{\alpha, \gamma} \in K[[u_1, \dots, u_n]] = \widehat{\mathcal{O}_{X, x}}$ put

$$\mu(h) := \inf\{|\alpha| + |\gamma|, |\gamma|, \gamma \mid \alpha \in \Delta, c_{\alpha, \gamma} \neq 0\} = (\beta, \gamma)$$

(with lexicographic order) and let $s := |\beta|$, and $t = |\gamma|$. Then for

$$\bar{h}_1 := h_0 - c_{\beta, \gamma} u^{0, \gamma} \sum_{\alpha \in \Delta^s} r_{\beta, \alpha} \tilde{f}_\alpha$$

we have $\mu(\bar{h}_1) > \mu(h)$.

Likewise since $\text{in}_x(D_{u_\beta} h_0) = c_{\beta, \gamma} u^{0, \gamma} + \sum_{\gamma' > \gamma} a_{\gamma'} u^{0, \gamma'}$ the same is true for the function

$$h_1 := h_0 - D_{u_\beta} h_0 \sum_{\alpha \in \Delta^s} r_{\alpha, \beta} \tilde{f}_\alpha.$$

The latter function remains to be in $\widehat{\mathcal{J}}^{\mu_i}(g_1, \dots, g_s)_x$ since $D_{u_\beta} h_0 \in \mathcal{J}^{\mu_i - s}$ and $\tilde{f}_\alpha := \tilde{u}^{\alpha'} \tilde{f}_i \in \mathcal{J}^s(g_1, \dots, g_s)_x$ for any $\alpha \in \Delta^s$.

This defines a convergent sequence $(h_n) \rightarrow h_\infty \in \widehat{\mathcal{J}}^{\mu_i}(g_1, \dots, g_s)_x$. Moreover the function $h_\infty \in \widehat{\mathcal{I}}_x$ has a form

$$h_\infty = u^{\alpha_j} + r_\infty$$

, with $\text{cosupp}(r_\infty) \in \Gamma \times \mathbb{N}^s$. By uniqueness of the standard basis (Corollary 6.3.7) we deduce that

$$f_j = u^{\alpha_j} + r_\infty = h_\infty \in \widehat{\mathcal{J}}^{d_j}(g_1, \dots, g_s)_x,$$

Likewise $f'_j \in \widehat{\mathcal{J}}^{d'_j}(g_1, \dots, g_s)_x = \widehat{\mathcal{J}}^{d'_j}(g'_1, \dots, g'_s)_x$ and that $f'_j \in \mathcal{J}^{d'_j}(g'_1, \dots, g'_s)_x$.

□

Definition 7.2.10. Let \mathcal{I} be a coherent sheaf of ideals on a smooth scheme X over a field K . Consider a Samuel stratum S of \mathcal{I} on X , and a weak standard pre-basis (f'_i, d'_i) of \mathcal{I} .

We shall call the (multiple) marked ideal $\mathcal{R}(\mathcal{I}) = \mathcal{R}(f'_1, \dots, f'_k)$ (respectively $\mathcal{J}(\mathcal{I})$) *the canonical Rees algebra along Samuel stratum* (respectively *canonical Rees ideal*) of \mathcal{I} along a Samuel stratum S .

Remark. Different approaches to Rees algebras were considered were considered by Hironaka, Villamayor, and Kawanoue-Matsuki. They considered, in particular, an additional saturation given by the integral closure of Rees algebra. Bravo-Garcia Escamilla-Villamayor show in [18] that the integral (and differential) closure determines a unique canonical Rees algebra defined by its equivalence class. This was then applied to Hironaka's construction of distinguished data as in [38]. On the other hand Hironaka [39], Bravo-Garcia Escamilla-Villamayor [18] and also Kawanoue-Matsuki [45], [46] show that such an algebra is finitely generated. Finally Bierstone-Milman [9], [10] do not consider any saturation and use equivalence relation instead in their inductive arguments.

7.3. Relative Rees algebras. From now on we consider the case of ground field of characteristic 0.

Theorem 7.3.1. *Let \mathcal{I} be any reduced ideal on (X, E) . Let $H = H_x(\mathcal{I})$ be the maximal value of the Hilbert function, and let $\mathcal{R}(\mathcal{I})$ denote canonical Rees ideal. Then any resolution of $\mathcal{R}(\mathcal{I})$ defines a resolution of (\mathcal{I}, H) . Moreover*

- (1) *The ideal $\mathcal{R}(\mathcal{I})$ is stable with respect to any étale morphisms and field extensions.*
- (2) *The centers of the admissible blow-ups of $\mathcal{R}(\mathcal{I})$ are contained in the Samuel strata of the ideals of the strict transforms of \mathcal{I} , and thus are normally flat.*

Proof. The Rees $\mathcal{R}(\mathcal{I})$ ideal is generated locally by the standard basis (f_i, d_i) . The support of the standard basis defines the Samuel stratum of \mathcal{I} for the value H of the Hilbert-Samuel function. Moreover the controlled transforms $(f'_i, d'_i) := \sigma^c(\{(f_i, d_i)\})$ of the standard basis (f_i, d_i) remain the standard basis of the strict transform \mathcal{I}' of \mathcal{I} . This implies that resolution of marked ideal $\{(f_i, d_i)\}$ defines a resolution of (\mathcal{I}, H) . On the other hand by Lemma 6.2.3

$$\sigma^c(\{(f_i, d_i)\}) \subseteq \sigma^c(\mathcal{R}(\mathcal{I})) \subseteq \mathcal{R}(\sigma^c(\{(f_i, d_i)\}))$$

The latter implies that the Samuel stratum S' of \mathcal{I}' can be described as

$$S' = \text{cosupp}\{(f'_i, d'_i)\} = \text{cosupp}(\mathcal{R}'(f'_i, d'_i)) = \text{cosupp}(\sigma^c(\mathcal{R}'(\mathcal{I})))$$

□

Definition 7.3.2. Let \mathcal{I} be any ideal on a smooth X , and $\mathcal{R}' = \mathcal{R}'(\mathcal{I})$ be its Rees Algebra along a Samuel stratum S . Let $E = \{D_1, \dots, D_k\}$ be a set of SNC divisors. By the *relative Rees* we shall mean the graded algebra $\mathcal{R}'(\mathcal{I}, E)$ generated by \mathcal{R}' and all $(\mathcal{I}_{D_i}, 1)$.

It follows immediately that

$$\text{cosupp}(\mathcal{R}'(\mathcal{I}, E)) = \text{cosupp}(\mathcal{R}'(\mathcal{I})) \cap E$$

Moreover this relation is preserved by the blow-ups with the center in $\text{cosupp}(\mathcal{R}'(\mathcal{I})) \cap E$.

Proposition 7.3.3. Let $X_1 \hookrightarrow X_2$ be a closed embedding of a smooth schemes over K , and $Y \hookrightarrow X_1$ be a closed embedding of reduced schemes. Assume that there exists a (possibly empty) set E_2 of SNC divisors on X_2 which is transversal to X_1 , and denote by E_1 its restriction to X_2 .

Let $\mathcal{I}_1 := \mathcal{I}_{Y, X_1}$, and $\mathcal{I}_2 := \mathcal{I}_{Y, X_2}$ be the sheaves of ideals of Y on X and X_2 , and suppose that X_1 is locally described by the vanishing locus of the set of parameters u_1, \dots, u_k . Then there is a certain étale extension of $X_2 \supset X_1$ the such that

- (1) There is an inclusion of the sheaves $\mathcal{O}_{X_1} \subset \mathcal{O}_{X_2}$
- (2) There is an inclusion of the relative Rees algebras $\mathcal{R}_1 := \mathcal{R}'(\mathcal{I}_1, E_1) \subset \mathcal{R}_2 := \mathcal{R}'(\mathcal{I}_2, E_2)$.
- (3) The Rees algebra R_2 is locally generated by $(u_1, 1), \dots, (u_k, 1)$ and $R_1 \subset R_2$, where $u_1 = 0, \dots, u_k = 0$ is a set of parameters describing X_1 on X_2 .
- (4) The restriction of Rees algebra R_2 to X_1 coincides with R_1 .
- (5) The relations above are preserved for the controlled transforms of R_2 , R_1 and u_1, \dots, u_k under admissible blow-ups of R_2 .

Proof. First we show the proposition in the case $E = \emptyset$.

Consider a system of parameters $u_1, \dots, u_k, u_{k+1}, \dots, u_n$ on X_2 , where u_1, \dots, u_k describe X_1 . By considering a division by u_1, \dots, u_k with respect to the standard monotone diagram Δ generated by the basis e_1, \dots, e_k we see that in the certain étale neighborhood \mathcal{O}_{X_2} can be identified with

$$\mathcal{O}_{X_1}^\Gamma = \{f \in \mathcal{O}_{X_1} \mid D_{u_i}(f) = 0, i = 1, \dots, k\} \subset \mathcal{O}_{X_1}.$$

(In the characteristic p , we shall use the condition is $D_{u_i^{p_j}}(f) = 0$).

Choosing the monotone order for the coordinates u_{k+1}, \dots, u_n one can find a monotone diagrams $\Delta_1 \in \mathbb{N}^{n-k}$ and $\Delta_2 = [0, 1]^k \times \Delta_1 \in \mathbb{N}^n$ defined by \mathcal{I}_1 and \mathcal{I}_2 . Then one can find a standard basis (passing to étale neighborhood) of \mathcal{I}_2 of the form

$$u_1, \dots, u_k, f_1(u_{k+1}, \dots, u_n), \dots, f_r(u_{k+1}, \dots, u_n),$$

with respect to Δ , such that $f_1(u_{k+1}, \dots, u_n), \dots, f_r(u_{k+1}, \dots, u_n)$ is a standard basis of \mathcal{I}_1 . Then it follows that the Rees $\mathcal{R}_1 := \mathcal{R}'(\mathcal{I}_2)$ algebra of \mathcal{I}_2 is generated by $(u_1, 1), \dots, (u_k, 1)$ and $\mathcal{R}_1 := \mathcal{R}'(\mathcal{I}_1) \subset \mathcal{R}_2$. Then \mathcal{R}_2 is nothing but the restriction of \mathcal{R}_2 to X_1 with the subscheme $X_1 \subset X_2$ described exactly by the support of $(u_1, 1), \dots, (u_k, 1)$. Moreover these relations between the standard bases and the Rees algebra are preserved under the blow-ups contained in the Samuel strata. The controlled transform $(u'_1, 1), \dots, (u'_k, 1)$ describes the strict transform X'_1 on X'_2 , and the controlled transform of the Rees algebra $(\mathcal{R}_1)'$ is the restriction of $(\mathcal{R}_2)'$ to X'_2 . Moreover $(\mathcal{R}_2)'$ is generated by $(u'_1, 1), \dots, (u'_k, 1)$ and $(\mathcal{R}_1)'$.

The general case follows since $\mathcal{R}'(\mathcal{I}_i, E_i)$, where $i = 1, 2$, is generated by $\mathcal{R}'(\mathcal{I}_i)$ and $(\mathcal{I}_{D_j}, 1)$. Moreover by part (1), the functions $x_j \in \mathcal{O}_{X_2}$ defining D_j on X_2 and their restrictions to X_1 can be identified under the inclusion $\mathcal{O}_{X_1} \subset \mathcal{O}_{X_2}$.

□

7.4. Minimal embedding spaces and Samuel stratum. When considering Samuel strata of an ideal \mathcal{I}_Y on a smooth variety X it is convenient to consider locally minimal embedding spaces. They define "optimal embedding" of the smallest dimension and are canonical in the sense of the following lemma

Definition 7.4.1. Let \mathcal{I} be any ideal on a smooth X , with a SNC divisor E and defined for a Samuel stratum S and E .

The maximal set of parameters $u_1, \dots, u_k \in \mathcal{R}_1$ transversal to $E = \{D_1, \dots, D_k\}$ defines locally a smooth *minimal embedding space* T containing S (of codimension k which is locally constant on $S \cap \bigcap D_i$).

The following lemma is a direct extension of Lemma 9.2.1.

Lemma 7.4.2. *Let \mathcal{I} be any ideal on a smooth X , with SNC divisor E and let $\mathcal{R} = \mathcal{R}(I, E)$ be its canonical Rees Algebra along a Samuel stratum S . For any point in $x \in S$ consider a maximal set of parameters $u_1, \dots, u_k \in \mathcal{R}^1$ which is a part of the standard basis in a neighborhood of x and is transversal to E . Then for any two sets of $u_1, \dots, u_k \in \mathcal{R}^1$ and $v_1, \dots, v_k \in \mathcal{R}^1$ there exist étale neighborhoods $\phi_u, \phi_v : \overline{X} \rightarrow X$ of $x = \phi_u(\overline{x}) = \phi_v(\overline{x}) \in X$, where $\overline{x} \in \overline{X}$, such that*

- (1) $\phi_u^*(\mathcal{R}) = \phi_v^*(\mathcal{R})$.
- (2) $\phi_u^*(E) = \phi_v^*(E)$.
- (3) $\phi_u^*(u_i) = \phi_v^*(v_i)$.

Proof. (0) (0) **Construction of étale neighborhoods** $\phi_u, \phi_v : U \rightarrow X$.

Let $U \subset X$ be an open subset for which there exist u_{k+1}, \dots, u_n which are transversal to u_1, \dots, u_k and v_1, \dots, v_k on U such that u_1, u_2, \dots, u_n and $v_1, \dots, v_k, u_{k+1}, \dots, u_n$ form two sets of parameters on U and divisors in E are described by some u_i , where $i \geq 2$. Let \mathbf{A}^n be the affine space with coordinates x_1, \dots, x_n . Construct first étale morphisms $\phi_1, \phi_2 : U \rightarrow \mathbf{A}^n$ with

$$\phi_1^*(x_i) = u_i \quad \text{for all } i \quad \text{and} \quad \phi_2^*(x_i) = v_i, \quad \text{for } i \leq k \quad \phi_2^*(x_i) = u_i \quad \text{for } i > k.$$

Then

$$\overline{X} := U \times_{\mathbf{A}^n} U$$

is a fiber product for the morphisms ϕ_1 and ϕ_2 . The morphisms ϕ_u, ϕ_v are defined to be the natural projections $\phi_u, \phi_v : \overline{X} \rightarrow U$ such that $\phi_1 \phi_u = \phi_2 \phi_v$. Set

$$w_i := \phi_u^*(u_i) = (\phi_1 \phi_u)^*(x_1) = (\phi_2 \phi_v)^*(x_i) = \phi_v^*(v_i), \quad \text{for } i \leq k$$

$$w_i = \phi_u^*(u_i) = \phi_v^*(u_i) \quad \text{for } i \geq k + 1.$$

One can extend the ground field algebraically and assume that the residue fields of the points above x and at x are the same. Then we can construct an automorphism $\widehat{\phi}_{uv} = \widehat{\phi}_{uv} \widehat{\phi}_u^{-1}$ such that $\widehat{\phi}_{uv}(u_i) = v_i$ for $i \leq k$.

(1) Let $h_i := v_i - u_i \in \mathcal{R}^1(\mathcal{I})$. For any $f \in \widehat{\mathcal{R}}^s$,

$$\widehat{\phi}_{uv}^*(f) = f(u_1 + h_1, \dots, u_k + h_k, u_{k+1}, \dots, u_n) = f(u_1, \dots, u_n) + \sum_{i \leq k} \frac{\partial f}{\partial u_i} \cdot h_i + \sum_{ij \leq k} \frac{1}{2!} \frac{\partial^2 f}{\partial u_i \partial u_j} \cdot h_i h_j + \dots +$$

The latter element belongs to

$$\widehat{\mathcal{R}}^s + \widehat{\mathcal{R}}_{s-1} \cdot \widehat{\mathcal{R}}^1 + \dots + \widehat{\mathcal{R}}_{s-2} \cdot \widehat{\mathcal{R}}^2 + \dots = \widehat{\mathcal{R}}^s.$$

Hence $\widehat{\phi}_{uv}^*(\widehat{\mathcal{R}}) \subset \mathcal{H}\widehat{\mathcal{R}}$. which implies that $\widehat{\phi}_u^*(\widehat{\mathcal{R}}) = \widehat{\phi}_v^*(\widehat{\mathcal{R}})$, and locally $\phi_u^*(\mathcal{R}) = \phi_v^*(\mathcal{R})$.

(2), (3) follow from the construction

(4) Let $h_i := v_i - u_i$ for $i \leq k$. By the above the morphisms ϕ_u and ϕ_v coincide on $\phi_u^{-1}(V(h_1, \dots, h_k)) = \phi_v^{-1}(V(h_1, \dots, h_k))$.

By (4) the blow-ups of the centers $C \subset \text{cosupp}(\mathcal{H}(\mathcal{I}))$ lifts to the blow-ups at the same center $\phi_u^{-1}(C) = \phi_v^{-1}(C)$. Thus (5), (6) follow. \square

7.5. Equivalence for marked ideals and Capacitors. Let us introduce the following equivalence relation for marked ideals:

Definition 7.5.1. We say that two (multiple) marked ideals $\{(X, \mathcal{I}_i, E, \mu_i)\}$ and $\{(X, \mathcal{J}_j, E, \mu_j)\}$ on a smooth variety X with SNC collection of divisor E , are equivalent:

$$\{(X, \mathcal{I}_i, E, \mu_i)\} \simeq \{(X, \mathcal{J}_j, E, \mu_j)\} \quad \text{if}$$

- (1) $\text{cosupp}\{(X, \mathcal{I}_i, E, \mu_i)\} = \text{cosupp}\{(X, \mathcal{J}_j, E, \mu_j)\}$

- (2) The sequences of admissible blow-ups $(X_i)_{i=0,\dots,k}$ are the same for both marked ideals and $\text{cosupp}\{(X, (\mathcal{I}_i)_k, E, \mu_i)\} = \text{cosupp}\{(X, (\mathcal{J}_j)_k, E, \mu_j)\}$

Definition 7.5.2. We write

$$\{(X, \mathcal{I}_i, E, \mu_i)\} \subseteq \{(X, \mathcal{J}_j, E, \mu_j)\} \quad \text{if}$$

- (1) $\text{cosupp}\{(X, \mathcal{I}_i, E, \mu_i)\} \supseteq \text{cosupp}\{(X, \mathcal{J}_j, E, \mu_j)\}$
(2) The sequences of admissible blow-ups $(X_i)_{i=0,\dots,k}$ of $\{(X, \mathcal{J}_j, E, \mu_j)\}$ is admissible for $\{(X, \mathcal{I}_i, E, \mu_i)\}$ and $\text{cosupp}\{(X, (\mathcal{I}_i)_k, E, \mu_i)\} \supseteq \text{cosupp}\{(X, (\mathcal{J}_j)_k, E, \mu_j)\}$

Example 7.5.3. For any $k \in \mathbf{N}$, $(\mathcal{I}, \mu) \simeq (\mathcal{I}^k, k\mu)$.

Remark. The marked ideals considered in this paper satisfy a stronger equivalence condition: For any smooth morphisms $\phi : X' \rightarrow X$, $\phi^*(\mathcal{I}, \mu) \simeq \phi^*(\mathcal{J}, \mu)$. This condition will follow and is not added in the definition.

Assume now that all marked ideals are defined for the smooth variety X and the same set of exceptional divisors E . Define the following operations of addition and multiplication of marked ideals:

- (1) $(\mathcal{I}_1, \mu_1) + \dots + (\mathcal{I}_m, \mu_m) := (\mathcal{I}_1^{\mu_2 \dots \mu_m} + \mathcal{I}_2^{\mu_1 \mu_3 \dots \mu_m} + \dots + \mathcal{I}_m^{\mu_1 \dots \mu_{k-1}}, \mu_1 \mu_2 \dots \mu_m)$.
(2) $(\mathcal{I}_1, \mu_1) \cdot \dots \cdot (\mathcal{I}_m, \mu_m) := (\mathcal{I}_1 \cdot \dots \cdot \mathcal{I}_m, \mu_1 + \dots + \mu_m)$

It follows from definition

- Lemma 7.5.4.** (1) $(\mathcal{I}_1, \mu_1) + \dots + (\mathcal{I}_m, \mu_m) \simeq \{(\mathcal{I}_1, \mu_1), \dots, (\mathcal{I}_m, \mu_m)\}$
(2) $(\mathcal{I}_1, \mu_1) \cdot \dots \cdot (\mathcal{I}_m, \mu_m) \subseteq \{(\mathcal{I}_1, \mu_1), \dots, (\mathcal{I}_m, \mu_m)\}$

Lemma 7.5.5. For any multiple marked ideal $\bar{\mathcal{I}} = \{\mathcal{I}_i, \mu_i\}$ the induced Rees algebra $\mathcal{R}(\bar{\mathcal{I}})$ is equivalent to $\bar{\mathcal{I}}$.

Proof. If $\text{ord}_x(\mathcal{I}_i) \geq \mu_i$ then $\text{ord}_x(\mathcal{R}^i) \geq i$, since the operations of differentiation and product preserve the relevant inequalities. Moreover by Lemma 6.2.3 the inequalities are preserved by the controlled transforms of Rees algebras. \square

One of the disadvantages of the canonical Rees algebra is that it consists of infinitely many marked ideals which is slightly inconvenient for our presentation of the resolution algorithm.

One can easily remedy this by introducing the capacitors of Rees algebras.

Definition 7.5.6. Let $R = \bigoplus R^i$ be a finitely generated Rees algebra on a smooth scheme X . By its *capacitor* we mean any gradation ideal $R^i \subset \mathcal{O}_X$, such that the marked ideal (R^i, i) is equivalent to R .

In practice it means that, when convenient, we can translate the problems of resolution of Rees algebras to their capacitors containing the essential information about Rees algebras.

Proposition 7.5.7. The following equivalence holds true:

Existence of resolution of finitely generated Rees algebras on smooth schemes



Existence of resolution of marked ideals on smooth schemes.

The following rather obvious but useful observation is made, in particular by Bravo-Garcia Escamilla-Villamayor in [18]:

Proposition 7.5.8. ([18]) Any finitely generated Rees algebra admits its capacitor.

Proof. If R is generated by a finitely many marked functions in gradation i_j then it is generated by R^{i_j} and thus equivalent to $\sum_j R^{i_j}$. But $\sum_j R^{i_j} \subset R^a$, where a is the product of i_j . This implies that R is equivalent to R^a . \square

There are various methods of finding some canonical or minimal capacitor of Rees algebras. In case of the Rees algebras along strata let n be the dimension of the minimal embedding spaces of \mathcal{I} along the stratum S . Let a be the maximal multiplicity of all possible vertices of all possible diagrams $\Delta \in \mathbf{N}^n$. It is finite by Corollary 6.5.2. Then its *canonical capacitor* is defined as $\bar{R} = R^a$.

8. STRONG HIRONAKA DESINGULARIZATION IN CHARACTERISTIC ZERO

8.1. Formulation of Hironaka's resolution theorems. .

We give a proof of the following version of the Hironaka non-embedded resolution

(1) Strong Canonical Resolution with normally flat centers

Theorem 8.1.1. *Let Y be an algebraic variety over a field of characteristic zero.*

There exists a canonical desingularization of Y that is a smooth variety \tilde{Y} together with a proper birational morphism $\text{res}_Y : \tilde{Y} \rightarrow Y$ such that

- (a) *res_Y is a composition of blow ups $Y = Y_0 \leftarrow Y_1 \leftarrow \dots \leftarrow Y_k = \tilde{Y}$ with smooth centers C_i , and exceptional divisors E_i .*
- (b) *The centers C_i are either contained in the set of singular points $\text{Sing}(Y_i)$ of Y_i , or if Y_i are smooth in the exceptional divisor D_i .*
- (c) *The centers C_i are normally flat on Y_i that is are contained in the Samuel stratum of Y_i .*
- (d) *The variety $\tilde{Y} = Y_k$ is nonsingular and the inverse image of the singular locus $\text{Sing}(Y)$ is a simple normal crossing divisor exceptional divisor is SNC divisor on \tilde{Y} .*
- (e) *res_Y is functorial with respect to smooth morphisms, and the field extensions, and it is equivariant with respect to any group action not necessarily preserving the ground field.*

(2) Strong Hironaka Embedded Desingularization

Theorem 8.1.2. *Let Y be a closed subvariety of a smooth variety X over a field of characteristic zero, and E be a (possibly empty) SNC divisor on X . There exists a sequence*

$$X_0 = X \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \leftarrow \dots \leftarrow X_i \leftarrow \dots \leftarrow X_r = \tilde{X}$$

of blow-ups $\sigma_i : X_{i-1} \leftarrow X_i$ of smooth centers $C_{i-1} \subset X_{i-1}$ such that

- (a) *The union E_i of the exceptional divisor of the induced morphism $\sigma^i = \sigma_1 \circ \dots \circ \sigma_i : X_i \rightarrow X$ and of the strict transform of the divisor E has only simple normal crossings and C_i has simple normal crossings with E_i .*
- (b) *Let $Y_i \subset X_i$ be the strict transform of Y . The centers C_i are either contained in the set of singular points $\text{Sing}(Y_i)$ of Y_i , or if Y_i are smooth, in the exceptional divisor E_i .*
- (c) *The strict transform $\tilde{Y} := Y_r$ of Y is smooth and has only simple normal crossings with the divisor E_r .*
- (d) *The morphism $(X, Y) \leftarrow (\tilde{X}, \tilde{Y})$ defined by the embedded desingularization commutes with smooth morphisms, field extensions, and embeddings of ambient varieties. It is equivariant with respect to any group action not necessarily preserving the ground K .*

(3) Canonical Principalization

Theorem 8.1.3. *Let \mathcal{I} be a sheaf of ideals on a smooth algebraic variety X , E be a possibly empty SNC divisor on X , and $Y \subset X$ be any closed subvariety of X . There exists a principalization of \mathcal{I} that is, a sequence*

$$X = X_0 \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \leftarrow \dots \leftarrow X_i \leftarrow \dots \leftarrow X_r = \tilde{X}$$

of blow-ups $\sigma_i : X_{i-1} \leftarrow X_i$ of smooth centers $C_{i-1} \subset X_{i-1}$ such that

- (a) *The union E_i of the exceptional divisor of the induced morphism $\sigma^i = \sigma_1 \circ \dots \circ \sigma_i : X_i \rightarrow X$ and of the strict transform of the divisor E has only simple normal crossings and C_i has simple normal crossings with E_i .*
- (b) *The total transform $\sigma^{r*}(\mathcal{I})$ is the ideal of a simple normal crossing divisor \tilde{E} which is a natural combination of the irreducible components of the divisor E_r .*

The morphism $(\tilde{X}, \tilde{\mathcal{I}}) \rightarrow (X, \mathcal{I})$ defined by the above principalization commutes with smooth morphisms and embeddings of ambient varieties. It is equivariant with respect to any group action not necessarily preserving the ground field K .

Remark. Note that the blow-up of codimension one components is an isomorphism. However it defines a nontrivial transformation of marked ideals. In the actual desingularization process this kind of blow-up may occur for some marked ideals induced on subvarieties of ambient varieties. Though they define isomorphisms of those subvarieties they determine blow-ups of ambient varieties which are not isomorphisms.

Remarks. (1) By the exceptional divisor of the blow-up $\sigma : X' \rightarrow X$ with a smooth center C we mean the inverse image $E := \sigma^{-1}(C)$ of the center C . By the exceptional divisor of the composite of blow-ups σ_i with smooth centers C_{i-1} we mean the union of the strict transforms of the exceptional divisors of σ_i . This definition coincides with the standard definition of the exceptional set of points of the birational morphism in the case when $\text{codim}(C_i) \geq 2$ (as in Theorem 8.1.2). If $\text{codim}(C_{i-1}) = 1$ the blow-up of C_{i-1} is an identical isomorphism and defines a formal operation of converting a subvariety $C_{i-1} \subset X_{i-1}$ into a component of the exceptional divisor E_i on X_i . This formalism is convenient for the proofs. In particular it indicates that C_{i-1} identified via σ_i with a component of E_i has simple normal crossings with other components of E_i .

(2) In the Theorem 8.1.2 we blow up centers of codimension ≥ 2 and both definitions coincide.

8.2. Hironaka resolution principle. The proof of the strong embedded and nonembedded Hironaka resolution with normal crossing centers builds upon the following theorem connecting resolution of marked ideals and strong desingularization of varieties.

Definition 8.2.1. Let K be a base field. By a *canonical resolution of marked ideals* we mean a functor which associate with any marked ideal (X, \mathcal{I}, E, μ) over K a unique resolution (X_i) . such that

- (1) For any smooth morphism $\phi : X' \rightarrow X$ the induced resolution $\phi^*(X_i)$ defines the canonical resolution of $\phi^*(X, \mathcal{I}, E, \mu)$.
- (2) (X_i) commutes with (separable) ground field extensions.

Remark. The canonical resolution of marked ideals constructed here does not commute with embedding of ambient varieties. Such a commutativity holds only for the marked ideals of the special type $(X, \mathcal{I}, \emptyset, 1)$.

Theorem 8.2.2. *Assume $\text{char}(K) = 0$. Then the following implications hold true:*

- (1) **Canonical resolution of marked ideals**
- \Downarrow
- (2) **Canonical strong Embedded Desingularization with normally flat centers**
- \Downarrow
- (3) **Canonical strong Desingularization with normally flat centers**

Remark. The proof of the first implication is more straightforward and shorter if we do not impose that the desingularization commutes with embeddings. This property is useful in the proof of the second implication. To ensure this condition we use the concept of the minimal embedding spaces.

Proof. (1) \Rightarrow (2) In the considerations below by resolving multiple marked ideal defined by Rees algebra R we shall mean the canonical resolution of their canonical capacitors \bar{R} . Let Y be a reduced closed subscheme of a smooth scheme X over a field K , and $E = \{D_1, \dots, D_k\}$ be SNC divisor on X . Denote by $\mathcal{I} := \mathcal{I}_Y \subset \mathcal{O}_X$ the coherent sheaf of ideals of Y . Let $H_1 := H_{\mathcal{I}}$ be the maximal value of the Hilbert-Samuel function of \mathcal{I} on X , and let S_1 be the corresponding stratum. Consider a minimal embedding space T_1 for Y on X (corresponding to S_1) and let $R(\mathcal{I})$ be the canonical Rees algebra. The divisors $E = \{D_1, \dots, D_k\}$ on X' might not be transversal to the locally defined minimal embedded space T_1 . We consider the relative Rees algebra $\mathcal{R}(\mathcal{I}, E)$, and its restriction to its minimal embedding space. Its resolution creates a strict transform \mathcal{I}' of \mathcal{I} with

$$S'_1 \cap \bigcup_{i=1}^k E'_i = \text{cosupp}(\mathcal{R}(\mathcal{I}')) \cap \bigcup_{i=1}^k E'_i = \emptyset$$

In other words it moves the intersection of k divisors (the strict transforms E'_i away from the the Samuel stratum $S_1 \text{cosupp}(\mathcal{R}(\mathcal{I}))$. By Lemma 7.4.2 and canonicity, the resolution is independent of the choice of the minimal embedded space. The strict transform of \mathcal{I}' describes the strict transform Y' of Y on X' and T' . That is $\mathcal{I}' = \mathcal{I}_{Y'}$. In particular all the centers are contained in $Y' \subset T'$.

Now, consider all possible intersections of $k - 1$ divisors in E' . Note that those intersections are disjoint and ordered. Then for each subset of $k - 1$ divisors E'' of E' apply the canonical resolution of the restriction of $\mathcal{R}(\mathcal{I}, E'')$ to its minimal embedding space. After that all the intersection of any $k - 1$ strict transforms of the divisors will be moved away from the Samuel stratum. Repeating the procedure for $k - 2, k - 3, \dots, 1$ will result in the variety X_{1A} with the Samuel stratum having no intersection with the strict transforms of E . Let T_{1A} be the strict transform of T_1 on X_{1A} and R_{1A} be the controlled transform of R_2 . Note that

passing from X_1 to X_{1A} produces "new" exceptional divisors E_{1A} from the centers contained in the strict transform of T_1 . They are transversal to T_{1A} , and their restrictions to T_{1A} define SNC divisors E_{1A}^T .

This defines a (multiple) marked ideal (X_{1A}, R_{1A}, E_{1A}) and its restriction $(T_{1A}, R_{1A}^T, E_{1A}^T)$. Resolving canonically R_{1A} restricted to T_{1A} defines a canonical resolution of $(X_1, \mathcal{I}_1, H_1)$, that is variety X_2 with the maximal value of the Hilbert-Samuel function $H_2 < H_1$. The centers are normally flat by Theorem 7.3.1.

After the resolution the maximal value of the Hilbert-Samuel function $H_{\mathcal{I}_2}$ of the strict transform \mathcal{I}_2 of \mathcal{I}_1 on the variety X_2 drops to the value $H_2 < H_1$ with the Samuel stratum S_2 .

We repeat the procedure for the the strict transform \mathcal{I}_2 on X_2 and H_2 with the induced divisor E_2 and continue the process until the Hilbert-Samuel attains its minimum on the strict transform Y' (or, in a reducible case on each component of the strict transform). This will be done in finitely many steps since by Theorem 6.5.1 the Hilbert-Samuel function has a d.c.c. property and the sequence $\dots < H_3 < H_2 < H_1$ shall stabilize. In the terminal case the Hilbert-Samuel function will attain the minimal value $H_\infty := H_{smooth}$ for a generic smooth point of Y' for all points of Y' . In other words Y' becomes smooth.

To ensure that the strict transform Y' has a SNC with the exceptional divisors we shall continue to run the algorithm for resolution of $(\mathcal{I}, H_{smooth})$ or alternatively its Rees ideal. At some point of the procedure the strict transform of Y will vanish. At this moment the algorithm prescribes to blow up the center which coincides (locally) with the (components) of the strict transform of Y . That is the strict transform has only SNC with the exceptional divisors. We shall stop the algorithm at this stage, and obtain the smooth subscheme Y' having SNC with the exceptional divisors.

It follows from Proposition 7.3.3, and Lemma 7.4.2 that the resulting resolution commutes with closed embedding étale morphisms and field extensions.

(2) \Rightarrow (3) A nonembedded canonical desingularization is obtained by gluing defined locally embedded canonical desingularizations.

We shall need the following:

Proposition 8.2.3. *For any affine variety U there is a smooth variety \tilde{U} along with a birational morphism $\text{res} : \tilde{U} \rightarrow U$ subject to the conditions:*

- (1) *For any closed embedding $U \subset X$ into a smooth affine variety X , there is a closed embedding $\tilde{U} \subset \tilde{X}$ into a smooth variety \tilde{X} which is a canonical embedded desingularization of $U \subset X$.*
- (2) *For any open embedding $V \hookrightarrow U$ there is an open embedding of resolutions $\tilde{V} \hookrightarrow \tilde{U}$ which is a lifting of $V \rightarrow U$ such that $\tilde{V} \rightarrow \text{res}_U^{-1}(V)$ is an isomorphism over V .*

Proof. (1) Consider a closed embedding of U into a smooth affine variety X (for example $X = \mathbf{A}^n$). The canonical embedded desingularization $\tilde{U} \subset \tilde{X}$ of $U \subset X$ defines the desingularization $\tilde{U} \rightarrow U$. This desingularization is independent of the ambient variety X . Let $\phi_1 : U \subset X_1$ and $\phi_2 : U \subset X_2$ be two closed embeddings and let $\tilde{U}_i \subset \tilde{X}_i$ be two embedded desingularizations. Find embeddings $\psi_i : X_i \rightarrow \mathbf{A}^n$ into the affine space \mathbf{A}^n . They define the embeddings $\psi_i \phi_i : U \rightarrow \mathbf{A}^n$. Recall

Lemma 8.2.4. (see [72] or [42]). *For any closed embeddings $\phi_1, \phi_2 : Y \subset \mathbf{A}^n$ there exist closed embeddings $\psi_1, \psi_2 : \mathbf{A}^n \rightarrow \mathbf{A}^{2n}$ such that $\psi_1 \phi_1 = \psi_2 \phi_2$.*

By Lemma 8.2.4, there are embeddings $\Psi_i : \mathbf{A}^n \rightarrow \mathbf{A}^{2n}$ such that $\Psi_1 \psi_1 \phi_1 = \Psi_2 \psi_2 \phi_2 : U \rightarrow \mathbf{A}^{2n}$. Since embedded desingularizations commute with closed embeddings of ambient varieties we see that the \tilde{U}_i are isomorphic over U .

(2) Let $V \rightarrow U$ be an open embedding of affine varieties. Assume first that $V = U_f = U \setminus V(f)$, where $f \in K[U]$ is a regular function on U . Let $U \subset X$ be a closed embedding into an affine variety X . Then $U_f \subset X_f$ is a closed embedding into an affine variety $X_f = X \setminus V(F)$ where F is a regular function on F which restricts to f . Since embedded desingularizations commute with smooth morphisms the open embedding $X_f \subset X$ defines the open embedding of embedded desingularizations $(\tilde{X}_f, \tilde{U}_f) \subset (\tilde{X}, \tilde{U})$ and the open embedding of desingularizations $\tilde{U}_f \subset \tilde{U}$.

Let $V \subset U$ be any open subset which is an affine variety. Then there are desingularizations $\text{res}_V : \tilde{V} \rightarrow V$ and $\text{res}_U : \tilde{U} \rightarrow U$. Suppose the natural birational map $\tilde{V} \rightarrow \text{res}_U^{-1}(V)$ is not an isomorphism over V . Then we can find an open subset $U_f \subset V$ such that $\text{res}_V^{-1}(U_f) \rightarrow \text{res}_U^{-1}(U_f)$ is not an isomorphism over U_f . But $U_f = V_f$ and by the previous case $\text{res}_V^{-1}(U_f) \simeq \tilde{U}_f = \tilde{V}_f \simeq \text{res}_U^{-1}(U_f)$. \square

To finish the proof of the second implication let Y be an algebraic variety over K . By the compactness of Y we find a cover of affine subsets U_i of Y such that each U_i is embedded in an affine space \mathbf{A}^n for $n \gg 0$. We can assume that the dimension n is the same for all U_i by taking if necessary embeddings of affine spaces $\mathbf{A}^{k_i} \subset \mathbf{A}^n$.

Let U_i be an open affine cover of X . For any two open subsets U_i and U_j set $U_{ij} := U_i \cap U_j$. For any U_i and U_{ij} we find canonical resolutions \widetilde{U}_i and \widetilde{U}_{ij} respectively. By the proposition \widetilde{U}_{ij} can be identified with an open subset of \widetilde{U}_i . We define \widetilde{X} to be a variety obtained by glueing \widetilde{U}_i along \widetilde{U}_{ij} . Then \widetilde{X} is a smooth variety and $\widetilde{X} \rightarrow X$ defines a canonical desingularization independent of the choice of U_{ij} . (For more details see [72]). \square

The proof of principalization follows from the following implication

Theorem 8.2.5. *The following implication holds true:*

$$(1) \text{ (Canonical) Resolution of relative marked ideals } (X, \mathcal{I}, E, \mu) \\ \downarrow \\ (2) \text{ (Canonical) Principalization of the sheaves } \mathcal{I} \text{ on } X$$

Proof. It follows immediately from the definition that a resolution of $(X, \mathcal{I}, E, 1)$ determines a principalization of \mathcal{I} . Denote by $\sigma : X \leftarrow \widetilde{X}$ the morphism defined by a resolution of $(X, \mathcal{I}, E, 1)$. The controlled transform $(\widetilde{\mathcal{I}}, 1) := \sigma^c(\mathcal{I}, 1)$ has empty support. Consequently, $V(\widetilde{\mathcal{I}}) = \emptyset$, and thus $\widetilde{\mathcal{I}}$ is equal to the structural sheaf $\mathcal{O}_{\widetilde{X}}$. This implies that the full transform $\sigma^*(\mathcal{I})$ is principal and generated by the sheaf of ideal of a divisor whose components are the exceptional divisors. The actual process of desingularization is often achieved before $(X, \mathcal{I}, E, 1)$ has been resolved (see [72]) \square

9. CANONICAL RESOLUTION OF MARKED IDEALS

9.1. Hypersurfaces of maximal contact. The concept of the *hypersurfaces of maximal contact* is one of the key points of this proof. It was originated by Abhyankhar, Hironaka, and Giraud and developed in the papers of Bierstone-Milman and Villamayor.

In our terminology we are looking for a smooth hypersurface containing the supports of marked ideals and whose strict transforms under multiple blow-ups contain the supports of the induced marked ideals. Existence of such hypersurfaces allows a reduction of the resolution problem to codimension 1.

First we introduce marked ideals which locally admit hypersurfaces of maximal contact.

Definition 9.1.1. (Villamayor (see [67])) We say that a marked ideal (\mathcal{I}, μ) is of *maximal order* (originally *simple basic object*) if $\max\{\text{ord}_x(\mathcal{I}) \mid x \in X\} \leq \mu$ or equivalently $\mathcal{D}^\mu(\mathcal{I}) = \mathcal{O}_X$. We say that a multiple marked ideal (in particular a Rees algebra) $\overline{\mathcal{I}} = \{(\mathcal{I}_i, \mu_i)\}$ is of *maximal order* if for any point $x \in X$ at least one of the ideals (\mathcal{I}_i, μ_i) is of maximal order in a neighborhood of x .

Any marked ideal of maximal order generates or diff-generates the Rees algebra of maximal order.

Lemma 9.1.2. (Villamayor (see [67])) *Let (\mathcal{I}, μ) be a marked ideal of maximal order and $C \subset \text{cosupp}(\mathcal{I}, \mu)$ be a smooth center. Let $\sigma : X \leftarrow X'$ be a blow-up at $C \subset \text{cosupp}(\mathcal{I}, \mu)$. Then $\sigma^c(\mathcal{I}, \mu)$ is of maximal order.*

Proof. If (\mathcal{I}, μ) is a marked ideal of maximal order then $\mathcal{D}^\mu(\mathcal{I}) = \mathcal{O}_X$. Then by Lemma 6.2.3, $\mathcal{D}^\mu(\sigma^c(\mathcal{I}, \mu)) \supset \sigma^c(\mathcal{D}^\mu(\mathcal{I}), 0) = \mathcal{O}_X$. \square

Lemma 9.1.3. (Villamayor (see [67])) *If (\mathcal{I}, μ) is a marked ideal of maximal order and $0 \leq i \leq \mu$ then $\mathcal{D}^i(\mathcal{I}, \mu)$ is of maximal order.*

Proof. $\mathcal{D}^{\mu-i}(\mathcal{D}^i(\mathcal{I}, \mu)) = \mathcal{D}^\mu(\mathcal{I}, \mu) = \mathcal{O}_X$. \square

Lemma 9.1.4. (Giraud (see [31])) *Let (\mathcal{I}, μ) be the marked ideal of maximal order. Let $\sigma : X \leftarrow X'$ be a blow-up at a smooth center $C \subset \text{cosupp}(\mathcal{I}, \mu)$. Let $u \in \mathcal{D}^{\mu-1}(\mathcal{I}, \mu)(U)$ be a function such that, for any $x \in V(u)$, $\text{ord}_x(u) = 1$. Then*

- (1) $V(u)$ is smooth.
- (2) $\text{cosupp}(\mathcal{I}, \mu) \cap U \subset V(u)$

Let $U' \subset \sigma^{-1}(U) \subset X'$ be an open set where the exceptional divisor is described by y . Let $u' := \sigma^c(u) = y^{-1}\sigma^*(u)$ be the controlled transform of u . Then

- (1) $u' \in \mathcal{D}^{\mu-1}(\sigma^c(\mathcal{I}_{|U'}, \mu))$.
- (2) $V(u')$ is smooth.
- (3) $\text{cosupp}(\mathcal{I}', \mu) \cap U' \subset V(u')$
- (4) $V(u')$ is the restriction of the strict transform of $V(u)$ to U' .

Proof. (1) $u' = \sigma^c(u) = u/y \in \sigma^c(\mathcal{D}^{\mu-1}(\mathcal{I})) \subset \mathcal{D}^{\mu-1}(\sigma^c(\mathcal{I}))$.

(2) Since u was one of the local parameters describing the center of blow-ups, $u' = u/y$ is a parameter, that is, a function of order one.

(3) follows from (2). □

Definition 9.1.5. We shall call a function

$$u \in T(\mathcal{I})(U) := \mathcal{D}^{\mu-1}(\mathcal{I}(U))$$

of multiplicity one a *tangent direction* of (\mathcal{I}, μ) on U .

As a corollary from the above we obtain the following lemma:

Lemma 9.1.6. (*Giraud*) Let $u \in T(\mathcal{I})(U)$ be a tangent direction of (\mathcal{I}, μ) on U . Then for any multiple blow-up (U_i) of (\mathcal{I}_U, μ) all the supports of the induced marked ideals $\text{cosupp}(\mathcal{I}_i, \mu)$ are contained in the strict transforms $V(u)_i$ of $V(u)$. □

Remarks. (1) Tangent directions are functions defining locally hypersurfaces of maximal contact.

(2) The main problem leading to complexity of the proofs is that of noncanonical choice of the tangent directions. We overcome this difficulty by introducing *homogenized ideals*.

Corollary 9.1.7. If R is a differential Rees algebra of maximal order then for any point $x \in X$, there is a tangent direction $u \in R^1(U)$ in a neighborhood U of x .

9.2. Properties of Rees algebra: Graded homogenization. An important properties of Rees algebras of maximal order is that for any tangent directions it "looks the same". This property allows to run induction and assures that the restriction of the Rees algebra to the maximal contact does not depend on the choice of tangent directions. From that perspective it can be considered as a more general "homogenization"- a technique which allows to run induction in a canonical way without using, so called, Hironaka's trick. On the other hand the technique of Rees algebra saturation is better suited for the case of multiple generators like in the case of ideals along stratum or in positive characteristic. Thus the notion of Rees algebra can be thus considered as graded homogenization.

Lemma 9.2.1. (Glueing Lemma) ([72]) Let R be a differential Rees algebra of maximal order on a smooth X . Let $u, v \in R^1(X)$ be two tangent directions at $x \in \text{cosupp}(\overline{R})$ which are transversal to the set of exceptional divisors E . Then there exist étale neighborhoods $\phi_u, \phi_v : \overline{X} \rightarrow X$ of $x = \phi_u(\overline{x}) = \phi_v(\overline{x}) \in X$, where $\overline{x} \in \overline{X}$, such that

- (1) $\phi_u^*(R) = \phi_v^*(R)$.
- (2) $\phi_u^{-1}(E) = \phi_v^{-1}(E)$.
- (3) $\phi_u^*(u) = \phi_v^*(v)$.

$$\text{Set } \overline{R} := \phi_u^*(R) = \phi_v^*(R), \quad \overline{E} := \phi_u^{-1}(E) = \phi_v^{-1}(E)$$

(4) For any $\overline{y} \in \text{cosupp}(\overline{X}, \overline{R}, \overline{E})$, $\phi_u(\overline{y}) = \phi_v(\overline{y})$.

(5) For any multiple blow-up (X_i) of (X, R, \emptyset, μ) the induced multiple blow-ups $\phi_u^*(X_i)$ and $\phi_v^*(X_i)$ of $(\overline{X}, \overline{R}, \overline{E})$ are the same (defined by the same centers).

$$\text{Set } (\overline{X}_i) := \phi_u^*(X_i) = \phi_v^*(X_i).$$

(6) For any $\overline{y}_i \in \text{cosupp}(\overline{X}_i, \overline{R}_i, \overline{E}_i)$ and the induced morphisms $\phi_{ui}, \phi_{vi} : \overline{X}_i \rightarrow X_i$, $\phi_{ui}(\overline{y}_i) = \phi_{vi}(\overline{y}_i)$.

Proof. We use the same strategy as applied already to Rees algebras in Lemma 7.4.2. (See more details in [72]) □

9.3. Properties of Rees algebra: Coefficient ideals. A differential Rees algebra has another important feature which is used in the inductive scheme. It has a good restriction properties and can serve as a coefficient ideal. This can be expressed by the following proposition:

Proposition 9.3.1. *Let (X, R, E) be a differential Rees algebra of maximal order. Assume that $S \subset X$ is a smooth subvariety which has only simple normal crossings with E . Then*

$$\text{cosupp}(R) \cap S = \text{cosupp}((\mathcal{R}')_{|S}).$$

Moreover let (X_i) be a multiple blow-up with centers C_i contained in the strict transforms $S_i \subset X_i$ of S . Then

- (1) The restrictions $\sigma_{i|S_i} : S_i \rightarrow S_{i-1}$ of the morphisms $\sigma_i : X_i \rightarrow X_{i-1}$ define a multiple blow-up (S_i) of $\mathcal{C}(\mathcal{I}, \mu)_{|S}$.
- (2) $\text{cosupp}(\bar{\mathcal{I}}_i, \mu) \cap S_i = \text{cosupp}(((\mathcal{R}'(\bar{\mathcal{I}}))_{|S})_i)$.
- (3) Every multiple blow-up (S_i) of $\text{cosupp}((\mathcal{R}'(\bar{\mathcal{I}}))_{|S})$ defines a multiple blow-up (X_i) of (\mathcal{I}, μ) with centers C_i contained in the strict transforms $S_i \subset X_i$ of $S \subset X$.

Proof. First observe that $\text{cosupp}((\mathcal{R}' \cap S)) \subseteq \text{cosupp}((\mathcal{R}')_{|S})$.

Let $x_1, \dots, x_k, y_1, \dots, y_{n-k}$ be local parameters at x such that $\{x_1 = 0, \dots, x_k = 0\}$ describes S . Then any function $f \in \mathcal{I}$ can be written as

$$f = \sum c_{\alpha f}(y) x^{\alpha},$$

where $c_{\alpha f}(y)$ are formal power series in y_i .

Now $x \in \text{cosupp}(\mathcal{R}' \cap S)$ iff $\text{ord}_x(c_{\alpha}) \geq \mu - |\alpha|$ for all $f \in R^{\mu}$ and $|\alpha| \leq \mu$. Note that

$$c_{\alpha f|S} = \left(\frac{1}{\alpha!} \frac{\partial^{|\alpha|}(f)}{\partial x^{\alpha}} \right)_{|S} \in \mathcal{D}^{|\alpha|}(\mathcal{I})_{|S}$$

and consequently $\text{cosupp}(R^{\mu}, \mu) \cap S = \bigcap_{f \in \mathcal{I}, |\alpha| \leq \mu} \text{cosupp}(c_{\alpha f|S}, \mu - |\alpha|) \supseteq \text{cosupp}((\mathcal{R}')_{|S})$.

The above relation is preserved by admissible multiple blow-ups of $\bar{\mathcal{I}}$. For the details see [72]. □

One can reformulate a useful corollary for the capacitor

Corollary 9.3.2. *Let (X, \mathcal{I}, E, μ) be a marked ideal of maximal order. Assume that S has only simple normal crossings with E . Then $R^c(\mathcal{I}, \mu)$ is capacitor for $R(\mathcal{I}, \mu)$, where $c := \mu!$, and*

$$\text{cosupp}(\mathcal{I}, \mu) \cap S = \text{cosupp}(R^c(\mathcal{I}, \mu)_{|S}).$$

Moreover let (X_i) be a multiple blow-up with centers C_i contained in the strict transforms $S_i \subset X_i$ of S . Then

- (1) The restrictions $\sigma_{i|S_i} : S_i \rightarrow S_{i-1}$ of the morphisms $\sigma_i : X_i \rightarrow X_{i-1}$ define a multiple blow-up (S_i) of $\mathcal{C}(\mathcal{I}, \mu)_{|S}$.
- (2) $\text{cosupp}(\mathcal{I}_i, \mu) \cap S_i = \text{cosupp}[R^c(\mathcal{I}, \mu)_{|S}]_i$.
- (3) Every multiple blow-up (S_i) of $R^c(\mathcal{I}, \mu)_{|S}$ defines a multiple blow-up (X_i) of (\mathcal{I}, μ) with centers C_i contained in the strict transforms $S_i \subset X_i$ of $S \subset X$.

These properties allow one to control and modify the part of support of (\mathcal{I}, μ) contained in S by applying multiple blow-ups of $\text{cosupp}((\mathcal{R}^c(\mathcal{I}, \mu))_{|S})$.

Proof. $R^c(\mathcal{I}, \mu)$ is generated as Rees algebra by $(\mathcal{D}^i(\mathcal{I}), \mu - i)$. The means that $R^c(\mathcal{I}, \mu) \subset \sum (\mathcal{D}^i(\mathcal{I}), \mu - i)$. As in the precious proof for any $f = \sum c_{\alpha f}(y) x^{\alpha} \in \mathcal{I}$, we have that $x \in \text{cosupp}(\mathcal{I} \cap S)$ iff $\text{ord}_x(c_{\alpha}) \geq \mu - |\alpha|$ for all $f \in R^{\mu}$ and $|\alpha| \leq \mu$. The rest of the reasoning is the same. □

9.4. Resolution algorithm. The presentation of the following Hironaka resolution algorithm builds upon Bierstone-Milman's, Villamayor's and Włodarczyk's algorithms which are simplifications of the original Hironaka proof. We also use Kollár's trick allowing to completely eliminate the use of invariants. It is possible to rewrite the algorithm in the language of Rees algebras and even make it independent of the dimension of embeddings. The relevant elegant notion of Rees algebra of marked ideals replaces the concept of the coefficient and homogenized ideals. The direct constructions lead to rather technical language of the multiple fractional marked ideals and Rees algebras with fractional grading. To avoid it one can consider, for

instance an integral subfiltration of Rees algebras, or simply capacitor as in our case. The latter is simpler conceptually and allows to work with (single) marked ideals, while using Rees algebras as auxilliary tool for the constructions.

Remarks. (1) Note that the blow-up of codimension one components is an isomorphism. However it defines a nontrivial transformation of marked ideals. The inverse image of the center is still called the exceptional divisor.

- (2) In the actual desingularization process this kind of blow-up may occur for some marked ideals induced on subvarieties of ambient varieties. Though they define isomorphisms of those subvarieties they determine blow-ups of ambient varieties which are not isomorphisms.
- (3) The blow-ups of the center C which coincides with the whole variety X is an empty set. The main feature which characterizes is given by the restriction property:

If X is a smooth variety containing a smooth subvariety $Y \subset X$, which contains the center $C \subset Y$ then the blow-up $\sigma_{C,Y} : \tilde{Y} \rightarrow Y$ at C coincides with the strict transform of Y under the blow-up $\sigma_{C,X} : \tilde{X} \rightarrow X$, i.e

$$\tilde{Y} \simeq \overline{\sigma_{C,X}^{-1}(Y \setminus C)}$$

Theorem 9.4.1. *For any marked ideal (X, \mathcal{I}, E, μ) such that \mathcal{I} there is an associated resolution $(X_i)_{0 \leq i \leq m_X}$, called canonical, satisfying the following conditions:*

- (1) *For any surjective étale morphism $\phi : X' \rightarrow X$ the induced sequence $(X'_i) = \phi^*(X_i)$ is the canonical resolution of $(X', \mathcal{I}', E', \mu) := \phi^*(X, \mathcal{I}, E, \mu)$.*
- (2) *Any étale morphism $\phi : X' \rightarrow X$ induces the sequence $(X'_i) = \phi^*(X_i)$ consisting of blow-ups of the canonical resolution of $(X', \mathcal{I}', E', \mu) := \phi^*(X, \mathcal{I}, E, \mu)$ and (possibly) some identical transformations.*

Proof. If $\mathcal{I} = 0$ and $\mu > 0$ then $\text{cosupp}(X, \mathcal{I}, \mu) = X$, and the blow-up of X is the empty set and thus it defines a unique resolution. Assume that $\mathcal{I} \neq 0$.

We shall use the induction on the dimension of X . If X is 0-dimensional, $\mathcal{I} \neq 0$ and $\mu > 0$ then $\text{cosupp}(X, \mathcal{I}, \mu) = \emptyset$ and all resolutions are trivial.

Step 1 Resolving a marked ideal (X, \mathcal{J}, E, μ) of maximal order.

Before performing the resolution algorithm for the marked ideal (\mathcal{J}, μ) of maximal order in Step 1 we shall replace it with the equivalent to the capacitor ideal $\mathcal{R}^c(\mathcal{J}, \mu)$. Resolving the ideal $\mathcal{R}^c(\mathcal{J}, \mu)$ defines a resolution of (\mathcal{J}, μ) at this step. To simplify notation we shall denote $\mathcal{R}^c(\mathcal{J}, \mu)$ by $(\overline{\mathcal{J}}, \overline{\mu})$.

Step 1a Reduction to the nonboundary case. Moving $\text{cosupp}(\overline{\mathcal{J}}, \overline{\mu})$ and H_α^s apart . For any multiple blow-up (X_i) of $(X, \overline{\mathcal{J}}, E, \overline{\mu})$ we shall identify (for simplicity) strict transforms of E on X_i with E .

For any $x \in X_i$, let $s(x)$ denote the number of divisors in E through x and set

$$s_i = \max\{s(x) \mid x \in \text{cosupp}(\overline{\mathcal{J}}_i)\}.$$

Let $s = s_0$. By assumption the intersections of any $s > s_0$ components of the exceptional divisors are disjoint from $\text{cosupp}(\overline{\mathcal{J}}, \overline{\mu})$. Each intersection of divisors in E is locally defined by intersection of some irreducible components of these divisors. Find all intersections H_α^s , $\alpha \in A$, of s irreducible components of divisors E such that $\text{cosupp}(\overline{\mathcal{J}}, \overline{\mu}) \cap H_\alpha^s \neq \emptyset$. By the maximality of s , the supports $\text{cosupp}(\overline{\mathcal{J}}|_{H_\alpha^s}) \subset H_\alpha^s$ are disjoint from $H_{\alpha'}^s$, where $\alpha' \neq \alpha$.

Set

$$H^s := \bigcup_{\alpha} H_\alpha^s, \quad U^s := X \setminus H^{s+1}, \quad \underline{H}^s := H^s \setminus H^{s+1}.$$

Then $\underline{H}^s \subset U_s$ is a smooth closed subset U_s . Moreover $\underline{H}^s \cap \text{cosupp}(\mathcal{I}) = H^s \cap \text{cosupp}(\mathcal{I})$ is closed.

Construct the canonical resolution of $\overline{\mathcal{J}}|_{\underline{H}^s}$. By Lemma 9.3.1, it defines a multiple blow-up of $(\overline{\mathcal{J}}, \overline{\mu})$ such that

$$\text{cosupp}(\overline{\mathcal{J}}_{j_1}, \overline{\mu}) \cap H_{j_1}^s = \emptyset.$$

In particular the number of the strict transforms of E passing through a single point of the support drops $s_{j_1} < s$. Now we put $s = s_{j_1}$ and repeat the procedure. We continue the above process till $s_{j_k} = s_r = 0$. Then $(X_j)_{0 \leq j \leq r}$ is a multiple blow-up of $(X, \overline{\mathcal{J}}, E, \overline{\mu})$ such that $\text{cosupp}(\overline{\mathcal{J}}_r, \overline{\mu})$ does not intersect any divisor in E .

Therefore $(X_j)_{0 \leq j \leq r}$ and further longer multiple blow-ups $(X_j)_{0 \leq j \leq m}$ for any $m \geq r$ can be considered as multiple blow-ups of $(X, \overline{\mathcal{J}}, \emptyset, \overline{\mu})$ since starting from X_r the strict transforms of E play no further role in the resolution process since they do not intersect $\text{cosupp}(\overline{\mathcal{J}}_j, \overline{\mu})$ for $j \geq r$. We reduce the situation to the "nonboundary case".

Step 1b. Nonboundary case

Let $(X_j)_{0 \leq j \leq r}$ be the multiple blow-up of $(X, \overline{\mathcal{J}}, \emptyset, \overline{\mu})$ defined in Step 1a.

For any $x \in \text{cosupp}(\overline{\mathcal{J}}, \overline{\mu}) \subset X$ find a tangent direction $u_\alpha \in \mathcal{D}^{\overline{\mu}-1}(\overline{\mathcal{J}})$ on some neighborhood U_α of x . Then $V(u_\alpha) \subset U_\alpha$ is a hypersurface of maximal contact. By the quasicompactness of X we can assume that the covering defined by U_α is finite. Let $U_{i\alpha} \subset X_i$ be the inverse image of U_α and let $H_{i\alpha} := V(u_\alpha)_i \subset U_{i\alpha}$ denote the strict transform of $H_\alpha := V(u_\alpha)$.

Set (see also [47])

$$\tilde{X} := \coprod U_\alpha \quad \tilde{H} := \coprod H_\alpha \subseteq \tilde{X}.$$

The closed embeddings $H_\alpha \subseteq U_\alpha$ define the closed embedding $\tilde{H} \subset \tilde{M}$ of a hypersurface of maximal contact \tilde{H} .

Consider the surjective étale morphism

$$\phi_U : \tilde{X} := \coprod U_\alpha \rightarrow X$$

Denote by $\tilde{\mathcal{J}}$ the pull back of the ideal sheaf $\overline{\mathcal{J}}$ via ϕ_U . The multiple blow-up $(X_i)_{0 \leq i \leq r}$ of $\overline{\mathcal{J}}$ defines a multiple blow-up $(\tilde{X}_{0 \leq i \leq r})$ of $\tilde{\mathcal{J}}$ and a multiple blow-up $(\tilde{H}_i)_{0 \leq i \leq r}$ of $\tilde{\mathcal{J}}|_{\tilde{H}}$.

Construct the canonical resolution of $(\tilde{H}_i)_{r \leq i \leq m}$ of the marked ideal $\tilde{\mathcal{J}}_{r|\tilde{H}_r}$ on \tilde{H}_r . It defines, by Lemma 7.5.5, a resolution $(\tilde{X}_{r \leq i \leq m})$ of $\tilde{\mathcal{J}}_r$ and thus also a resolution $(\tilde{X}_i)_{0 \leq i \leq m}$ of $(\tilde{X}, \tilde{\mathcal{J}}, \emptyset, \overline{\mu})$. Moreover both resolutions are related by the property

$$\text{cosupp}(\tilde{\mathcal{J}}_i) = \text{cosupp}(\tilde{\mathcal{J}}_{i|\tilde{H}_i}).$$

Consider a (possible) lifting of ϕ_U :

$$\phi_{iU} : \tilde{X}_i := \coprod U_{i\alpha} \rightarrow X_i,$$

which is a surjective locally étale morphism. The lifting is constructed for $0 \leq i \leq r$.

For $r \leq i \leq m$ the resolution \tilde{X}_i is induced by the canonical resolution $(\tilde{H}_i)_{r \leq i \leq m}$ of $\overline{\mathcal{J}}_{r|\tilde{H}_r}$.

We show that the resolution $(\tilde{X}_i)_{r \leq i \leq m}$ descends to the resolution $(X_i)_{r \leq i \leq m}$.

Let $\tilde{C}_{j_0} = \coprod C_{j_0\alpha}$ be the center of the blow-up $\tilde{\sigma}_{j_0} : \tilde{X}_{j_0+1} \rightarrow \tilde{X}_{j_0}$. The closed subset $C_{j_0\alpha} \subset U_{j_0\alpha}$ defines the center of an extension of the canonical resolution $(H_{j\alpha})_{r \leq j \leq m}$.

If $C_{j_0\alpha} \cap U_{j_0\beta} \neq \emptyset$ then by the canonicity and condition (2) of the inductive assumption, the subset $C_{j_0\alpha\beta} := C_{j_0\alpha} \cap U_{j_0\beta}$ defines the center of an extension of the canonical resolution $H_{j\alpha\beta} := ((H_{j\alpha} \cap U_{j\beta}))_{r \leq j \leq m}$. On the other hand $C_{j_0\beta\alpha} := C_{j_0\beta} \cap U_{j_0\alpha}$ defines the center of an extension of the canonical resolution $(H_{j\beta\alpha} := H_{j\beta} \cap U_{j\alpha})_{r \leq j \leq m}$.

By Glueing Lemma 9.2.1 for the tangent directions u_α and u_β we find there exist étale neighborhoods $\phi_{u_\alpha}, \phi_{u_\beta} : \overline{U}_{\alpha\beta} \rightarrow U_{\alpha\beta} := U_\alpha \cap U_\beta$ of $x = \phi_u(\bar{x}) = \phi_v(\bar{x}) \in X$, where $\bar{x} \in \overline{X}$, such that

- (1) $\phi_{u_\alpha}^*(\mathcal{J}) = \phi_{u_\beta}^*(\mathcal{J})$.
- (2) $\phi_{u_\alpha}^*(E) = \phi_{u_\beta}^*(E)$.
- (3) $\phi_{u_\alpha}^{-1}(H_{j\alpha\beta}) = \phi_{u_\beta}^{-1}(H_{j\beta\alpha})$.
- (4) $\phi_{u_\alpha}(\bar{x}) = \phi_{u_\beta}(\bar{x})$ for $\bar{x} \in \text{cosupp}(\phi_{u_\alpha}^*(\mathcal{J}))$.

Moreover all the properties lift to the relevant étale morphisms $\phi_{u_{\alpha i}}, \phi_{u_{\beta i}} : \overline{U}_{\alpha\beta i} \rightarrow U_{\alpha\beta i}$. Consequently, by canonicity $\phi_{u_{\alpha j_0}}^{-1}(C_{j_0\alpha\beta})$ and $\phi_{u_{\beta j_0}}^{-1}(C_{j_0\beta\alpha})$ define both the next center of the extension of the canonical resolution $\phi_{u_\alpha}^{-1}(H_{j_0\alpha\beta}) = \phi_{u_\beta}^{-1}(H_{j_0\beta\alpha})$ of $\phi_{u_\alpha}^*(\mathcal{J}|_{H_{\alpha\beta}}) = \phi_{u_\beta}^*(\mathcal{J}|_{H_{\beta\alpha}})$.

Thus

$$\phi_{u_\alpha}^{-1}(C_{j_0\alpha\beta}) = \phi_{u_\beta}^{-1}(C_{j_0\beta\alpha}),$$

and finally, by property (4),

$$C_{j_0\alpha\beta} = C_{j_0\beta\alpha}.$$

Consequently \widetilde{C}_{j_0} descends to the smooth closed center $C_{j_0} = \bigcup C_{j_0\alpha} \subset X_{j_0}$ and the resolution $(\widetilde{X}_i)_{r \leq i \leq m}$ descends to the resolution $(X_i)_{r \leq i \leq m}$.

Step 2. Resolving of marked ideals (X, \mathcal{I}, E, μ) .

For any marked ideal (X, \mathcal{I}, E, μ) write

$$I = \mathcal{M}(\mathcal{I})\mathcal{N}(\mathcal{I}),$$

where $\mathcal{M}(\mathcal{I})$ is the *monomial part* of \mathcal{I} , that is, the product of the principal ideals defining the irreducible components of the divisors in E , and $\mathcal{N}(\mathcal{I})$ is a *nonmonomial part* which is not divisible by any ideal of a divisor in E . Let

$$\text{ord}_{\mathcal{N}(\mathcal{I})} := \max\{\text{ord}_x(\mathcal{N}(\mathcal{I})) \mid x \in \text{cosupp}(\mathcal{I}, \mu)\}.$$

Definition 9.4.2. (Hironaka, Bierstone-Milman, Villamayor, Encinas-Hauser) By the *companion ideal* of (\mathcal{I}, μ) where $I = \mathcal{N}(\mathcal{I})\mathcal{M}(\mathcal{I})$ we mean the marked ideal of maximal order

$$O(\mathcal{I}, \mu) = \begin{cases} (\mathcal{N}(\mathcal{I}), \text{ord}_{\mathcal{N}(\mathcal{I})}) & + & (\mathcal{M}(\mathcal{I}), \mu - \text{ord}_{\mathcal{N}(\mathcal{I})}) & \text{if } \text{ord}_{\mathcal{N}(\mathcal{I})} < \mu, \\ (\mathcal{N}(\mathcal{I}), \text{ord}_{\mathcal{N}(\mathcal{I})}) & & & \text{if } \text{ord}_{\mathcal{N}(\mathcal{I})} \geq \mu. \end{cases}$$

In particular $O(\mathcal{I}, \mu) = (\mathcal{I}, \mu)$ for ideals (\mathcal{I}, μ) of maximal order.

Step 2a. Reduction to the monomial case by using companion ideals

By Step 1 we can resolve the marked ideal of maximal order $(\mathcal{J}, \mu_{\mathcal{J}}) := O(\mathcal{I}, \mu)$. By Lemma 7.5.4, for any multiple blow-up of $O(\mathcal{I}, \mu)$,

$$\begin{aligned} \text{cosupp}(O(\mathcal{I}, \mu))_i &= \text{cosupp}[\mathcal{N}(\mathcal{I}), \text{ord}_{\mathcal{N}(\mathcal{I})}]_i \cap \text{cosupp}[\mathcal{M}(\mathcal{I}), \mu - \text{ord}_{\mathcal{N}(\mathcal{I})}]_i = \\ &= \text{cosupp}[\mathcal{N}(\mathcal{I}), \text{ord}_{\mathcal{N}(\mathcal{I})}]_i \cap \text{cosupp}(\mathcal{I}_i, \mu). \end{aligned}$$

Consequently, such a resolution leads to the ideal (\mathcal{I}_{r_1}, μ) such that $\text{ord}_{\mathcal{N}(\mathcal{I}_{r_1})} < \text{ord}_{\mathcal{N}(\mathcal{I})}$. Then we repeat the procedure for (\mathcal{I}_{r_1}, μ) . We find marked ideals $(\mathcal{I}_{r_0}, \mu) = (\mathcal{I}, \mu), (\mathcal{I}_{r_1}, \mu), \dots, (\mathcal{I}_{r_m}, \mu)$ such that $\text{ord}_{\mathcal{N}(\mathcal{I}_{r_0})} > \text{ord}_{\mathcal{N}(\mathcal{I}_{r_1})} > \dots > \text{ord}_{\mathcal{N}(\mathcal{I}_{r_m})}$. The procedure terminates after a finite number of steps when we arrive at the ideal (\mathcal{I}_{r_m}, μ) with $\text{ord}_{\mathcal{N}(\mathcal{I}_{r_m})} = 0$ or with $\text{cosupp}(\mathcal{I}_{r_m}, \mu) = \emptyset$. In the second case we get the resolution. In the first case $\mathcal{I}_{r_m} = \mathcal{M}(\mathcal{I}_{r_m})$ is monomial.

Step 2b. Monomial case $\mathcal{I} = \mathcal{M}(\mathcal{I})$.

Let x_1, \dots, x_k define equations of the components $D_1^x, \dots, D_k^x \in E$ through $x \in \text{cosupp}(X, \mathcal{I}, E, \mu)$ and \mathcal{I} be generated by the monomial x^{a_1, \dots, a_k} at x . In particular

$$\text{ord}_x(\mathcal{I})(x) := a_1 + \dots + a_k.$$

Let $\rho(x) = \{D_{i_1}, \dots, D_{i_l}\} \in \text{Sub}(E)$ be the maximal subset satisfying the properties

- (1) $a_{i_1} + \dots + a_{i_l} \geq \mu$.
- (2) For any $j = 1, \dots, l$, $a_{i_1} + \dots + \check{a}_{i_j} + \dots + a_{i_l} < \mu$.

Let $R(x)$ denote the subsets in $\text{Sub}(E)$ satisfying the properties (1) and (2). The maximal components of the $\text{cosupp}(\mathcal{I}, \mu)$ through x are described by the intersections $\bigcap_{D \in A} D$ where $A \in R(x)$. The maximal locus of ρ determines at most one maximal component of $\text{cosupp}(\mathcal{I}, \mu)$ through each x .

After the blow-up at the maximal locus $C = \{x_{i_1} = \dots = x_{i_l} = 0\}$ of ρ , the ideal $\mathcal{I} = (x^{a_1, \dots, a_k})$ is equal to $\mathcal{I}' = (x^{a_1, \dots, a_{i_j-1}, a, a_{i_j+1}, \dots, a_k})$ in the neighborhood corresponding to x_{i_j} , where $a = a_{i_1} + \dots + a_{i_l} - \mu < a_{i_j}$. In particular the invariant $\text{ord}_x(\mathcal{I})$ drops for all points of some maximal components of $\text{cosupp}(\mathcal{I}, \mu)$. Thus the maximal value of $\text{ord}_x(\mathcal{I})$ on the maximal components of $\text{cosupp}(\mathcal{I}, \mu)$ which were blown up is bigger than the maximal value of $\text{ord}_x(\mathcal{I})$ on the new maximal components of $\text{cosupp}(\mathcal{I}, \mu)$. The algorithm terminates after a finite number of steps. \square

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