PRINCIPALIZATION OF IDEALS, AND CANONICAL DESINGULARIZATION ON LOGARITHMIC VARIETIES

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ABSTRACT. We prove canonical principalization for ideals on varieties with a generalized type of smooth logarithmic structures. We do this by adapting the methods of [Wło05], discarding steps which become redundant. This provides an alternative, self-contained approach to our work [ATW16a, ATW16b], circumventing Deligne–Mumford stacks.

We deduce desingularization of varieties with logarithmic structures, which is functorial with respect to logarithmically smooth morphisms. The result extends Hironaka's desingularization theorem to the logarithmic category in a functorial manner. In particular this work establishes a new, more efficient, resolution algorithm even in the classical situation. This work is the first step in our program to apply logarithmic desingularization to a morphism $X \to S$, aiming to prove functorial semistable reduction theorems.

1. INTRODUCTION

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1.1. The new algorithm: logarithmic aspects.

1.1.1. *Main results of the paper.* We construct logarithmic analogues of the classical algorithms that apply to logarithmic varieties. In Theorem 1.3.2 we establish principalization of ideals on stably toroidal varieties, naturally generalizing toroidal (logarithmically smooth) varieties, and, similarly to the classical case, we deduce non-embedded desingularization of logarithmic varieties in Theorem 1.3.8. Both algorithms are functorial with respect to logarithmically smooth morphisms.

The present paper aims to be self contained, minimizing pre-requisites. Our previous work [ATW16a, ATW16b] relies on both Kato's theory of logarithmic structures and the theory of algebraic stacks, in addition to the methods of resolution of singularities. There has been a persistent resistance in the algebraic geometry community to the adoption of algebraic stacks[1]. This work aims to circumvent the use of algebraic stacks in our work and provide an entry point for those members of the community. For this purpose we work systematically with varieties carrying a new type of logarithmic structures, named here *stably toroidal varieties*.

In particular, classical ideas and results of logarithmic geometry - mostly due to Fontaine, Illusie, and Kato - are revisited and extended to this context, requiring no prior mastery of logaritmic geometry. The new class of logarithmic varieties used here specifically aims to meet the needs of the logarithmic resolution algorithm.

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[1] See, e.g., [Kol08, p. 367]: "The study of stacks is strongly recommended to people who would have been flagellants in earlier times."

The remaining part of this section introduces our logarithmic resolution of singularities results in this new framework of stably toroidal varieties.

1.1.2. *What we gain in the classical case.* In the classical setting one principalizes ideals on toroidal varieties (Y, E) with a smooth Y and SNC divisor E. The algorithm is functorial with respect to smooth morphisms $f\colon Y' \to Y$ with $E' = f^{-1}(E)$. We will see below that our algorithm has less stages, a simpler invariant, and is more efficient, but its main new feature is functoriality with respect to arbitrary logarithmically smooth morphisms $(Y', E') \to (Y, E)$.

1.1.3. *Algorithms and smooth functoriality.* Hironaka's original proof of canonical desingularization was existential, but latter works refined this to canonical (i.e. depending only on X) algorithms, e.g. see [Vil89, BM97, Wło05, Kol07, BM08]. In fact, the latter algorithms possess the following stronger functoriality property: for any smooth morphism $Y \to X$ the obtained desingularizations $Y' \to Y$ and $X' \to X$ are compatible, in particular, $Y' = X' \times_X Y$. This property was first emphasized in [Wło05], and since then it plays an important role in desingularization theory and its applications. For example, it allows to extend the desingularization to stacks and formal schemes, see [Tem12, Section 5], and in fact, it significantly simplifies some arguments in the construction.

1.1.4. *General context.* This paper opens a large project whose aim is to develop a new generation of desingularization algorithms that apply to morphisms. New algorithms should use logarithmic smoothness instead of smoothness and be functorial with respect to arbitrary logarithmically smooth morphisms. Furthermore, the natural category to work with is the category of arbitrary fs logarithmic varieties (or, more generally, schemes) without further restrictions on monoids of the logarithmic structures.

1.1.5. *This paper and the sequels.* This paper is devoted to constructing a new algorithm as above in the simplest possible case, the case of varieties. We expect that the same algorithm will apply to schemes of finite type over valuations rings, and even over a general base S after a fine enough modification of S. This is a work in progress to be worked out in sequel papers. The main reason to restrict the generality here is to avoid technical issues related to non-noetherianity. Even for varieties, our approach contains many new ideas and techniques, that we prefer to develop avoiding unnecessary technical burden.

We aim to revisit a functorial principalization and functorial resolution of singularities in more subtle regimes, such as qe logarithmic schemes, schemes over valuation rings, and logarithmic analytic spaces, in a future manuscript.

1.2. **The new algorithm and stably toroidal varieties.**

main example

1.2.1. *A motivating example.* We start with a simple example. Consider $X_1 = \mathrm{Spec}(k[x, u])$ with logarithmic structure (u) and ideal $I_1 = (x^2 + u^2)$. It is natural to expect that any reasonable functorial algorithm principalizes I_1 by blowing up a single isolated point (x, u), and this will be the case with our algorithm, defining a birational morphism $X_1' \to X_1$ between toroidal varieties.

Consider the toroidal morphism $X_1 = \mathrm{Spec}(k[x, u]) \to X_2 = \mathrm{Spec}(k[x, w])$, the Kummer covering set by $w \mapsto u^2$. It takes $I_2 = (x^2 + w)$ to I_1. By functoriality of principalization we must proceed by blowing up the "ideal" $(x, w^{1/2})$ or alternatively its "power" (x^2, w) on X_2, as it is the transformation of X_2 corresponding to the blow-up of (x, u). The variety X_2' resulting from this blowing up is not toroidal. However the natural morphism $X_1' \to X_2'$ is, in a sense, a Kummer covering by the toroidal variety X_1'.

This leads us to consider non-coherent logarithmic varieties locally possessing Kummer coverings which are toroidal - in other words, varieties which are toroidal locally in the Kummer topology we consider here. It also leads to ideals with fractional monomials as those defined in the Kummer topology.

We will see here that the challenge of functoriality for logarithmically smooth morphisms dictates its own solution.

1.2.2. *Stably toroidal varieties.* Example 1.2.1 motivates the definition of a class of "smooth" varieties more general than logarithmically smooth varieties. This notion is anchored in the *Kummer topology* generated by *Kummer étale* morphisms. Then *stably toroidal varieties* are those logarithmic varieties which are logarithmically smooth locally in the Kummer topology.

This class allows a greater flexibility, while the main technical aspects of the algorithm, and the local considerations, remain nearly unchanged, since they are carried out on logarithmically smooth charts. The class of coherent ideals living on such a Kummer site is substantially larger than the class of ideals on the Zariski or étale sites on a toroidal variety.

In particular, this allows us to work with generators which are *Kummer monomials*, namely monomials with fractional powers. Each time we work with a Kummer monomial we use an appropriate neighborhhod which is a Kummer covering where this element is a usual monomial.

1.3. **Statements of main results.**

1.3.1. *Logarithmic principalization.* Here is the main result of the paper.

Th:principalization

Theorem 1.3.2 (Principalization)**.** *Let (X, U) be a stably toroidal variety and $\mathcal{I}$ a nonzero ideal sheaf on X. There is a sequence $X_n \to X_{n-1} \to \cdots \to X_0 = X$ of blow-ups of smooth-monomial centers $\mathcal{J}_i$ on stably toroidal varieties X_i, all supported over the vanishing locus $V(\mathcal{I})$, such that $\mathcal{I}\mathcal{O}_{X_n}$ is locally monomial (in the Kummer topology). The process is functorial for normally logarithmically smooth base change morphisms $Y \to X$, in the sense that the sequence of Kummer blowing up for Y is the saturated normalized pullback of the sequence for X, with trivial blowings up removed.*

The smooth-monomial centers are those which can be, locally in the Kummer topology, described as $\mathcal{J}_i = (u_{i1}, \ldots, u_{is_i}, m_{i1}, \ldots, m_{ik_i})$, where $u_{i1}, \ldots, u_{is_i}$ are regular parameters, and $m_{i1}, \ldots, m_{ik_i}$ are monomials with respect to the logarithmic structure. In each stage the toroidal structure is enriched by the exceptional locus. Normally logarithmically smooth maps of stably toroidal varieties are the appropriate generalization of logarithmically smooth maps of toroidal varieties.

The saturated normalized pullback in the statement is the closure of $U_Y \times_X X_n$ in the normalization of the usual pullback $Y \times_X X_n$. It forms the pullback in the category of stably toroidal varieties.

The precise meaning of functoriality with "trivial blowings up removed" is spelled out in Theorem 8.2.1 as well as Theorem 1.3.8 below.

1.3.3. *Non-embedded toroidal desingularization.* Classical ideas reducing resolution to principalization apply in the logarithmic setting. In Section 8.4 we obtain the following:

Th:nonembedded 2 **Theorem 1.3.4.** *Let X be a fine and saturated logarithmic variety of finite type over K and assume that X is generically logarithmically smooth and locally equidimensional.*

Then there is a projective morphism $X' \to X$ which is an isomorphism over the logarithmically smooth locus of X and such that X is stably toroidal. The process is functorial with respect to arbitrary log smooth (and, in particular, smooth) morphisms.

1.3.5. *Torification.* Theorem 1.3.4 can be combined with the following result:

Th:Torification **Theorem 1.3.6** (Torification)**.** *Let (X, U) be a stably toroidal variety. There exists a canonical sheaf $\mathcal{I}$ on X, called a* **torific ideal***, such that the normalized blow-up of $\mathcal{I}$ transforms (X, U) into a toroidal variety. Moreover $\mathcal{I}$ is defined functorially with respect to smooth morphisms and products with log smooth varieties.*

In fact, the torification process is functorial with respect to a more general class of isotropical log smooth morphisms described in Section 5.2.26.

1.3.7. *Non-embedded logarithmic desingularization.* Combining stably toroidal desingularization with torification we obtain functorial desingularization by logaritmically smooth varieties.

Th:nonembedded **Theorem 1.3.8** (Non-embedded desingularization)**.** *Let X be a fine and saturated logarithmic variety over K and assume that X is generically logarithmically smooth and locally equidimensional. Then there is a projective morphism $X' \to X$ which is an isomorphism over the logarithmically smooth locus of X such that X' is logarithmically smooth. The process is functorial with respect to isotropical logarithmically smooth morphisms, in particular, with respect to products with logarithmically smooth varieties.*

Functoriality implies, in particular, that the loci where X is smooth with normal crossings divisor are untouched. Functoriality for logarithmically smooth morphisms does not hold in the classical algorithms, and the task of maintaining the normal crossings locus has been considered an important challenge. See [BM97, Section 12],[BM12, Theorem 1.5], [BDMV14, Theorem 1.4] for results on *simple* normal crossings and further discussion.[2]

1.3.9. *Resolution of toroidal varieties.* After completing log-smooth resolution one might also want to resolve the toroidal singularities. This is a classical problem with various solutions. There are algorithms based on the functorial resolution of general varieties, see [Niz06, Theorem 5.10] and [IT14, Theorem 1.1], or a more efficient version [BM03]. These methods work by embedding into a smooth ambient variety. Combinatorial (non-embedded) methods of toric geometry provide much

[2]We emphasize that pinch points do not induce fine and saturated logarithmic structures, hence do not contradict Theorem 1.3.8, see also [Tem09, Appendix A.7].

faster solutions, e.g. see [KKMSD73, II.2 Theorem 11*], [ACMW14, Theorem 4.4.2], [Wło17]. The latter is in fact functorial for smooth morphisms.

1.4. **Smooth, toroidal and stably toroidal varieties.** Our paper links three different kinds of smoothness. They have a manifestation in the languge of differential forms. On a smooth variety the sheaf of differential forms is locally free on the Zariski site; on a logarithmically smooth variety the sheaf of logarithmic differential forms is locally free in the étale site. Finally on a stably toroidal variety the sheaf of logarithmic differentials is locally free on the finest of all three - the Kummer étale site.

Those three different kinds of smoothness determine three types of canonical resolutions:

The most general -*stably toroidal resolution* - is functorial with respect to arbitrary logarithmically smooth morphisms. It implies directly, via a short torification argument, the *logarithmically smooth resolution* which is functorial with respect to smooth morphisms, isotropical log smooth morphisms and products with log smooth varieties. Finally the *Hironaka desingularization* is functorial with respect to smooth morphisms. It can be deduced from desingularization of toroidal varieties.

1.5. **The structure of the paper.** In Sections 2, 3 we lay the foundations of the language of logarithmic geometry in the context of the resolution problem. We study the language of toroidal varieties X (logarithmically smooth varieties) as ambient schemes in the resolution. We introduce the most basic notions here, like logarithmic order, smooth-monomial blow-ups, and study the sheaves of differentiations which live on the toroidal varieties. This is an extension of the theory developed in the simplified proofs of the Hironaka resolution [Vil89, BM97, Wło05, Kol07, BM08]. The language of the ambient toroidal varieties is more universal and flexible then the language of the smooth ambient schemes with SNC divisor. Also the functoriality of the resolution algorithm is considered with respect to the larger class of the log smooth morphisms. Consequently the key logarithmic notions introduced in the algorithm like, for instance, the logarithmic order of ideal, sheaf of differentials and others are necessarily need to be preserved by the logarithmically smooth morphisms, as opposed to the classical situation, where the functoriality is considered for smooth morphisms. To add more functionality on X we consider the, so called Kummer topology, consisting of finite log étale maps (Kummer étale maps) over X. It allows to interpret the monomials with fractional powers on étale site as regular monomials on the corresponding Kummer neighborhoods. As it was mentionned earlier, this is necessary to guarantee log smooth functionality, and it significantly simplifies and optimizes the algorithm avoiding many unnecessary steps.

In the next section 4 we study the notion of stably toroidal varieties, naturally extending the definition of toroidal varieties. Roughly speaking these are varieties which are étale locally isomorphic to finite abelian quotients of toroidal varieties. Looking somewhat differently these are exactly the varieties which are toroidal (or log smooth) locally in the Kummer topology.

Stably toroidal and toroidal varieties share many common properties. In particular, similarly to toroidal varieties their logarithmic structures are defined by a certain smooth open subset $U \subset X$ whose complement is a Weil divisor $D = X \setminus U$.

The stratifictaion defined by the intersection components of this Weil divisor coincides with the natural logarithmic stratification on X, defined by the rank of the monoids. Recall that toroidal varieties (X, U) are pairs, where $U \subset X$ is an open subvariety, locally modeled by pairs (X_σ, T), where T is the open torus in a toric variety X_σ. In particular, the pairs of toric varieties $(X_\sigma \times \mathbf{A}^n, T \times \mathbf{A}^n)$ also provide a toroidal structure. Then stably toroidal varieties are locally modeled by pairs of toric varieties which are finite toric quotients of the toric varieties of the form $(X_\sigma \times \mathbf{A}^n, T \times \mathbf{A}^n)$, where one must assume that the quotient of the open toric variety $T \times \mathbf{A}^n$ is smooth.

As before, we consider the Kummer topology consisting of finite log étale maps over stably toroidal X. It allows one to think of sufficiently small neighborhoods of X to be toroidal. Thus all the considerations are, in practice, done locally for toroidal neighborhoods. The advantage of the language of stably toroidal varieties over the class of toroidal varieties is that stably toroidal varieties are preserved by the Kummer smooth-monomial blow-ups used in the algorithm. This is no longer true for toroidal varieties (See Section 4.1).

In the following section 7 we develop the language of marked ideals on stably toroidal varieties in their new logarithmic context. Consequently the concepts of admissible blow-ups, maximal contact, coefficient ideals, and homogenized ideals are now rewritten in the logarithmic language.

Finally, in the last section 8 we prove the resolution of marked ideals, and deduce the main results on the canonical principalization and resolution.

logarithmic

2. Logarithmic geometry and toroidal varieties

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2.1. Preliminaries on logarithmic structures. [3]

2.1.1. *Resolution of singularies and logarithmic strutures.* When studying the resolution algorithms on smooth ambient varieties we often consider the boundary, which is a simple normal crossing divisor occcuring in the desingularization process. This enriches naturally the structure of the variety. Now, any progress in the resolution is considered with respect to this logaritmic structure. In other words the monomial part of the singularities is, to some extent, neglected or factored out.

Here we further extend this concept. First of all we can generalize the notion of SNC divisors by allowing *toroidal schemes* with more general singuarities.

2.1.2. *Toric varieties.* Recall that an affine toric variety X contains a torus T and has a form $X = X_\sigma := \mathrm{Spec}(K[\sigma^\vee \cap M])$, where $M = Hom(T, K^*)$ is the lattice of characters, $N = Hom(K^*, T) = Hom(M, \mathbb{Z})$ is its dual, and σ is a rational cone in $N^Q = N \otimes Q = Hom(M, \mathbb{Q})$, and $\sigma^\vee := \{F \in M^Q \mid F_{|\sigma} \geq 0\}$ be its dual in $M^Q = M \otimes \mathbb{Q}$. Alternatively, we represent toric varieties $X_P = \mathrm{Spec}(K[P])$ by means of monoids $P_\sigma := \sigma^\vee \cap M$. (See [KKMSD73],[Ful93]) Here monoids are commutative semigroups with 1.

2.1.3. *Toroidal varieties.* A pair (X, U) of a normal variety X and its open subset U is called *toroidal* (respectively *strict toroidal*) if étale locally (respectively Zarski locally) it admits an étale morphism to a toric variety (X_σ, T), see [KKMSD73,

[3] (Dan) references needed throughout

Chapter 2]. This means that there exists an étale (respectively, Zariski) neighborhood $f : V \to X$, with an open subset $U_V := f^{-1}(U)$ and an étale morphism $\phi : (V, U_V) \to (X_\sigma, T)$ for which $U_V = f^{-1}(T)$. Such a morphism $\phi : (V, U_V) \to (X_\sigma, T)$ can be defined by the map of monoids $P_\sigma \to \mathcal{O}_X$ (with respect to multiplication) from the monoid $P_\sigma := \sigma^\vee \cap M$. This monoid P_σ is saturated, finitely generated and integral (see definitions below). It contains the submonoid P^*_σ of the invertible elements, which is, in fact, a lattice corresponding to the orbit O_σ. The monoid P_σ is is called *sharp* if its "invertible part" is trivial. The quotient $\overline{P}_\sigma := P_\sigma / P^*_\sigma$ is thus a sharp monoid. It induces a smooth map $(V, U_V) \to X_{\overline{P}_\sigma} = \mathrm{Spec}(K[\overline{P}_\sigma])$. Such a map is, in fact, highly non-canonical, and is defined up to multiplication by invertible elements. So the toroidal structure is locally determined by the map $P_\sigma \to O_X/O^*_X$ or even $P_\sigma/P^*_\sigma \to O_X/O^*_X$.

2.1.4. *Log structures.* All the above motivates the Kato definition of *log and pre log structures.*

Definition 2.1.5. ([Kat89b]) Let X be a scheme. By a *pre-log-structure* we mean a sheaf of monoids $\mathcal{M}_X$ on the étale site $X_{\text{ét}}$, together with a map of sheaf of monoids $\alpha : \mathcal{M}_X \to \mathcal{O}_X$, called *the structure morphism.* A pre-log structure is called *log structure* if $\alpha_{|\alpha^{-1}(\mathcal{O}^*_X)} : \alpha^{-1}(\mathcal{O}^*_X) \to \mathcal{O}^*_X$ is an isomorphism via α. The pair $(X, \mathcal{M}_X)$ is called *a log scheme.* It will be called a Zariski log scheme if the log-structure is defined on the Zariski open site.

In particular, in the toroidal situation, one can represent $\mathcal{M}_X$, equivalently, by a pair (X, U), where the open subscheme U is the locus where the logarithmic structure is trivial. Then the sheaf of monoids can be reconstructed from the open embedding $i : U \to X$ by $M_X := \mathcal{O}_{X_{\text{ét}}} \cap i_*(\mathcal{O}^*_{U_{\text{ét}}})$.

Any pre-log structure $\mathcal{M}_X$ induces an associated log structure, which is denoted by $\mathcal{M}^a_X$. If the pre-log structure $\mathcal{M}_X$ is defined by a subsheaf of monoids of $\mathcal{O}_X$ the induced log structure $\mathcal{M}^a_X$ is generated by the subsheaves $\mathcal{M}_X$ and and $\mathcal{O}^*_X$. In general, it is defined by the push-out of $\mathcal{O}^*_X \leftarrow \alpha^{-1}(\mathcal{O}^*_X) \to \mathcal{M}_X$.

Definition 2.1.6. A map $\mathcal{M} \to \mathcal{N}$ of two pre-log structures on X is a map of sheaves of monoids which is compatible with the structure morphism.

2.1.7. *Logarithmic structures on toric varieties.* Toric varieties (X, T) posses the natural logarithmic structure, which is described by the sheaf on the étale site $X_{\text{ét}}$ as :

$$\mathcal{M} := \mathcal{O}_{X_{\text{ét}}} \cap i_*(\mathcal{O}^*_{T_{\text{ét}}}).$$

containing the subsheaf of the invertible elements $\mathcal{O}^*_{X_{\text{ét}}}$. Similarly we can define $\overline{\mathcal{M}} := \mathcal{M}/\mathcal{O}^*_{X_{\text{ét}}}$.

By Lemma 1, p. 60, [KKMSD73] the sheaf of the monoids $\mathcal{M}$ on $X_{\text{ét}}$ is, in fact, generated on the Zariski site. We have the following corollary.

toroidal-cartier **Lemma 2.1.8.** *Let $\phi : (Y, U) \to (X_\sigma, T)$ be an étale map mapping a certain $x \in Y$ to a point t in the closed orbit $O_\sigma \subset X_\sigma$ then the Cartier divisors supported on $Y \smallsetminus U$ are the pull-backs of the relevant toric Cartier divisors supported on $X_\sigma \smallsetminus T$. In particular $\mathcal{M}_{x,Y} = \phi^*(\mathcal{M}_{t,X_\sigma}) \cdot \mathcal{O}^*_{x,Y}$.*

toric2 **Lemma 2.1.9.** *Let X_Σ be a toric variety. Consider the natural orbits stratification $\{O_\sigma\}_{\sigma\in\Sigma}$. Denote by $\mathcal{M}$ the natural logarithmic structure associated with the pair (X_Σ, T), where T is an open torus.*

(1) *For any open subset X_σ, where $\sigma \in \Sigma$, the natural map*
$$P_\sigma \to \mathcal{O}(X_\sigma) = K[P_\sigma]$$
defines an isomorphism of the monoids: $P_\sigma \to \mathcal{M}(X_\sigma)$.

(2) *If $V \subset X_\sigma$ is an open subset then $\mathcal{M}(V) = P_\sigma \cdot \mathcal{O}(V)^*$, so we get an isomorphism of the sheaves $\mathcal{M}_{|V} \simeq P^a_\sigma = P_\sigma \mathcal{O}^*_V$, and the isomorphism of the monoids:*
$$\overline{P}_\sigma \to \overline{\mathcal{M}(X_\sigma)}$$

(3) *If $x \in O_\sigma$ then $\mathcal{M}_x = P_\sigma \mathcal{O}^*_{X,x}$ and*
$$\overline{\mathcal{M}}_x = \mathcal{M}_x / O^*_{X,x} = \overline{P}_\sigma = P_\sigma / P^*_\sigma$$

The lemma follows easily from the definition of toric varieties. These properties extend immediately to the toroidal varieties.

toric3 **Lemma 2.1.10.** *Let (X, U) be a toroidal variety with the induced logarithmic structure $\mathcal{M}$. Let (Y, U_Y) be an étale neighborhood of (X, U) admitting an étale (or smooth) map $f : Y \to X_\sigma$. Then f defines an isomorphism of the monoids of the effective Cartier divisors* $\mathrm{Car}(Y, U_Y)$ *on Y supported on $Y \smallsetminus U_Y$:*
$$\mathrm{Car}(Y, U_Y) \simeq \mathrm{Car}(X_\sigma, T) = \overline{P_\sigma} \simeq \mathcal{M}(Y)/\mathcal{O}^*(Y).$$
(see [KKMSD73]*). Thus there is an isomorphism of sheaves of the monoids on Y: $\mathcal{M}_{|Y} = P^a_\sigma = P_\sigma \mathcal{O}^*_Y$.*

The above lemmas motivate the definition of *charts* discussed in the next section.

2.2. **Fine and saturated logarithmic structures.**

2.2.1. *Groupification, and induced cones.* If P is a monoid then the abelian group generated by P, is its *groupification* P^{gp}, and it comes with the natural map of monoids $P \to P^{gp}$. On the other hand the monoid P contains the group of invertible elements denoted by P^*. Then the submonid P^* is a subgroup of P^{gp}. The quotient monoid $\overline{P} := P/P^*$ is *sharp*, and it does not contain non-identical invertible elements.

2.2.2. *Fine and saturated monoids.* The log structures are very general. They add new elements to the structures of varieties into play, like the transformations defined by monoids, log-differential forms preserving monoids, and more general definition of logarithmic smoothness, with locally free sheaf of logarithmic differentials.

The natural general category of log-structures consists of those locally induced by fine and saturated monoids. These are locally equipped with charts as discussed briefly for toric and toroidal varieties in Lemmas 2.1.9, 2.1.10.

Definition 2.2.3. ([Kat89b]) A monoid P is *integral* if it "has no zero divisors". To be precise, if $ca = cb$ implies $a = b$. Equivalently, P is integral if the natural map $P \to P^{gp}$ is injective. We define the *integral closure* P^{int} of P to be the image of P in P^{gp}.

A monoid P is called *saturated* if it is integral and saturated in its groupification P^{gp} so if $a^n \in P$ for $n \geq 1$ and $a \in P^{gp}$ then $a \in P$. Any monoid P defines its *saturated closure* $P^{sat} := \{a \in P^{gp} \mid a^n \in P\}$.

A monoid is *fine* if it is integral and finitely generated.

2.2.4. *Charts of fine and saturated log structures.* By a *chart* of a logarithmic structure $\mathcal{M}$ on a scheme X we mean a map $P \to \mathcal{M}(X) = \Gamma(X, \mathcal{M})$ from a monoid P to $\mathcal{M}(X) = \Gamma(X, \mathcal{M})$, generating the pre-log-structure of a locally constant sheaf $P_X \to \mathcal{M} \to \mathcal{O}_X$, and defining the associated log structure P^a which is isomorphic to $\mathcal{M}_X$. A log scheme is called *coherent*, *fine* or *fine and saturated* (or *fs*) if étale locally there a chart $P \to \mathcal{M}(X)$, with P being respectively a finitiely generated, fine, or, fine and saturated monoid. (see [Kat89b])

log-stratification

2.2.5. *Logarithmic stratification defined by the logarithmic structure.* Let X be a toric variety. Observe that, by Lemma 2.1.9, for any point p in an orbit O_σ of dimension i on a toric variety X the rank of the induced sharp monoid at p is equal to $\mathrm{rk}(\overline{M}_p^{\mathrm{gp}}) = n - i$. This interprets the natural stratification by the orbits of codimension i in the logarithmic language. The construction extends immediately to a toroidal variety X: We define a *logarithmic stratification* by locally closed smooth subvarieties $X(i)$ such that $\mathrm{rk}(\overline{M}_p^{\mathrm{gp}}) = i$ for any $p \in X(i)$. Logarithmic stratifications are preserved by étale, in fact, even Kummer logarithmically étale morphism. Moreover one can generalize this definition to any fine logarithmic schemes. [AT17, §2.2.10]. In the sequel by *irreducible strata* we shall mean the irreducible components of the strata of the logaritmic stratification. The stratification by the irreducible components will be called *irreducible logarithmic stratification.*

2.3. Maps of logarithmic schemes.

2.3.1. *Log structures induced by morphisms.* If $f : X \to Y$ is a morphism of schemes, and Y is a logarithmic scheme with the logartithmic structure $i : \mathcal{M}_Y \to \mathcal{O}_Y$ then by pulling back this $\mathcal{M}_Y$ we obtain the natural logaritmic structure $f^*(\mathcal{M}_Y)$ on X. It is a log-structure associated with the pre-log structure defined by a composition of maps of sheaves of monoids

$$f^{-1}(\mathcal{M}_Y) \to f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$$

(see [Kat89b])

2.3.2. *Making coherent log schemes into fine and saturated.* Any log scheme with coherent (and thus locally finitely generated) log structure can be made fine by applying functor "int", such that any log scheme X with a global chart $P \to \mathcal{M}(X)$ transforms to $(X^{\mathrm{int}}, P^{\mathrm{int}})$, where

$$X^{\mathrm{int}} := X \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[P^{\mathrm{int}})],$$

with the log structure $\mathcal{M}^{\mathrm{int}}_{X^{\mathrm{int}}}$, obtained by applying the functor int to $\mathcal{M}_{X^{\mathrm{int}}} := \pi^*(\mathcal{M}_X)$, where $\pi : X^{\mathrm{int}} \to X$ is the standard projection. Moreover X^{int} has a chart $P^{\mathrm{int}} \to \mathcal{M}(X^{\mathrm{int}})$, induced by $P \to \mathcal{M}(X)$. Similarly we define the functor "sat" which takes the coherent log scheme (X, P) to the fine and saturated log structure $(X^{\mathrm{sat}}, P^{\mathrm{sat}})$, with

$$X^{\mathrm{sat}} = X \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[P^{\mathrm{sat}}])$$

with the chart $P^{\mathrm{sat}} \to X^{\mathrm{sat}}$. (see [Kat89b])

Observe $\mathrm{Spec}(\mathbb{Z}[P^{\mathrm{sat}}])$ consists of the irreducible components of the normalization $\mathrm{Spec}(\mathbb{Z}[P])^{\mathrm{nor}}$, of $\mathrm{Spec}(\mathbb{Z}[P])$ where no monomials in P vanish.

2.3.3. *Logarithmic maps.*

Definition 2.3.4. A *morphisms of log schemes* (see [Kat89b]) $f : X \to Y$ is a morphism of the underlying schemes and a map of the logarithmic structures $f^*(\mathcal{M}_Y) \to \mathcal{M}_X$.

A morphism of log schemes $(Y, V) \to (X, U)$ can described as a morphism $f : Y \to X$ such that $f(V) \subseteq U$. Both definitions are equivalent in this case.

A morphism of log schemes $f : X \to Y$ will be called *strict* if it induces an isomorphism $f^*\mathcal{M}_X \xrightarrow{\sim} \mathcal{M}_Y$.

In particular, the morphism of log schemes $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is smooth (respectively étale) if $f : X \to Y$ is smooth (respectively étale) and at the same time f is strict, that is $f^*(\mathcal{M}_Y) = \mathcal{M}_X$.

2.3.5. *Charts of logarithmic morphisms.* One can extend the definition of charts to morphisms.

Definition 2.3.6. (see [Kat89b]) Let $f : X \to Y$ be a logarithmic morphism of log schemes. A *chart for the morphism* f is a triple $(P_X \to \mathcal{M}_X, Q_Y \to \mathcal{M}_Y, Q \to P)$, where $P_X \to \mathcal{M}_X$ and $Q_Y \to \mathcal{M}_Y$ are the charts of $\mathcal{M}_X$, and $\mathcal{M}_Y$, and the map of monoids $Q \to P$ induces the map of constant sheaves $Q_X \to P_X$ which commutes with the map of sheaves of monoids $f^*(\mathcal{M}_Y) \to \mathcal{M}_X$.

fiber

2.3.7. *Fiber products of log schemes with fine and saturated log structures.* In the category of coherent log schemes the fiber products exist, and its underlying scheme coincides with $X \times_Z Y$. The logarithmic structure on $X \times_Z Y$ is defined by the tensor product

$$\mathcal{M} := (\pi_X^{-1}(\mathcal{M}_X) \otimes_{\pi_X^{-1}(\mathcal{M}_Z)} \pi_X^{-1}(\mathcal{M}_Y))^a$$

(see [Kat89b]). Such a fiber product is no longer fine and saturated. In order to obtain the product in the fs category we apply the functors "sat" to $(X \times_Z Y, \mathcal{M})$.

2.3.8. *Logaritmically smooth maps.* The log smooth (respectivley log étale) morphisms $f : X \to Y$ of log schemes are roughly those which are étale locally induced by generically smooth (resp. étale) monomial maps defined by maps of monoids. In some important cases, the condition means that the sheaf of log-differentials $\Omega_{X/Y}$ is locally free in étale topology. In general, they are more complex then ordinary smooth and étale maps.

Definition 2.3.9. (see [Kat89b]) Let $f : X \to Y$ be a morphism of fine log schemes. Assume we have a chart $Q \to \mathcal{M}_Y$, where Q is a finitely generated integral monoid. Then f is log smooth (resp. log étale) if étale locally on X, there exists a chart $P_X \to \mathcal{M}_X, Q_Y \to \mathcal{M}_Y, Q \to P$ extending the chart $Q_Y \to \mathcal{M}_Y$, satisfying the following properties.

(1) The kernel and the torsion part of the cokernel (resp. the kernel and the cokernel) of $Q^{gp} \to P^{gp}$ are finite groups of order invertible on X.
(2) The induced morphism from $f : X \to Y \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P]$ is étale in the classical sense.

Remark 2.3.10. The condition (1) can be formulated equivalently as

(1') The induced map

$$\operatorname{Spec}(\mathcal{O}_X[Q^{gp}]) \to \operatorname{Spec}(\mathcal{O}_X[P^{gp}])$$

is smooth (resp. étale). This can be easily verified computing the relevant Jacobians. (see [Niz]) Let $Y' \subset Y$ and $X' \subset X$ be the loci where the log-structure is trivial, and all Q^{gp}, respectively P^{gp} are regular (equivalenty Q, nad respectivley P are invertible). Then the morphism $X \to Y$ factors through $X' \to Y'$. Moreover the condition (1') implies that $Y' \times_{\mathrm{Spec}(\mathbb{Z}[Q^{gp}])} \mathrm{Spec}(\mathbb{Z}[P^{gp}]) \to Y'$ is smooth (respectively étale). By condition (2) the morphism $X' \to Y' \times_{\mathrm{Spec}(\mathbb{Z}[Q^{gp}])} \mathrm{Spec}(\mathbb{Z}[P^{gp}])$ is étale. Thus logarithmically smooth (log étale) maps $X \to Y$ are those induced by the maps of the monoids, and which are smooth (étale) on the logarithmically trivial locus $X' \to Y'$.

It is a classical result by Kato (see [Kat89b]) that the composition of log smooth maps of fs log schemes is log smooth, and that the log smooth maps of fs log schemes are stable under base changes. (We give a proof of a more general result for normally log smooth maps in sections 4.6.3, and 4.6.3).

2.3.11. *Toroidal varieties.* In particular case of $X \to Y = \mathrm{Spec}(K)$, where K is the base field, and X is fine and saturated log scheme, the condition for log smoothness is equivalent to the condition that étale locally there exists an étale morphism

$\phi : X \to \mathrm{Spec}(K[P]) = X_\sigma$ to the toric variety (X_σ, T) over K. This defines locally an open subset $U = \phi^{-1}(T)$, so that the logarithmic structure on X is defined by the pair (X, U) representing a toroidal variety.

2.3.12. *Toroidal and log smooth maps between toroidal varieties.*

Definition 2.3.13. A morphism $f : (X, U) \to (Y, V)$ is *toroidal* if étale locally on X, there exists an étale maps $Y \to X_\sigma$, and $X \to X_\delta$, and a toric morphism $X_\sigma \to X_\delta$, such that $f : X \to Y \times_{X_\sigma} X_\delta$ is étale.

If additionally the induced dual map $M_\delta \to M_\sigma$ has the cockernel whose torsion parts are finite groups of order invertible on X, and the morphism is dominant then the morphism is log smooth. Note that for dominant toroidal morphisms the induced dual maps of the monoids are injective (so the condition on the kernel is satisfied).

One can easily see that the definition of log-smooth and dominant toroidal maps are equivalent for toroidal varieties in characteristic zero. Thus, the category of toroidal varieties with toroidal maps coincides with the category of fs log smooth varieties with logarithmically smooth morphisms .

3. Toroidal varieties as ambient schemes

logarithm3c2

3.1. Logarithmic differential operators on toroidal varieties. [4] ←4

logparamsec

3.1.1. *Logarithmic parameters on toroidal varieties.* Assume that X is a strictly toroidal variety and $p \in X$ is a point, and let $S = s_p$ be the logarithmic stratum through p. By *logarithmic parameters* or *coordinates* at a point p of X we mean a family $x_1, \ldots, x_n \in \mathcal{O}_{X,p}$ that reduces to a regular family of parameters of $\mathcal{O}_{S,p}$ and a monoidal chart $u \colon \overline{M}_p \hookrightarrow \mathcal{O}_{X,p}$. We will say that the parameters x_i are *free* or *ordinary*, while the elements of $u(\overline{M}_p \setminus \{0\})$ will be called *monomial parameters* and denoted $u^\alpha = u(\alpha)$.

[4](Dan) Add cross references to other paper

Similarly, an element $x \in \mathcal{O}_{X,p}$ is an *ordinary parameter* at p if it reduces to a parameter in $\mathcal{O}_{S,p}$. This happens if and only if $V(x)$ is strictly logarithmically smooth (strictly toroidal) at p. Usually, we will simply say that x is a *local parameter* at p.

formalrem

Remark 3.1.2. (i) Once parameters are fixed we obtain a formal-local description of X at p via

$$\widehat{\mathcal{O}}_{X,p} = K(x)[\![\overline{M}_p, x_1, \ldots, x_n]\!].$$

(ii) Furthermore, thanks to the characteristic zero assumption, if $f\colon Y \to X$ is a morphism of strictly toroidal varieties and q is closed in the fiber $f^{-1}(p)$ then f is logarithmically smooth at q if and only if $\overline{M}_p \hookrightarrow \overline{M}_q$ and the $\widehat{\mathcal{O}}_{X,p}$-algebra $\widehat{\mathcal{O}}_{Y,q}$ is of the form

$$K(x)[\![\overline{M}_q, x_1, \ldots, x_n, y_1, \ldots, y_m]\!].$$

In fact, one can take any $y_1, \ldots, y_m \in \mathcal{O}_{Y,q}$ whose images form a basis of $\Omega^1_{s_q/s_p,q}$, where s_p and s_q are the logarithmic strata through p and q, respectively. Furthermore, after replacing Y by a strictly étale neighborhood of q the composed homomorphism $\overline{M}_p \stackrel{u}{\to} \mathcal{O}_{X,p} \to \mathcal{O}_{Y,q}$ can be extended to a monoidal chart $u'\colon \overline{M}_q \to \mathcal{O}_{Y,q}$, thus extending the coordinates (x, u) at p to coordinates (x, y, u') at q.

3.1.3. *Logarithmic derivations.* Let $X = (X, D)$ be a toroidal K-variety. In the sheaf $\mathcal{D}er_K(X, \mathcal{O}_X)$ of all K-derivations $\mathcal{O}_X \to \mathcal{O}_X$ consider the subsheaf $\mathcal{D}er_K((X, D), \mathcal{O}_X) = \mathcal{D}er_{\mathbf{Log}}(X, \mathcal{O}_X)$ of *logarithmic derivations*, i.e. derivations that take the ideal defining the toroidal divisor D to itself. In other words, these are derivations preserving D, or tangent vectors on X tangent to D. We call $\mathcal{D}er_K((X, D), \mathcal{O}_X)$ the *logarithmic tangent sheaf* of (X, D).

Furthermore, if $f\colon Y \to X$ is an étale strict morphism then the isomorphism $\mathcal{D}er_K(Y, \mathcal{O}_Y) = f^*\mathcal{D}er_K(X, \mathcal{O}_X)$ induces an isomorphism $\mathcal{D}er_K((Y, f^{-1}(D)), \mathcal{O}_Y) = f^*\mathcal{D}er_K((X, D), \mathcal{O}_X)$. Therefore, the formation of logarithmic tangent sheaves extends to a toroidal variety .

For brevity, we denote the logarithmic tangent sheaf by $\mathcal{D}^1_X$.

3.1.4. *Local description of logarithmic derivations.*

logderiv

Lemma 3.1.5. *(see [ATW16a]) Assume that X is a strictly toroidal scheme, $p \in X$ is a closed point, and $x_1, \ldots, x_n \in \mathcal{O}_{X,p}$ and $u\colon \overline{M}_p \hookrightarrow \mathcal{O}_{X,p}$ are logarithmic parameters at p. Then,*

(1) For each $1 \leq i \leq n$ there exists a unique element $\frac{\partial}{\partial x_i} \in \mathcal{D}^1_{X,p}$ vanishing on $u(\overline{M}_p)$ and satisfying $\frac{\partial}{\partial x_i} x_j = \delta_{ij}$.

(2) For each element L of $N_p := \mathrm{Hom}_{\mathbb{Z}}(\overline{M}^{gp}_p, \mathcal{O}_{X,p})$ there exists a unique derivation $\mathbb{D}_L \in \mathcal{D}^1_{X,p}$ such that $\mathbb{D}_L(u^\alpha) = L(\alpha)u^\alpha$ for any monomial coordinate and $\mathbb{D}_L(x_i) = 0$ for $1 \leq i \leq n$.

(3) The construction of (2) provides an embedding $\mathbb{D}\colon N_p \hookrightarrow \mathcal{D}^1_{X,p}$ and then

$$\mathcal{D}^1_{X,p} = \mathbb{D}(N_p) \oplus \left(\oplus_{i=1}^n \mathcal{O}_{X,p} \frac{\partial}{\partial x_i}\right).$$

(4) For any $\partial \in \mathcal{D}^1_{X,_p}$ and a monomial $m \in M_p$ we have that $\partial m \in m\mathcal{O}_{X,_p}$. In particular, logarithmic derivations preserve monomial ideals, including the ideals of logarithmic strata.

Proof. Locally at p the coordinates induce a logarithmically smooth morphism to $X_0 = \mathrm{Spec}(k[x_1,\dots,x_n][\overline{M}_p])$. This reduces the claim to the case of X_0, which can be done by a direct inspection. In particular, in this case $\mathcal{D}^1_{X,p}$ is freely generated by $\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}$ and $u_1\frac{\partial}{\partial u_1},\dots,u_l\frac{\partial}{\partial u_l}$ where u_i form a basis for $\overline{M}^{\mathrm{gp}}_p$. (In fact, since $\mathrm{char}(k)=0$ it suffices to take u_i that form a basis of $\overline{M}^{\mathrm{gp}}_p\otimes\mathbb{Q}$.) ♣

logderlem

3.1.6. *Basic properties.* It follows from Lemma 3.1.5(4) that without choices one can say about $\mathcal{D}^1_{X,p}$ the following:

(1) A logarithmic K-derivation of $\mathcal{O}_{X,p}$ preserves the ideal of the logarithmic stratum $S = s_p$ through p and hence restricts to a K-derivation of $\mathcal{O}_{S,p}$. This provides a surjective $\mathcal{O}_{S,p}$-homomorphism $\mathcal{D}^1_{X,p}\otimes_{\mathcal{O}_{X,p}}\mathcal{O}_{S,p}\to\mathcal{D}^1_{S,p}$, which does not possess, however, a natural lift to a homomorphism of $\mathcal{O}_{X,p}$-modules $\mathcal{D}^1_{X,p}\to\oplus_i\mathcal{O}_{X,p}\frac{\partial}{\partial x_i}$.

(2) For any $\partial\in\mathcal{D}^1_{X,_p}$ and any monomial $m\in M_p$ the element $\frac{\partial m}{m}\in\mathcal{O}_{X,p}$ is uniquely defined. By Leibnitz rule, sending m to $\frac{\partial m}{m}$ one obtains a homomorphism of monoids $M_p\to(\mathcal{O}_{X,p},+)$, whose extension to M^{gp}_p will be denoted ϕ_∂. In particular, a homomorphism $\phi\colon\mathcal{D}^1_{X,_p}\to\mathrm{Hom}_{\mathbb{Z}}(M^{\mathrm{gp}},\mathcal{O}_{X,p})$ arises. So, any monoidal chart $u\colon\overline{M}_p\hookrightarrow M_p$ induces by composing ϕ with the dual of u an epimorphism $\mathcal{D}^1_{X,_p}\to N_p$. To split the latter one also has to fix free coordinates at p.

tangentsheaf

Lemma 3.1.7. *(see [ATW16a])*

The logarithmic tangent sheaf $\mathcal{D}^1_X$ on a srictly toroidal variety X is locally free of rank $\dim(X)$.

Proof. Follows from Lemma 3.1.5(3). ♣

Remark 3.1.8. The lemma also follows from the fact that $\mathcal{D}^1_X$ is dual to the logarithmic cotangent sheaf $\Omega^{\log}_{(X,D)/K}$, which is locally free of rank $\dim(X)$ by results of Kato.

3.1.9. *Higher order differential operators.* Since $\mathrm{char}\,K=0$, a nontrivial monomial ideal sheaf $\mathcal{I}$ is stable under $\mathcal{D}^1_X$, namely $\mathcal{D}^1_X(\mathcal{I})=\mathcal{I}$. This does not hold for the unit ideal sheaf (e.g. take $X=\mathrm{Spec}\,K[M]$), and to by-pass this inconvenience we will work with the sheaf $\mathcal{D}^{(\le 1)}_X$, which does stabilize every monomial ideal sheaf. It can be defined as the subsheaf of the total sheaf of differential operators generated as an $\mathcal{O}_X$-module by $\mathcal{O}_X$ and the logarithmic tangent sheaf $\mathcal{D}^1_X$. We will also make use of the subsheaf $\mathcal{D}^{(\le n)}_X$ generated by the images of $(\mathcal{D}^1_X)^{\otimes i}$ for $0\le i\le n$. In particular, we will consider the quasi-coherent algebra of logarithmic differential operators $\mathcal{D}^\infty_X=\cup_i\mathcal{D}^{(\le i)}_X$.

The following lemma shows the full analogy with the derivations on the smooth variety.

Remark 3.1.10. One can interpret $\mathcal{D}^{(\le i)}_X$ as the sheaf of the log differential operators on X of order at most i. Note that we use here that the characteristic is zero and hence the algebra of differential operators is generated by operators of order 1.

3.1.11. *Functoriality.*

Lem:toroidal-logetale **Lemma 3.1.12.** *(see [ATW16a])*

If $f : Y \to X$ is a logarithmically smooth morphism, then the natural homomorphisms $\mathcal{D}_Y^{(\leq i)} \to f^*(\mathcal{D}_X^{(\leq i)})$ and $\mathcal{D}_Y^\infty \to f^*(\mathcal{D}_X^\infty)$ are surjections and the kernels act on $f^{-1}(\mathcal{O}_X)$ by 0. Furthermore, if f is logarithmically étale, for example, a toroidal modification or an étale morphism, then all these maps are isomorphisms.

Proof. We can work locally on X and étale-locally on Y, so by Remark 3.1.2 we can choose compatible coordinates $x_1,\ldots,x_n$ and $u\colon \overline{M}_p \to \mathcal{O}_{X,p}$ at $p \in X$ and $x_1,\ldots,x_n,y_1,\ldots,y_m$ and $u'\colon \overline{M}_q \to \mathcal{O}_{Y,q}$ at $q \in f^{-1}(p)$. Fix a basis $u_1,\ldots,u_r$ of $\overline{M}_p^{\text{gp}}$ and let $u_1,\ldots,u_r,u'_1,\ldots,u'_s$ be its extension to a basis of $\overline{M}_q^{\text{gp}} \otimes \mathbb{Q}$. By Lemma 3.1.5, $\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n},u_1\frac{\partial}{\partial u_1},\ldots,u_r\frac{\partial}{\partial u_r}$ form a basis of $\mathcal{D}^1_{X,p}$ and

$$\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial y_1},\ldots,\frac{\partial}{\partial y_m},u_1\frac{\partial}{\partial u_1},\ldots,u_r\frac{\partial}{\partial u_r},u'_1\frac{\partial}{\partial u'_1},\ldots,u'_s\frac{\partial}{\partial u'_s}$$

form a basis of $\mathcal{D}^1_{Y,q}$. In addition, it is clear that $\frac{\partial}{\partial y_i}$ and $u'_j\frac{\partial}{\partial u'_j}$ vanish on $f^*\mathcal{O}_{X,p}$, hence $f^*(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i}$ and $f^*(u_j\frac{\partial}{\partial u_j}) = u_j\frac{\partial}{\partial u_j}$. Finally, note that if the morphism is logarithmically étale then $m = s = 0$. The assertion for $i = 1$ follows.

The case of $i = 1$ implies that the maps $\mathcal{D}_Y^{(\leq i)} \to f^*(\mathcal{D}_X^{(\leq i)})$ are surjective, and, since the kernel is the two-sided ideal generated by $\frac{\partial}{\partial y_i}$ and $u'_j\frac{\partial}{\partial u'_j}$, it vanishes on $f^*(\mathcal{O}_{X,p})$. This proves the assertion for any finite i, and hence also for ∞. ♣

Cor:logsmooth **Corollary 3.1.13.** *(see [ATW16a])*

If $f\colon Y \to X$ is a logarithmically smooth morphism and $\mathcal{I}$ is an ideal on X then $(\mathcal{D}_X^{(\leq i)}\mathcal{I})\mathcal{O}_Y = \mathcal{D}_Y^{(\leq i)}(\mathcal{I}\mathcal{O}_Y)$ and $(\mathcal{D}_X^\infty\mathcal{I})\mathcal{O}_Y = \mathcal{D}_Y^\infty(\mathcal{I}\mathcal{O}_Y)$.

Indeed, the ideals $(\mathcal{D}_X^{(\leq i)}\mathcal{I})\mathcal{O}_Y$ and $\mathcal{D}_Y^{(\leq i)}(\mathcal{I}\mathcal{O}_Y)$ have the same sets of generators.

Deriving ideals on strictly toroidal varieties is compatible with restriction onto the logarithmic strata. Namely, §3.1.6(1) immediately implies the following

diffonstrata **Lemma 3.1.14.** *Assume that X is a strictly toroidal variety, $S \hookrightarrow X$ is a logarithmic stratum, and $\mathcal{I}$ is an ideal on X. Then for any $i \geq 0$ we have that $\mathcal{D}_X^{(\leq i)}(\mathcal{I})_{|S} = \mathcal{D}_S^{(\leq i)}(\mathcal{I}_{|S})$.*

Sec:monomial-blowup

3.2. **Blowing up of smooth-monomial ideals.**

Sec:centers

3.2.1. *Smooth monomial centers.* Let (X,U) be a strictly toroidal variety. We shall generalize smooth centers in nonsingular varieties.

Definition 3.2.2. By a *smooth-monomial* center on X we mean the center which Zariski locally can be written as

$$\mathcal{J} = (x_1,\ldots,x_r,m_1,\ldots,m_s),$$

where $m_i = u^{\alpha_i}$ are monomials, and $r \geq 1$. If the center $\mathcal{J}$ can be written locally as

$$\mathcal{J} = (m_1,\ldots,m_s),$$

then it will be called the *monomial center*.

Sec:Kummer-charts

3.2.3. *Local equations for smooth-monomial blowings up.* In particular, we assume given a monoidal chart $u : \overline{M}_p \to \mathcal{O}_{X,p}$ and a system of regular parameters $(x_1, \ldots, x_n)$ completing the system appearing in $\mathcal{J}$. We say that such system is *compatible* with the center $\mathcal{J}$.

Consider the effect of the blow-up $\sigma : X' \to X$ of a smooth-monomial center $\mathcal{J}$ on a strictly toroidal variety (X, U). Consider the logarithmic structure (X', U'), with $U' = \sigma^{-1}(U) \smallsetminus E$, where E is the exceptional divisor.

(1) The local parameters at $p' \in \sigma^{-1}(p)$ *in the chart associated with* x_r are

$$x'_l = \frac{x_l}{x_r}, \quad l < r, \quad x'_j = x_j, \quad r \leq j \leq n$$

and we denote by $y := x_r = x'_r$ the local equation of the exceptional divisor. The new monoid $M' = \overline{M}_{p'}$ is the saturation of the monoid generated by $\sigma^*(M)$, y, and the elements $m'_i = m_i/x_r$. The monoid M' describes the logarithmic structure of the toroidal variety (X', U').

(2) The local parameters at $p' \in \sigma^{-1}(p)$ *in the chart associated with* m_1 are

$$x'_l = x_l/y, \quad 1 \leq l \leq r, \quad x'_j = x_j, \quad r < j \leq n$$

with $y = m_1$ being the equation of the exceptional divisor. The new monoid is the saturation of the monoid generated by $\sigma^*(M)$ and the elements m_i/y.

Again we get locally the logarithmic structure of the toroidal variety (X', U'). Coordinates in the other charts transform similarly.

As a corollary from the above we get the following

Blow-up1

Proposition 3.2.4. *(see [ATW16a]) Let (X, U) be a strictly toroidal variety. Let $\mathcal{J}$ be a smooth monomal ideal on X. Then the normalized blow-up $X' = bl_{\mathcal{J}}(X)$ of X, defines a map $f : X' \to X$, such that:*

(1) X' is a strictly toroidal variety, with respect to the log structure (X', U'), where $U' = f^{-1}(U) \smallsetminus E$, where E is the exceptional divisor.

(2) $\mathcal{J}\mathcal{O}_{X'}$ is invertible.

(3) If $\mathcal{J}$ is a monomial ideal on X then the normalized blowing up $\pi : X' \to X$ of $\mathcal{J}$ is a log smooth morphism.

(4) If $X_1 \to X$ is log smooth and $X' \to X$ is the normalized blowing up of $\mathcal{I}\mathcal{O}_X$, then $X'_1 = X' \times_X X_1$, the product taken in the category of toroidal logarithmic schemes, and is given by the closure of $U \times_X X_1$ in the normalization of the usual pullback $X' \times_X X_1$.

Proof. (1) follows from the previous considerations for charts in Section 3.2.3. (2) is obvious (3) follows from theorem of [KKMSD73]. (4) Follows from the universal properties of the product and the blow-up. Note that the normalized product $X' \times_X X_1$ is the disjoint union of a component which is the closure of $U \times_X X_1$, and other components mapping to $X \smallsetminus U$. The latter components posses some vanishing functions defined by the logarithmic structure, and are removed by the process of saturated closure. ♣

saturationrem

Remark 3.2.5. Inspecting monoidal charts and using [KKMSD73] it is easy to see that the *normalized* blowing up $X' \to X$ of a monomial ideal $\mathcal{I}$ on X coincides with the *normalized* blowing up of its saturation $\mathcal{I}^{\text{sat}}$. In addition, $\mathcal{I}\mathcal{O}_{X'} = \mathcal{I}^{\text{sat}}\mathcal{O}_{X'}$.

Recall that if $\mathcal{I}$ corresponds to $\overline{\mathcal{I}} \subseteq \overline{\mathcal{M}}$ then $\mathcal{I}^{\rm sat}$ corresponds to the ideal $\overline{\mathcal{I}}^{\rm sat}$ consisting of all elements $x \in \overline{\mathcal{M}}$ such that $lx \in l\overline{\mathcal{I}}$ for some $l > 0$.

3.3. **Logarithmic derivations on a smooth-monomial blowing up.**

33 **Lemma 3.3.1.** *Let $\sigma : X' \to X$ be the smooth-monomial blowing up at center $\mathcal{J}$. Then sections of $\mathcal{I}_E \cdot \sigma^*(\mathcal{D}^1_X)$ form logarithmic derivations on X', that is $\mathcal{I}_E \cdot \sigma^*(\mathcal{D}^1_X) \subseteq \mathcal{D}^1_{X'}$. In general $\mathcal{I}^i_E \cdot \sigma^*(\mathcal{D}^{(\leq i)}_X) \subseteq \mathcal{D}^{(\leq i)}_{X'}$.*

Proof. As in the classical case, this can be reduced to a direct inspection based on the chain rule, see Lemma 3.3.2 and Remark 3.3.2 below. ♣

Some key elements landing in the subsheaf $\mathcal{D}^i_{X'}$ are very useful. We use the notation in Section 3.2.3.

Lem:formulas **Lemma 3.3.2.** *In any chart other than the chart associated to x_1, we have the equality $y\sigma^*(\frac{\partial}{\partial x_1}) = \frac{\partial}{\partial x'_1}$.*

We emphasize that notation such as $\frac{\partial}{\partial x_1}$ only makes sense when a complete set of parameters as well as a monoidal chart are given. In particular it is very much chart-dependent.

Proof. These operators are characterized by the way they act on parameters and monomials, so we check that the two sides agree on these. Without loss of generality the chart belongs to either x_r or m_s. We suppress the notation σ^*, in essence working on the open set where σ is an isomorphism.

On either type of chart, we have $y\frac{\partial x_j}{\partial x_1} = 0 = \frac{\partial(yx'_j)}{\partial x'_1} = \frac{\partial x_j}{\partial x'_1}$ for all $2 \leq j \leq n$.

Similarly $y\frac{\partial m_j}{\partial x_1} = 0 = \frac{\partial(ym'_j)}{\partial x'_1} = \frac{\partial m_j}{\partial x'_1}$ for the transformed monomials, and more directly $y\frac{\partial u_k}{\partial x_1} = 0 = \frac{\partial u_k}{\partial x'_1}$ for the other monomials.

Finally we have $y\frac{\partial x_1}{\partial x_1} = y = \frac{\partial(yx'_1)}{\partial x'_1} = \frac{\partial x_1}{\partial x'_1}$. ♣

Lemmas 3.3.1 and 3.3.2 suggest the following notion of *controlled transforms* of sheaves of derivations:

Not:controlled **Notation 3.3.3.** In the situation with local coordinates as above, for any $i \leq r$ and in all charts other than the x_i-chart, with exceptional variable y, we will write $\sigma^c(x_i, 1) := x'_i = x_i/y$ and $\sigma^c(\frac{\partial}{\partial x_i}) := \frac{\partial}{\partial x'_i} = y\sigma^*(\frac{\partial}{\partial x_i})$ for $i \leq r$.

We define analogues which do not depend on charts for sheaves. First we set

$$\sigma^c(\mathcal{D}_X) := \mathcal{I}_E(\sigma^*(\mathcal{D}_X)) \subseteq \mathcal{D}_{X'}.$$

Furthermore, for any $i > 0$ we set

$$\sigma^c\left(\mathcal{D}^{(\leq i)}_X\right) := \mathcal{I}^i_E\left(\sigma^*\left(\mathcal{D}^{(\leq i)}_X\right)\right) \subseteq \mathcal{D}^{(\leq i)}_{X'}.$$

monomsatursec

3.4. **The monomial saturation of an ideal.**

Definition 3.4.1. Let X be a strictly toroidal variety and $\mathcal{I}$ an ideal sheaf. Define the *monomial saturation of $\mathcal{I}$* to be

$$\mathcal{M}(\mathcal{I}) := \bigcap_{\substack{\tilde{\mathcal{I}} \supseteq \mathcal{I} \\ \tilde{\mathcal{I}} \text{ monomial}}} \tilde{\mathcal{I}}.$$

Clearly $\mathcal{M}(\mathcal{I})$ is a monomial ideal containing $\mathcal{I}$, and if $\mathcal{I}$ is monomial then $\mathcal{M}(\mathcal{I})=\mathcal{I}$.

Th:monomial-part **Theorem 3.4.2.** *Let $\mathcal{X}$ be a strictly toroidal variety and $\mathcal{I}$ an ideal sheaf.*

(1) $\mathcal{I}$ is monomial if and only if $\mathcal{D}_X^{(\leq 1)}\mathcal{I}=\mathcal{I}$.
(2) $\mathcal{D}_X^{\infty}\mathcal{I}=\mathcal{M}(\mathcal{I})$
(3) If $Y\to X$ is logarithmically smooth then $\mathcal{M}(\mathcal{I}\mathcal{O}_Y)=\mathcal{M}(\mathcal{I})\mathcal{O}_Y$.

Proof. (1) One can reduce the situation to the completion of the local ring. Acting by the monomial derivations $u\frac{\partial}{\partial u}$ on the functions in the ideal we see that if $f=\sum c_{\alpha,f}(x)u^\alpha\in\widehat{\mathcal{O}}_{p,X}$ is in the ideal $\widehat{\mathcal{I}}_p$ then each $c_{\alpha,f}(x)u^\alpha$ is in $\widehat{\mathcal{I}}_p$, and thus acting by $\frac{\partial}{\partial x}$ we get that all $u^\alpha\in\widehat{\mathcal{I}}_p$. (For details see [ATW16a])

(2) Since $\mathcal{D}_X^{(\leq 1)}\mathcal{D}_X^{\infty}\mathcal{I}=\mathcal{D}_X^{\infty}\mathcal{I}$, the ideal $\mathcal{D}_X^{\infty}\mathcal{I}$ is a monomial ideal by (1). Since $\mathcal{I}\subseteq\mathcal{D}_X^{\infty}\mathcal{I}$ we have $\mathcal{M}(\mathcal{I})\subseteq\mathcal{M}(\mathcal{D}_X^{\infty}\mathcal{I})=\mathcal{D}_X^{\infty}\mathcal{I}$. On the other hand $\mathcal{D}_X^{\infty}\mathcal{I}\subseteq\mathcal{D}_X^{\infty}\mathcal{M}(\mathcal{I})=\mathcal{M}(\mathcal{I})$, giving the equality.

(3) We pass to completions and argue as in (1) ♣

Sec:logord 3.5. **Logarithmic order and differential logarithmic order.** Let $\mathcal{I}$ be an ideal on a strictly toroidal variety X. By the *logarithmic order* of $\mathcal{I}$ at a point $p\in X$ we mean

$$\operatorname{logord}_p(\mathcal{I})=\operatorname{ord}_p(\mathcal{I}_{|s_p})$$

where s_p is the logarithmic stratum through p. As we remark below, logord_p is a logarithmic analogue of the classical order ord_p on a smooth variety. Note also that logord_p is a monomial order where we put infinite weights on monomial coordinates and weight 1 on free coordinates. An element x is a local parameter at p if and only if $\operatorname{logord}_p(x)=1$.

Define the *differential logarithmic order* at a point $p\in X$ to be

$$\mathcal{D}\mathrm{ord}_p(\mathcal{I})=\min\{a\in\mathbb{N}\mid\mathcal{D}_X^{(\leq a)}(\mathcal{I})_p=\mathcal{D}_X^{\infty}(\mathcal{I})_p\}$$

We will use the logarithmic differential order to detect the logarithmic order and classify ideals in some situations. The logarithmic order, in general, is not preserved by log smooth morphisms. However it will be used mostly in the situations where it is indepenedent of logarithmic modifications. First, we have

11 **Lemma 3.5.1.** *(see [ATW16a])*

$$\mathcal{D}\mathrm{ord}_p(\mathcal{I})\leq\operatorname{logord}_p(\mathcal{I}),$$

with equality whenever $\operatorname{logord}_p(\mathcal{I})<\infty$.

Proof. Fix $x_1,\dots,x_r\in\mathcal{O}_{X,p}$ which reduce to a system of regular parameters $\bar{x}_1,\dots,\bar{x}_r$ on s_p. If $\operatorname{logord}_p(\mathcal{I})=m<\infty$ there is an element $f\in I$ with $\bar{f}\in\mathcal{I}|_{s_p}$ and some differential operator on the stratum $\frac{\partial^m}{\partial\bar{x}_{i_1}\dots\partial\bar{x}_{i_m}}$ such that $\frac{\partial^m}{\partial\bar{x}_{i_1}\dots\partial\bar{x}_{i_m}}\bar{f}(p)\neq 0$, as we are working in characteristic 0. This implies that $\frac{\partial^m f}{\partial x_{i_1}\dots\partial x_{i_m}}(p)\neq 0\in\mathcal{O}_{X,p}$, giving the inequality.

♣

Remark 3.5.2. (1) The two logarithmic orders logord_p and $\mathcal{D}\mathrm{ord}_p$ are closely related to the usual order ord_p. In particular if there are no divisors at p then $\operatorname{ord}_p=$

$\mathrm{logord}_p = \mathcal{D}\mathrm{ord}_p$. In the case of clean ideals (see next section) both logarithmic order coincide.

(2) In our principalization algorithm, the logarithmic order logord_p plays a role similar to the role of ord_p in the non-logarithmic algorithm of [Wło05]: it keeps track of the progress during all the stages.

(3) The differential logarithmic order $\mathcal{D}\mathrm{ord}_p(\mathcal{I})$ will be used to get additional information when the logarithmic order is infinite and to simplify proofs.

The logarithmic order is compatible with logarithmically smooth morphisms:

ordfunctor

Lemma 3.5.3. *(see [ATW16a])*

Assume that $f\colon X' \to X$ is a logarithmically smooth morphism between strictly toroidal varieties, $\mathcal{I}$ is an ideal on X with $\mathcal{I}' = \mathcal{I}\mathcal{O}_{X'}$, and $p' \in X'$ is a point with $p = f(p')$. Then

$$\mathrm{logord}_{p'}(\mathcal{I}') = \mathrm{logord}_p(\mathcal{I}).$$

Proof. Note that f induces smooth morphisms between logarithmic strata. Therefore, the claim reduces to the classical fact that the order of ideals on smooth varieties is compatible with smooth morphisms. ♣

Sec:classify-ideals

3.6. Clean ideals, balanced ideals, and their orders.

Def:classify-ideals

Definition 3.6.1. (see [ATW16a])

A nowhere zero ideal $\mathcal{I}$ is

- *balanced* if the monomial ideal $\mathcal{M}(\mathcal{I})$ is invertible.
- *clean* if $\mathcal{M}(\mathcal{I}) = 1$.

Given a balanced ideal $\mathcal{I}$ we define its *clean part*

$$\mathcal{I}^{cln} := (\mathcal{M}(\mathcal{I}))^{-1}\mathcal{I}.$$

In particular, $\mathcal{I}$ factors as $\mathcal{I}^{cln} \cdot \mathcal{M}(\mathcal{I})$ and this is compatible with differentiation:

commute

Lemma 3.6.2. *Let $\mathcal{M}$ be a monomial ideal and $\mathcal{I}$ an arbitrary ideal. Then*

$$\mathcal{D}_X^{(\leq i)}(\mathcal{M}\mathcal{I}) = \mathcal{M}\mathcal{D}_X^{(\leq i)}(\mathcal{I}) = \mathcal{M}(\mathcal{I}).$$

Proof. $\mathcal{D}_X^{(\leq i)}(\mathcal{D}_X^{\infty}(\mathcal{I})) = \mathcal{D}_X^{\infty}(\mathcal{I})$ ♣

This lemma has the following effect on the two orders:

order reduced

Lemma 3.6.3. *(1) If $\mathcal{I}$ is a balanced ideal then $\mathcal{D}\mathrm{ord}_p(\mathcal{I}) = \mathcal{D}\mathrm{ord}_p(\mathcal{I}^{cln})$ for any $p \in X$.*
(2) If $\mathcal{I}$ is clean then $\mathrm{logord}_p(\mathcal{I}) = \mathcal{D}\mathrm{ord}_p(\mathcal{I}) < \infty$ for any $p \in X$.
(3) If $\mathcal{I}$ is balanced then $\mathrm{logord}_p(\mathcal{I}^{cln}) = \mathcal{D}\mathrm{ord}_p(\mathcal{I})$ for any $p \in X$.

Proof. (1) Write $\mathcal{I} = \mathcal{M}(\mathcal{I}) \cdot \mathcal{I}^{cln}$. Then by Lemma 3.6.2 $\mathcal{D}_X^{(\leq i)}(\mathcal{I}) = \mathcal{M}(\mathcal{I})\mathcal{D}_X^{(\leq i)}(\mathcal{I}^{cln})$, and

$$\mathcal{D}_X^{(\leq i)}(\mathcal{I}) = \mathcal{M}(\mathcal{I}) \quad \Longleftrightarrow \quad \mathcal{D}_X^{(\leq i)}(\mathcal{I}^{cln}) = (1).$$

(2) This follows from the equality in Lemma 3.5.1.

(3) follows from (1) and (2). ♣

Sec:monomial-saturation

3.6.4. *Making an ideal balanced by associated blowings up.*

Absolute

Proposition 3.6.5 (see [ATW16a], Compare [Kol16, Proposition 20],)**.** *Let $\sigma : X' \to X$ be the he normalized blowing up $X' \to X$ of the ideal $\mathcal{M}(\mathcal{I})$. Then $\mathcal{I}\mathcal{O}_{X'}$ is balanced.*

Proof. By Theorem 3.4.2(3) $\mathcal{M}(\mathcal{I}\mathcal{O}_{X'}) = \mathcal{M}(\mathcal{I})\mathcal{O}_{X'}$. By the defining property of blowing up this is an invertible monomial ideal, so $\mathcal{I}\mathcal{O}_{X'}$ is balanced. ♣

3.7. **Embeddings of logarithmic varieties.** Let X be a fine and saturated Zariski logarithmic variety and $p \in X$ a closed point. Let $u : M = \overline{M}_p \to \mathcal{O}_{X,p}$ be a sharp monoidal chart at p and let $M^+ = M \smallsetminus \{0\}$ be the maximal ideal of M. Then, locally at p, the stratum s_p through p is given by the ideal $u(M^+)\mathcal{O}_{X,p}$.

3.7.1. *Existence of local embedding.* The following lemma shows how coordinates can be used to construct strict closed embeddings of logarithmic varieties into strictly toroidal varieties.

emblem

Lemma 3.7.2. *Fix a monoidal chart $u\colon M \to \mathcal{O}_{X,p}$ and elements $t_1, \ldots, t_n \in \mathcal{O}_{X,p}$ whose images generate the cotangent space to s_p at p. Consider a neighborhood X_0 of p on which $u(M)$ and $t_1, \ldots, t_n$ are global functions so that a morphism of logarithmic schemes $f = (u, t)\colon X_0 \to \operatorname{Spec}(K[M]) \times \mathbb{A}^n_K$ arises. Then*

(1) f is strict at p as a morphism of logarithmic schemes.
(2) f is unramified at p.
(3) On a small enough neighborhood X_1 of p the morphism f factors into a composition of a closed immersion $X_1 \hookrightarrow Y$ and an étale morphism $Y \to \operatorname{Spec}(K[M]) \times \mathbb{A}^n_K$.

Proof. Set $S = s_p$, $Z = \operatorname{Spec}(K[M]) \times \mathbb{A}^n_K$ and $q = f(p)$ for shortness.

(1) The sharp monoid at q is, by the construction, M.

(2) Since p is closed $K(p)$ is finite over $K(q) = K$, and since $\operatorname{char}(K) = 0$ the extension $K(p)/K$ is separable. By definition, $\mathcal{O}_{S,p} = \mathcal{O}_{X,p}/u(M^+)$ and $t_1, \ldots, t_n$ generate the maximal ideal of $\mathcal{O}_{S,p}$. Therefore $m_q\mathcal{O}_{X,p} = m_p$, and f is unramified at p by [Sta, Tag:02GF].

(3) This is a general fact about unramified morphisms, see [Sta, Tag:0395]. ♣

3.7.3. *The embedding dimension.*

embcor

Corollary 3.7.4. *Let X be a Zariski logarithmic variety, $p \in X$ a closed point, $r = \operatorname{rk}(\overline{M}_p)$ and d the dimension of the cotangent space to the logarithmic stratum s_p at p. Then $d + r$ is the minimal natural number n such that there exists a neighborhood X_0 of p and a strict closed immersion $X_0 \hookrightarrow Y$, where Y is a strictly toroidal variety of dimension n.*

Proof. By Lemma 3.7.2, there exists a neighborhood X_0 of p admitting a strict closed immersion into an étale covering of $\operatorname{Spec}(K[\overline{M}_p]) \times \mathbb{A}^d_K$. So, $n \le d + r$. Conversely, a strict closed immersion $i\colon X_0 \hookrightarrow Y$ into a strictly toroidal variety, induces a closed immersion of the logarithmic strata $s_p \hookrightarrow s_q$ through p and $q = i(p)$. Therefore, $\dim(s_q) \ge d$ and hence $\dim_q(Y) \ge r + d$. ♣

3.7.5. *Compatibility of embeddings with logarithmically smooth morphisms.*

emblem2 **Lemma 3.7.6.** *Assume that $f\colon X' \to X$ is a logarithmically smooth morphisms and $i\colon X \hookrightarrow Y$ is a strict closed immersion of logarithmic varieties. Then for any point $p \in X'$ there exist an étale neighborhood X'_0 with induced logarithmic structure, a strict closed immersion $X'_0 \hookrightarrow Y'_0$, and a logarithmically smooth morphism $Y'_0 \to Y$ such that $X'_0 = X \times_Y Y'_0$.*

Proof. The question is local on X at the image $q = f(p)$. Hence we can assume that X possesses a global chart $X \to \operatorname{Spec}(K[M])$ with $M = \overline{M}_q$. By [Kat89b, Theorem 3.5] we can find X'_0 such that the morphism $X'_0 \to X$ factors through a strict étale morphism $X'_0 \to X \otimes_{K[M]} K[N]$, where $M \hookrightarrow N$ are fs monoids. Shrinking Y around $i(q)$ we can assume that the chart of X lifts to a global chart $Y \to \operatorname{Spec}(K[M])$ of Y. The morphism $X \otimes_{K[M]} K[N] \to X$ lifts to a logarithmically smooth morphism $Y \otimes_{K[M]} K[N] \to Y$. So, it remains to lift the strict étale morphism $X'_0 \to X \otimes_{K[M]} K[N]$ to a strict étale morphism $Y'_0 \to Y \otimes_{K[M]} K[N]$. But this is the problem of lifting a usual étale morphism from a closed subscheme, which is easily seen to be possible locally. (For example, one can use the explicit local description of étale morphisms from [Sta, Tag:00UE].) ♣

3.7.7. *Étale equivalence of embeddings in varieties of the same dimension.* Our main result about embeddings of a logarithmic variety X into strictly toroidal varieties is that locally at p such an embedding $i\colon X \hookrightarrow Y$ is determined by the dimension of Y at $i(p)$ up to an étale morphism of the target. In fact, this result is almost obvious formally-locally since each such embedding corresponds to a homomorphism of completed local rings of the form $K(p)[\![\overline{M}_p, t_1, \dots, t_n]\!] \twoheadrightarrow \widehat{\mathcal{O}}_{X,p}$. Using Lemma 3.7.2 we will obtain a more refined étale-local version.

embth **Theorem 3.7.8.** *Assume that X is a logarithmic variety, $p \in X$ is a point, and $i\colon X \hookrightarrow Y$, $i'\colon X \hookrightarrow Y'$ are two strict closed immersions whose targets are irreducible strictly toroidal varieties of the same dimension. Then there exists neighborhoods X_0, Y_0 and Y'_0 of p, $i(p)$ and $i'(p)$, and étale morphisms $f\colon Z \to Y_0$ and $f'\colon Z \to Y'_0$ with the same source, such that*

(1) i and i' restrict to closed immersions $i_0\colon X_0 \hookrightarrow Y_0$ and $i'_0\colon X_0 \hookrightarrow Y'_0$,

(2) f and f' restrict to isomorphisms over $i(X_0)$ and $i'(X_0)$, respectively.

Loosely speaking, the theorem asserts that locally at p both i and i' factor through a closed immersion $X \hookrightarrow Z$, where Z is étale over both Y and Y'.

Proof. Fix coordinates $t_1, \dots, t_n \in \mathcal{O}_{X,p}$ and $u\colon M = \overline{M}_p \hookrightarrow \mathcal{O}_{X,p}$ at p. Set $q = i(p)$ and $q' = i'(p)$, and lift $t_1, \dots, t_n$ to elements $x_1, \dots, x_n \in \mathcal{O}_{Y,q}$ and $x'_1, \dots, x'_n \in \mathcal{O}_{Y',q'}$. Furthermore, complete the latter families to families $x_1, \dots, x_m \in \mathcal{O}_{Y,q}$ and $x'_1, \dots, x'_m \in \mathcal{O}_{Y',q'}$ of ordinary parameters such that the images of x_i and x'_i in $\mathcal{O}_{X,p}$ vanish for $n < i \le m$. The two latter families are of the same size by the assumption on the dimensions.

Next, we claim that u can be lifted to a monoidal chart $v\colon M = \overline{M}_q \hookrightarrow \mathcal{O}_{Y,q}$. Indeed, it suffices to choose $m_1, \dots, m_r \in M$ that form a basis of $M^{\mathrm{gp}} = \mathbb{Z}^r$, and lift $u(m_i)$ to elements $v_i \in M_q \subset \mathcal{O}_{Y,q}$. Since $\overline{M}_q = M$, for any element $\sum_{i=1}^r n_i m_i$ with $n_i \in \mathbb{Z}$ the element $\prod_{i=1}^r v_i^{n_i}$ of M_q^{gp} actually lies in M_q, and hence there exists a unique homomorphism $v\colon M \to \mathcal{O}_{Y,q}$ sending $m_1, \dots, m_r$ to $v_1, \dots, v_r$. Clearly, v is a lifting of u. In the same way, fix a lifting $v'\colon M \to \mathcal{O}_{Y',q'}$ of u.

Taking appropriate neighborhoods X_0, Y_0 and Y'_0 of p, q and q', respectively, we can assume that (1) is satisfied and all elements we have constructed are global functions. Consider morphisms $g\colon Y_0 \to T = \mathrm{Spec}(K[M]) \times \mathbb{A}^m_K$ and $g'\colon Y'_0 \to T$ induced by $(x_1, \dots, x_n, \dots, x_m, v)$ and $(x'_1, \dots, x'_n, \dots, x'_m, v')$, respectively. Since g and g' are étale at q and q', shrinking X_0, Y_0 and Y'_0 further, we can assume that g and g' are étale everywhere. By the construction, both g and g' restrict to the morphism $g\colon X_0 \to T$ induced by $(x_1, \dots, x_n, 0, \dots, 0, u)$. Set $Z = Y_0 \times_T Y'_0$ and note that the composition $X_0 \hookrightarrow X_0 \times_T X_0 \hookrightarrow Z$ is a closed immersion $j\colon X \hookrightarrow Z$ that lifts i and i_0 to Z. Since the projections $f\colon Z \to Y_0$ and $f'\colon Z \to Y'_0$ are étale, $j(X_0)$ is a connected component of both $f^{-1}(i(X_0))$ and $f'^{-1}(i'(X_0))$, and we accomplish the proof by removing all other components of these preimages from Z. ♣

Kummer
Kummer site

3.8. Kummer topology on toroidal varieties. [5]

←5

3.8.1. *Étale site.* We shall refine the Zariski topology on a toroidal variety X. First, consider the strictly toroidal étale site $X_{\text{ét}}$. It consists of étale maps $\pi_Y : Y \to X$, from the strictly toroidal varieties Y to X. These objects form a category with the étale maps $\alpha_{12} : Y_1 \to Y_2$ commuting with projections π_{Y_i}. This category is closed with respect to fibers squares. One considers the *coverings* on $X_{\text{ét}}$, that is families of the maps $\{\alpha_i : Y_i \to Y\}_{i\in I}$ with taget Y, which are jointly surjective, that is $\bigcup \alpha_i(Y_i) = Y$.

This determines the étale site that is a category with coverings satisfying the following conditions:

(1) The identity $V \to V$ defines a covering.
(2) If $\{\phi_i : V_i \to V\}$ is a covering, and $\{\phi_{ji} : V_{ji} \to V_i\}$ are covering then the induced family $\{V_{ij} \to V\}$ is a covering.
(3) If $V' \to V$ is a map and $\{\phi_i : V_i \to V\}$ is a covering then $V' \times_V V_i \to V'$ is a covering.

The étale site determines the topology which allows to define presheaves and sheaves on $X_{\text{ét}}$. Recall that $\mathcal{F}$ is a presheaf on $X_{\text{ét}}$ if it is a contravariant functor from $X_{\text{ét}}$ to the category of sets. It is a *sheaf*, if additionally it satisfies the following glueing condition. For any cover $X_0 := \{Y_i \to Y\}_{i\in I}$ of $Y \in X_{\text{ét}}$ there is a exact sequence with respect to the two maps

$$\mathcal{F}(Y) \to \prod_{i\in I} \mathcal{F}(Y_i) \rightrightarrows \prod_{(i_0,i_1)\in I\times I} \mathcal{F}(Y_{i_0} \times_Y Y_{i_1}),$$

with right arrows defined by the two natural projections of

$$X_1 = X_0 \times_Y X_0 \to X_0$$

Kummer.

3.8.2. *Kummer topology.* The construction of the étale site can be further generalized. Recall that *a Kummer map* or *a map of Kummer type* of monoids $i : P \to Q$ is an injective map of monoids such that for any $a \in Q$ there is $n \in \mathbb{N}$, such that $a^n \in i(P)$.

A map of log schemes $f : X \to Y$ will be called of *Kummer type* or *Kummer* if for any $x \in X$, the induced homomorphism of monoids $(\mathcal{M}/\mathcal{O}^*)_{Y,\overline{f(x)}} \to (\mathcal{M}/\mathcal{O}^*)_{X,\overline{x}}$

5 (Dan) References when appropriate

is of Kummer type. We shall additionally assume that the cokernel of $P^{gp} \to Q^{gp}$ is finite invertible on X.

One shows (for instance in [Nak97]) that Kummer maps are stable under base changes and composition. (See also Sections 4.6.12, 4.6.15 for the proofs in a more general setting).

By the *Kummer log étale map* $f : X \to Y$ of fs log schemes we mean the log étale map which is of Kummer type. (see Section 4.6.15)

One can show that étale locally Kummer log étale morphisms are exactly log étale maps described by the charts $(P \to M_X, Q \to M_Y, P \to Q$, with $P \to Q$ Kummer

The *Kummer étale site* on a toroidal variety X is the category $X_{\text{két}}$ of the Kummer étale maps $Y \to X$. The Kummer étale site defines the Kummer topology, and allows to consider substantially larger set of sheaves of ieals, and regular elements (See, for instance [Niz]. Similar arguments are used in Section 4.7.1.) The basic idea of the Kummer topology is to refine the Zariski topology of toroidal variety X so that the fractional monomials are defined locally. In other words any fractional monomials on $X_{\text{két}}$ "live" on corresponding Z-schemes of $X_{\text{két}}$ over X, and are defined on Z as ordinary monomials.

3.8.3. *Kummer ideals and ordinary ideals.* The Kummer topology is finer than the étale topology, (has more sheaves) which is finer than Zariski topology. The presheaves $\mathcal{O}_{X_{\text{ét}}}$ and $\mathcal{O}_{X_{\text{két}}}$ on the respectively étale and Kummer étale sites are given by the functor $Y \mapsto \Gamma(Y, \mathcal{O}_Y)$ which is in fact a sheaf of rings.

We shall consider its coherent sheaves of ideals $\mathcal{I} \subset \mathcal{O}_{X_{\text{két}}}$ and call them *Kummer ideals*. Similarly we shall use the sheaves of differentials $\mathcal{D}^n_{X_{\text{ét}}}$ on $X_{\text{ét}}$ (respectively on $X_{\text{két}}$).

Any ideal $\mathcal{I} \subseteq \mathcal{O}_X$ (where X is considered with Zariski topology) defines a Kummer ideal $\mathcal{I}_{\text{két}}$ on $X_{\text{két}}$ by assigning $\mathcal{I}_{X'} := \mathcal{I}\mathcal{O}_{X'}$ to objects X' on $X_{\text{két}}$ (or $X_{\text{ét}}$). If a Kummer ideal is of the form $\mathcal{I}_{\text{két}}$ we will call it *an ordinary ideal,* and for brevity denote it by $\mathcal{I}$ instead of $\mathcal{I}_{\text{két}}$.

3.8.4. *Étale descent of the ideals.* Conversely let $\mathcal{I}_{X_0}$ be an ideal on an étale covering $X_0 \to X$ of X and set $X_1 = X_0 \times_X X_0$. If for the two different natural projections $\pi_i : X_1 \to X_0$, $i = 0, 1$ the compatibility condition holds: $\pi_1^*(\mathcal{I}_{X_0}) = \pi_2^*(\mathcal{I}_{X_0})$ then $\mathcal{I}_{X_0}$ descends to the ideal $\mathcal{I}$ on X, such that $\mathcal{I}_{X_0} = \mathcal{O}_{X_0}\mathcal{I}$. The condition is valid, in general for the flat topology.

If $\mathcal{I}_{X_0}$ is an ideal sheaf on a certain Kummer étale covering $X_0 \to X$ which satisfies the compatibility condition then it determines a Kummer ideal $\mathcal{I}$ on $X_{\text{két}}$.

The descent conditions are critical for the algorithm since they allow to work and run (the canonical) algorithm in étale and ket topology.

Kummer topology

3.8.5. *Refinements of the Kummer site and Kummer topology.* The Kummer site defines the topology on the scheme which is the category of the coherent sheaves. The same topology can be defined by certain subcategories (*sieves*). If we consider any Kummer cover of X and their all coverings we shall have the same sheaves and thus the same topology.

Let $\mathcal{I}$ be a Kummer ideal on $X_{\text{két}}$ or $X_{\text{két}}$. By its *Kummer domain* of $\mathcal{I}$ we mean the smallest subsite $X_{\text{két}}(\mathcal{I})$ of $X_{\text{két}}$, such that for any $Y \in X_{\text{két}}(\mathcal{I})$, the global sections $\Gamma(Y, \mathcal{I}_Y)$ generate $\mathcal{I}_{Y_{\text{két}}}$ on $Y_{\text{két}}$. It is the smallest site such that for any $Y_1 \to Y_2$, in $X_{\text{két}}(\mathcal{I})$ we have the compatibility condition $\mathcal{I}_{Y_1} = \mathcal{O}_{Y_1}\mathcal{I}_{Y_2}$. One could think of the neighborhoods in $X_{\text{két}}(\mathcal{I})$ as of "local" neighborhoods where the ideal is defined. The other remaining neighborhoods are in a sense "global" as they do not have enough sections to generate the ideal sheaf.

Example 3.8.6. Consider the Kummer monomial $(u^{1/2})$ on $X = \operatorname{Spec}(k[u])$ with the logaritmic structure defined by the variable u. Then $X[u^{1/2}] :=$ $\operatorname{Spec}((\mathcal{O}[X])[u^{1/2}]) = \operatorname{Spec}(K[u^{1/2}]) \to X$ is a Kummer étale map, in fact a Kummer cover where $u^{1/2}$ is defined. In this case $X_{\text{két}}(u^{1/2})$ consists of the refinements of $X[u^{1/2}]$. Note that the global sections $\Gamma(X, (u^{1/2})) = (u)$ on X do not generetae the ideal $(u^{1/2})$ on $X_{\text{két}}$. Since there is no enough sections on X we need to pass to a "smaller neighborhood" $X[u^{1/2}]$ to see the generating section $u^{1/2}$.

Kummer varieties

4. Stably toroidal varieties

The goal of this section is to extend the class of toroidal varieties to a more general and flexible category of *stably toroidal varieties*, which is closed with respect to Kummer blow-ups. The stably toroidal varieties in many aspects are very similar to the toroidal ones. Once a very basic setup is established, the formalism reduces to the toroidal case. Any stably toroidal variety has a toroidal neighborhoods in the Kummer topology so local considerations are nearly the same.

In Section 3.8 we recalled how to construct Kummer topology on toroidal varieties. This allows to consider Kummer ideals generated, in particular, by rational powers of monomials. Introducing Kummer blow ups significantly optimizes the resolution algorithm.

The main problem which rises is that the class of toroidal varieties is not preserved by Kummer blow ups. The stably toroidal varieties, discussed in this section are equipped with the (normal) Kummer topology and are preserved by the Kummer blow-ups.

examkumsec

4.1. **Motivating example for stably toroidal varieties.** Consider an affine space $X = \operatorname{Spec}(\mathbb{C}[x, u])$ with the logarithmic structure defined by the open subset $u \neq 0$. It is a toric variety X_σ associated with the cone $\sigma = \operatorname{span}((1,0)(0,1))$, with the log structure of toroidal variety corresponding to an open subset $X_\tau = \operatorname{Spec}(\mathbb{C}[x, u, u^{-1}])$, where $\tau = \operatorname{span}((0,1))$.

Consider the Kummer site $X_{\text{két}}$, and the *Kummer ideal sheaf* $\mathcal{I} = (x, u^{1/2})$, which is induced on on a Kummer covering $X[u^{1/2}] := \operatorname{Spec}(\mathcal{O}(X)[u^{1/2}])$ of X, generated by $w = u^{1/2}$.

Observe that there is a natural group action of the group $\mu_2 = \{\pm 1\}$ on a Kummer cover $X[u^{1/2}] = \operatorname{Spec}(\mathbb{C}[x, w])$ defined by the multiplication of $\mu_2 = \{\pm 1\}$ on the generator $u^{1/2}$. The variety X can be represented as the quotient $X = X[u^{1/2}]/\mu_2$, with $\mu_2 = \{\pm 1\}$ acting toroidally on X.

The variety $X[u^{1/2}] = \operatorname{Spec}(\mathbb{C}[x, w])$ is toroidal, and moreover the (normalized) blow-up of $\mathcal{I} = (x, w)$ on $\operatorname{Spec}(\mathbb{C}[x, w])$ transforms it, by Lemma 3.2.4, into a toroidal variety (Z, U_Z) defined by the pair of a toric variety Z and its open toric nonsingular subvariety U_Z.

The blow-up of $\mathcal{I}$ descends to the normalized blow-up X' of X at (x^2, u): whose open affine neighborhood (in chart $u^{1/2}$) is $X'_{u^{1/2}} = \operatorname{Spec} \mathbb{C}[w^2, wy, y^2]$.

By the universal property of the blow-up we get that Z is the normalization of $X'[u^{1/2}]$ (indicated below by 'nor') :

$$(X'_{u^{1/2}}[u^{1/2}])^{\mathrm{nor}} = (\mathbb{C}[w^2, wy, y^2][y])^{\mathrm{nor}} = \mathbb{C}[w^2, wy, y]^{\mathrm{nor}} = \mathbb{C}[w, y].$$

Moreover X' is the quotient of Z by the action on μ_2.

$$\operatorname{Spec} \mathbb{C}[w, y]/\mu_2 = \operatorname{Spec} \mathbb{C}[w^2, wy, y^2]$$

In other words the resulting logarithmic structure (X', U') on the toric variety X' is the quotient of toric variety Z with the toroidal logarithmic structure (Z, U_Z). Moreover the open subset $U' \subset X$ defining the logarithmic structure on the quotient $(X', U') = (Z, U_Z)/\mu_2$ is a nonsingular toric variety by construction.

The variety (X', U') is not toroidal anymore. Instead it is an example of a *stably toroidal* variety. Its affine toric neighborhoods is exactly a *toric doubleton.* Each such a doubleton is represented by the pair of affine toric varieties (X_σ, X_τ), where τ is a regular face of σ. Moreover there is a Kummer map $N_Z \subset N$ such that $(X_{\sigma,N_Z}, X_{\tau,N_Z})$ is toroidal, with the nonsingular open subset X_{τ,N_Z}. This implies that the variety is of the form

$$(X_{\sigma,N_Z}, X_{\tau,N_Z}) = (X_\delta \times X_\tau, T \times X_\tau),$$

for its two faces δ and τ. Since the Kummer map defines the isomorphism of the vector spaces we see that σ as the cone is the sum of its faces and is isomorphic to their product

$$\sigma = \delta + \tau \simeq \delta \times \tau.$$

Moreover on the level of the lattices we get the Kummer inclusion $N_Z = N_\delta \times N_\tau \to N_\sigma$, which represents the toric doubleton (X_σ, X_τ)as the quotient of $(X_\delta \times X_\tau, T \times X_\tau)$.

4.2. **Toric doubleton.** The logarithmic structures on toroidal varieties are modeled by toric varieties containg open torus. The *stably toroidal varieties* introduced here (and in greater extend in [FW17]) are modeled by more general *toric doubletons.*

For a cone σ in $N^{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ with lattice N, denote by N_σ the sublattice of N spanned by $\sigma \cap N$. Then $X_{\sigma,N} = X_{\sigma,N_\sigma} \times T$, where $T = \operatorname{Spec}(K[(N/N_\sigma)^\vee])$ is a torus with the lattice N/N_σ, and $N \simeq N_\sigma \times N/N_\sigma$.

Recall that smooth affine toric varieties $X_\sigma = X_{\sigma,N}$ correspond to *regular cones* σ, namely cones generated by a basis of the lattice $N_\sigma \subset N$ associated with σ.

As in Section 3.8.2, a *Kummer map of saturated monoids* $P \to Q$ is induced by a finite index inclusion of lattices.

Definition 4.2.1. By a *toric doubleton* we mean a pair of toric varieties $(X_{\sigma,N}, X_{\tau,N})$ such that

(1) $X_{\tau,N} \subset X_{\sigma,N}$ is a smooth open toric subvariety of X_σ cooresponding to a regular face τ of σ.
(2) There exists a face δ of σ, inducing a $\mathbb{Q}$-isomorphism $i_\sigma : \tau \times \delta \to \sigma = \tau + \delta$ of cones, which restricts to a finite-index inclusion of lattices $N_\tau \times N_\delta \to N_\sigma$.

4.2.2. *The monoids and logarithmic structures on toric doubletons.*

doubleton-structure

Lemma 4.2.3. *Let $(X_{\sigma,N_\sigma}, X_{\tau,N_\sigma})$ is a doubleton. Consider the natural projection*

$$\pi : (\sigma,\tau) \to (\sigma/\tau, N_\sigma/N_\tau)$$

and let

$$N^0_\delta := \pi(N_\sigma) = N_\sigma/N_\tau$$

(1) The restriction of π:

$$\pi_{|(\delta,N_\delta)} : (\delta, N_\delta) \to (\sigma/\tau, N_\sigma/N_\tau) \simeq (\delta, N^0_\delta),$$

is a Kummer homomorphism.

(2) The map $(X_{\delta,N_\delta} \times X_{\tau,N_\tau}, T_\delta \times X_{\tau,N_\tau}) \to (X_{\sigma,N_\sigma}, X_{\tau,N_\sigma})$ is the quotient of a toroidal variety by the group action

$$\Gamma_{\sigma,\tau} := N_\sigma/\pi(N_\delta \times N_\tau) = N^0_\delta/N_\delta.$$

(3) There exists a normalized fiber diagram

$$\begin{array}{ccccc}
X_{\delta,N_\delta} \times X_{\tau,N_\tau} \times T & \longrightarrow & X_{\delta,N_\delta} \times X_{\tau,N_\tau} & \longrightarrow & X_{\delta,N_\delta} \\
\downarrow & & \downarrow & & \downarrow \\
X_{\sigma,N} = X_{\sigma,N_\sigma} \times T & \longrightarrow & X_{\sigma,N_\sigma} & \longrightarrow & X_{\delta,N^0_\delta}
\end{array}$$

(4) The reduced fibers of the map $X_\sigma \to X_\delta$ have abelian quotient singularities, and are all of the same dimension. The reduced preimages of the strata are locally toric with $\mathbb{Q}$-factorial singularities.

Proof. (1) Follows from the definition of the toric doubleton. The map $N_\delta = (N_\tau \times N_\delta)/N_\tau \to N_\sigma/N_\tau$ is Kummer. (2) Follows from the definition of the toric doubleton and the properties of the toric groups defined by the lattice extensions (see also Section 5.1.1).

(3) The fiber product of lattices $N_\delta \times_{N^0_\delta} N_\sigma$ contains N_δ and N_τ. Moreover there is a projection to N_δ which splits so it contains $N_\delta + N_\tau = N_\delta \times N_\tau$. By the universal property we get the equality. This implies that the $X_{\delta,N_\delta} \times X_{\tau,N_\tau}$ is the fiber product $X_{\sigma,N_\sigma} \times_{X_{\delta,N^0_\delta}} X_{\delta,N_\delta}$ in the category of (normal) toric varieties.

(4) By (2), and (3), the fibers of $X_\sigma \to X_\delta$ are the quotients of the fibers $X_\tau \times T$ of the projection $\pi : X_\delta \times X_\tau \times T \to X_\delta$. ♣

doubleton2

Lemma 4.2.4. *With notation as above,*

(1) Set

$$N^1_\delta := N/N_\tau$$

For the logaritmic structure defined by the doubleton $(X_{\sigma,N}, X_{\tau,N})$ the maps of monoids define the isomorphisms:

$$P_{\delta,N^1_\delta} = (\delta \cap N^1_\delta)^\vee \to \mathcal{M}(X_{\sigma,N}, X_{\tau,N})$$

$$P_{\delta,N^0_\delta} = (\delta \cap N^0_\delta)^\vee \to \overline{\mathcal{M}}(X_{\sigma,N}, X_{\tau,N})$$

induced by $X_{\sigma,N_\sigma} \to X_{\delta,N^0_\delta}$ defines the canonical isomorphism.

(2) If x is in the orbit O_σ then $\overline{\mathcal{M}}_x = P_{\delta,N^0_\delta}$.

Proof. (1) By the definition $\mathcal{M}(X_{\sigma,N}, X_{\tau,N}) = \{F \in P_{\sigma,N} \mid F_{|\tau} = 0\}$. This defines the submonoid P_{δ,N^1_δ} of $P_{\sigma,N}$ which is dual to $(\delta, N^1_\delta) = (\sigma/\tau, N/N_\tau)$. Similar for P_{δ,N^0_δ}.

(2) If $x \in O_\sigma$, then for any function $F \in \mathcal{O}(X_\sigma)$ such that $F(x) \neq 0$, and open subsets $U = (X_\sigma)_F$, and $V = (X_{\tau,N})_F$ we have that

$$\mathcal{M}(U,V) = \mathcal{M}(X_{\sigma,N}, X_{\tau,N}) \cdot \mathcal{O}(U)^*.$$

So $\overline{\mathcal{M}}(U,V) = P_{(\delta,N^0_\delta)}$, and $\overline{\mathcal{M}}_{(X_{\sigma,N},X_{\tau,N})),x} \simeq P_{(\delta,N^0_\delta)}$.

♣

4.2.5. *Example.* Consider the toric doubleton (X_σ, X_τ), from Example 4.1. There are natural maps

$$X_{\delta,N_\delta} \times X_{\tau,N_\tau} = \operatorname{Spec}(\mathbb{C}[y]) \times \operatorname{Spec}(\mathbb{C}[w]) \to X_{\sigma,N_\sigma} = \operatorname{Spec}\mathbb{C}[w^2, wy, y^2].$$

$$X_{\sigma,N_\sigma} = \operatorname{Spec}\mathbb{C}[w^2, wy, y^2] \to X_{\delta,N^0_\delta} = \operatorname{Spec}\mathbb{C}[y^2]$$

4.2.6. *Logarithmic stratification on toric doubletons.* As in the logarithmically smooth case, we stratify the logarithmic structures of toric doubletons by the rank $\operatorname{rk}(\overline{M}^{\mathrm{gp}}_p)$, see Section 2.2.5. The reader is warned that, as the logarithmic structure is not coherent, the monoid itself is not constant along strata, though it is constant after a Kummer extension.

restriction **Lemma 4.2.7.** *The restriction of the toric doubleton $(X_{\sigma,N}, X_{\tau,N})$ to an open toric subvariety $X_{\sigma',N}$ is the toric doubleton $(X_{\sigma',N}, X_{\tau',N})$, where $\tau' := \tau \cap \sigma'$.*

D-strata **Lemma 4.2.8.** *Let $(X_{\sigma,N}, X_{\tau,N})$ be a toric doubleton. Consider the natural projection $\pi_{X_\sigma} : X_{\sigma,N} \to X_{\delta,N^0_\delta}$*

1. *There is a bijective correspondence between the irreducible strata defined by the logarithmic structure $\mathcal{M}$ of the toric doubleton, and the orbits in X_δ, (or the faces δ' of δ):*
$$\delta' \mapsto s(\delta') := \pi^{-1}_{X_\sigma}(O_{\delta'})$$
2. *For any $x \in O_{\sigma'} \subseteq s(\delta')$ we have that $\overline{\mathcal{M}}_x \simeq P_{\delta',N^0_{\delta'}}$, where $N^0_{\delta'} = N_{\sigma'}/N_{\tau'}$, with $x \in O_{\sigma',N_\sigma}$, and $\tau' = \tau \cap \sigma'$.*
3. *The map of the induced sheaves of monoids of the associated logarithmic structures $P^a_{\delta,N^0_\delta} \to \mathcal{M}$ is Kummer.*
4. *The irreducible logarithmic stratification S coincides with the stratification defined by the intersection of the irreducible components of $D_{\sigma,\tau} := X_{\sigma,N} \smallsetminus X_{\tau,N}$.*

Proof. (2) Consider the stratification S by the strata $s(\delta')$. If $x \in O_{\sigma'} \subseteq s(\delta')$, with $\sigma' = \delta' + \tau'$, where τ' is a face of τ then by Lemmas 4.2.4, and 4.2.7 we get that $\overline{\mathcal{M}}_x \simeq P_{\delta',N^0_{\delta'}}$ is dual to σ'/τ'. Moreover $P_{\delta',N_{\delta'}} \to \overline{\mathcal{M}}_x = P_{\delta',N^0_{\delta'}}$ is Kummer.

(1) It follows from the above that the stratification S coincides with the logarithmic stratification by the ranks of the monoids.

(3) The logarithmic structure P^a_{δ,N^0_δ} is induced by the canonical logarithmic structure on the toric variety X_{δ,N^0_δ}. That is $\overline{P}^a_{\delta,N^0_\delta,x} = P_{\delta',N_{\delta'}}$, with $N_{\delta'} \subset N^0_{\delta'}$. We use (2).

(4) Follows from the fact that the stratification is determined by the strata on X_δ satisfying this condition.

♣

4.2.9. *Toroidal structure on toric doubletons.*

Lemma 4.2.10. *(1) The intersection $X_{\delta,N} \cap X_{\tau,N}$ is the torus T of N, hence the logarithmic structure on the doubleton $(X_{\sigma,N}, X_{\tau,N})$ restricts to the toric logarithmic structure $(X_{\delta,N}, T)$.*

(2) The set of the points $X^{trd}_{\delta,N}$ on the doubleton $X_{\delta,N} \cap X_{\tau,N}$ where the log structure is toroidal is open. Any stratum in the logarithmic stratification on $X_{\delta,N} \cap X_{\tau,N}$ intersects $X^{trd}_{\delta,N}$.

4.3. **Regular doubletons.** A toric doubleton is said to be *regular* if i_σ preserves the lattice structure, namely $i_\sigma(N_\tau \times N_\delta) = N_\sigma$

We conclude from the definition that

doubleton **Lemma 4.3.1.** *(1) If $(X_{\sigma,N}, X_{\tau,N})$ is a regular doubleton then $i_\sigma : (\tau \times \delta, N_\tau \times N_\delta) \to (\sigma, N_\sigma)$ defines an isomorphism of toric varieties $X_{\delta,N_\delta} \times X_{\tau,N_\tau} \to X_{\sigma,N_\sigma}$, and the map $X_{\sigma,N_\sigma} \to X_{\delta,N_\delta}$ is smooth.*

(2) The doubleton $(X_{\sigma,N}, X_{\tau,N})$ defines a toroidal variety if and only if it is regular.

Proof. (1) follows from the definition, noting that the lattice N/N_σ does not interfere in the product structure. The 'if' implication in (2) follows since

$$X_{\sigma,N} = X_{\delta,N_\delta} \times \mathbb{A}^r \times \mathbb{G}_m^k$$

and

$$X_{\tau,N} = \mathbb{G}_m^l \times \mathbb{A}^r \times \mathbb{G}_m^k,$$

where $r = \operatorname{Rank} N_\tau$, $k = \operatorname{Rank} N/N_\sigma$, and l is the rank of N_σ, so that $k+l$ is the rank N/N_τ. The 'only if' part can be deduced from Lemma 4.2.4: First note that, by Lemma 1, p. 60, [KKMSD73] (see Lemma 2.1.8 above), the logarithmic structure on $(X_{\sigma,N_\sigma}, X_{\tau,N_\sigma})$ is induced on the Zariski site of X_{σ,N_σ}. By (2), the logarithmic structure on the open (toroidal) subset $X_{\delta,N}$ is defined by $P_\delta = (\delta, N_\delta)^\vee$.

By Lemma 4.2.4

$$\overline{\mathcal{M}}(X_\sigma, X_\tau) = \mathcal{O}(X_\sigma) \cap \mathcal{O}(X_\tau)^* = P'_\delta := (\delta, N^0_\delta)^\vee.$$

If the log variety (X_σ, X_τ) were toroidal then $P'_\delta \to \mathcal{M}(X_\sigma, X_\tau)$ is a chart in a neighborhood of $x \in O_\sigma$, so $\overline{(P'_\delta)}^a$ coincides with the log structure $\overline{\mathcal{M}}$ defined by (X_σ, X_τ).

The restriction of $\overline{(P'_\delta)}^a$ to an open subset $X_{\delta,N}$ is equal to P'_δ. So, since $P'_\delta \subset P_\delta$ is Kummer, we get that $P'_\delta = P_\delta$ and $N_\sigma = N_\delta + N_\tau$.

♣

4.4. **Stably toroidal varieties.**

4.4.1. *Rational toric charts and stably toroidal varieties.* Recall that one associates with toric doubletons $(X_{\sigma,N}, X_{\tau,N})$ the map $X_{\sigma,N} \to X_{\delta,N^1_\delta}$, where $N^0_\delta := N_\sigma/N_\tau$, and $N^1_\delta := N/N_\tau = N^0_\delta \times N/N_\sigma$.

Definition 4.4.2. A normal logarithmic variety (X, U) over a field K will be called *stably toroidal* (respectively *strictly stably toroidal*), if every point $x \in X$ has an étale neighborhood $Y \to X$ (respectively Zariski neighborhood $Y \to X$), with $U_Y = U \times_X Y$, and an étale map $(Y, U_Y) \to (X_\sigma, X_\tau)$ to a toric doubleton (X_σ, X_τ), with $U_Y = X_\tau \times_{X_\sigma} Y$, such that the quotient N_δ/N^0_δ of the groups recalled above is of order relatively prime to the characteristic of the ground field.

By *rational toric chart* we mean the induced morphism $Y \to X_{\delta,N^1_\delta}$, which is a composistion of the étale morphism $Y \to X_{\sigma,N}$ and the projection $X_{\sigma,N} \to X_{\delta,N^1_\delta}$, defined by the doubleton. Alternatively the rational toric chart is the induced map of monoids $P_{\delta,N^1_\delta} \to \mathcal{M}_{(Y,U_Y)}$, defined by the composition $Y \to X_{\delta,N^1_\delta}$ above.

4.4.3. *Properties of stably toroidal varieteies.* The notion of stably toroidal varieties is a natural extension of the notion of toroidal varieties. In particular we have:

Proposition 4.4.4. *If (X, U) is stably toroidal then*

1. *There is a stratification S of X by locally closed strata with abelian quotient singularities induced by the divisor $X \smallsetminus U$. It coincides with the logarithmic stratification on (X, U).*
2. *There exists étale locally an étale rational chart $Y \to X_\sigma \to X_\delta$ such that*
   - *The map $Y \to X_\delta$ is equidimensional. The reduced fibers of the map $Y \to X_\delta$, have abelian quotient singularities.*
   - *The logarithmic stratification on Y is induced by the orbit stratification on X_δ.*
   - *There is a Kummer map of the log structures $P^a \to \mathcal{M}_{(Y,U_Y)}$.*
3. *The strata of S admit abelian quotient singularities.*
4. *There is an open subvariety $X^{tord} \supseteq U$ of X which is the set of points of X where (X, U) is toroidal.*
5. *The subset X^{tord} contains generic points of the strata.*

Proof. The proof is a consequence of the properties for toric doubletons showed in Lemmas 4.2.3, 4.2.4, and 4.3.1. ♣

In the strictly toroidal case one has a similar combinatorial complex description as for the toroidal varieties, and their birational modifications ([FW17]).

4.5. Log schemes with rational logarithmic structures.

4.5.1. *Rational logarithmic structures.*

Definition 4.5.2. By *the rational chart* on a log scheme X we mean a map of monoids $P \to \mathcal{M}(Y)$ from a fs monoid P on an étale neighborhood $Y \to X$, inducing a Kummer map of sheaves of the logarithmic structures $P^a \to \mathcal{M}_Y$, (Section 3.8.2). A log scheme with *rational logarithmic structure* is a *normal* log scheme admitting étale locally rational charts.

Note that the assumption of normality is critical.

Example 4.5.3. Stably toroidal varieties admit locally toric rational charts. Moreover, by Lemma 4.2.3, the toric rational charts on stably toroidal varieties is a particular case of rational charts.

Example 4.5.4. Normal fs log-schemes posses rational logaithmic structure. The charts of fs log schemes are rational. On the other hand not all rational charts are ordinary charts defined for fs log schemes.

In the sequel we shall use only rational charts, referring often to them as *charts*.

4.5.5. *Localized and optimized charts.* Let $Y \to X$ be an étale neighborhood of X. By the *localization* of the chart $j : P \to \mathcal{M}(Y)$ at a point $x \in Y$, we mean the induced chart

$$P^{loc}_x := P \cdot (j^{-1}(\mathcal{O}^*_{Y,x}))^{-1} \to \mathcal{M}(Y'),$$

where Y' is a Zariski neighborhood of $x \in Y$. The chart $j : P \to \mathcal{M}(Y)$ is called *localized* at $x \in Y$ if the induced map $P/P^* \to (\mathcal{M}/\mathcal{O}^*)_{Y,x}$ is Kummer.

localization

Lemma 4.5.6. *Let $P \to \mathcal{M}_X$ be a rational chart, and P^{loc}_x be its localization an $x \in Y$. Then*

(1) $P^{loc}_x \to \mathcal{M}_X$ is a rational chart on an étale neighborhood Y of x.
(2) $P^{loc}_x/(P^{loc}_x)^* \to (\mathcal{M}_X/\mathcal{O}^*)_x$ is Kummer
(3) $X \to (X \times_{\operatorname{Spec}(\mathbb{Z}[P])} \operatorname{Spec}(\mathbb{Z}[P^{loc}_x])^{\mathrm{nor}}$ is an open embedding.

Proof. (1) The map $P^{loc}_x \to \mathcal{M}_X$ is induced when passing to the Zariski localization. It induces the same logaritmic structure in a Zariski neighborhood of x.

(2) By definition the image of $P^{loc}_x \to (\mathcal{M}_X/\mathcal{O})_x$ is Kummer.

(3) By the universal property there exist morphisms $X \to (X \times_{\operatorname{Spec}(\mathbb{Z}[P])} \operatorname{Spec}(\mathbb{Z}[P^{loc}_x])^{\mathrm{nor}}$, and its right inverse $(X \times_{\operatorname{Spec}(\mathbb{Z}[P])} \operatorname{Spec}(\mathbb{Z}[P^{loc}_x])^{\mathrm{nor}} \to X$. Moreover X and $X \times_{\operatorname{Spec}(\mathbb{Z}[P])} \operatorname{Spec}(\mathbb{Z}[P^{loc}_x])^{\mathrm{nor}}$ are of the same dimension and normal, and X is irreducible. This implies that X is a connected component of $(X \times_{\operatorname{Spec}(\mathbb{Z}[P])} \operatorname{Spec}(\mathbb{Z}[P^{loc}_x])^{\mathrm{nor}}$.

♣

4.5.7. *Optimized charts.* Let $Y \to X$ be an étale neighborhood of X. A rational chart $P \to \mathcal{M}(Y)$ will be called *optimized for $x \in Y$* if $P/P^* \to (\mathcal{M}/\mathcal{O}^*)_{X,x}$ is an isomorphism. By an *optimization* of $P \to \mathcal{M}(Y)$ at $x \in Y$ we mean a chart $P' \to \mathcal{M}(Y')$, where $P^{loc}_x \subseteq P'$, is Kummer and Y' is an étale neighborhood of a geometric point $\overline{x}$ over x, such that $P'/(P')^* \to (\mathcal{M}/\mathcal{O}^*)_{Y',\overline{x}}$ is an isomorphism.

optimization

Lemma 4.5.8. *Let X be a logarithmic variety with a rational structure. Let $P \to \mathcal{M}_X$ be a rational chart, in an étale neighborhood of a point $x \in X$.*

(1) There exists an optimization of $P' \to \mathcal{M}_X$ at x.
(2) $X' \to X \times_{\operatorname{Spec}(\mathbb{Z}[P])} \operatorname{Spec}(\mathbb{Z}[P'])^{\mathrm{nor}}$ is an étale morphism.

Proof. (1) We can replace P with P^{loc}_x, so that $P/P^* \to (\mathcal{M}/\mathcal{O}^*)_{X,\overline{x}}$ is Kummer map of the index relatively prime to the charateristic of X . Since the quotient group $(P/P^*)^{gp} = P^{gp}/P^*$ is torsion free there is an injective map $P^{gp}/P^* \to P^{gp}$, which is the right inverse (section) of the projection $P^{gp} \to P^{gp}/P^*$. This defines the section $P/P^* \to P$, of the projection $P \to P/P^*$. Let P' be the push out of $P \leftarrow P/P^* \to (\mathcal{M}/\mathcal{O}^*)_{X,\overline{x}}$. By definition, $P \subseteq P'$ is Kummer, and there is an induced map $P' \to (\mathcal{M}/\mathcal{O}^*)_{X,\overline{x}}$ defining the isomorphism $\overline{\beta}: P'/(P')^* \to (\mathcal{M}/\mathcal{O}^*)_{X,\overline{x}}$. One needs to show that there is a map $\beta: P' \to \mathcal{M}_{X,\overline{x}}$ on an étale neighborhood of $\overline{x}$, which induces the map $\overline{\beta}$.

Any generator $a \in P'$, maps to the class $[y] \in (\mathcal{M}/\mathcal{O}^*)_{X,\overline{x}}$ of $y \in (\mathcal{M}/\mathcal{O}^*)_{X,\overline{x}}$. Thus there is a power $a^n \in P$, which maps to $[a^n] \in (\mathcal{M}/\mathcal{O}^*)_{X,\overline{x}}$, so that $[a^n] = [y^n]$, $a^n = y^n\alpha$, where $\alpha \in \mathcal{O}^*_{X,\overline{x}}$. Passing to an étale neighborhood $X' = X \times_{Spec(\mathbb{Z}[\alpha])} (\mathbb{Z}[\sqrt[n]{\alpha}])$, we get that $a = y\sqrt[n]{\alpha} \in \mathcal{M}(X')$.

Repeating the process for all the generators of P' we may assume that $P' \supseteq \mathcal{M}(X')$ is a rational chart on an étale neighborhood X' of X, which is optimized at $x \in X$.

(2) By the construction

$$(X' \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[P'])^{\mathrm{nor}} \to (X \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[P'])^{\mathrm{nor}}$$

is étale. Moreover, since

$$(\mathrm{Spec}(\mathbb{Z}[P'] \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[P']))^{\mathrm{nor}}$$

contains $\mathrm{Spec}(\mathbb{Z}[P'])^{\mathrm{nor}}$ as its connected component so

$$\mathrm{Spec}(\mathbb{Z}[P'])^{\mathrm{nor}} \to \mathrm{Spec}(\mathbb{Z}[Q] := (\mathrm{Spec}(\mathbb{Z}[P'] \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[P']))^{\mathrm{nor}}$$

is étale (open embedding), and we conclude that

$$X' = X' \times_{\mathrm{Spec}(\mathbb{Z}[P'])} \mathrm{Spec}(\mathbb{Z}[P'])^{\mathrm{nor}} \to X' \times_{\mathrm{Spec}(\mathbb{Z}[P'])} \mathrm{Spec}(\mathbb{Z}[Q])^{\mathrm{nor}} \to$$

$$X' \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[P'])^{\mathrm{nor}}$$

is also étale.

♣

4.5.9. *Rational charts of the morphisms.*

Definition 4.5.10. Let $f: Y \to X$ be a morphism log schemes with rational structures.

A *rational chart for the morphism* f is a triple $(P_X \to \mathcal{M}_X, Q_Y \to \mathcal{M}_Y, P \to Q)$, where $P_X \to \mathcal{M}_X$ and $Q_Y \to \mathcal{M}_Y$ are the rational charts of $\mathcal{M}_X$, and $\mathcal{M}_Y$, and the map of monoids $P \to Q$ induces the map of constant sheaves $P_Y \to Q_Y$ which is compatible with the map of sheaves of monoids $f^*(\mathcal{M}_X) \to \mathcal{M}_Y$.

Definition 4.5.11. A rational chart $(P \to M_X, Q \to M_Y, P \to Q)$ of the map of log schemes $Y \to X$ is *optimized* at $y \in Y$ if $Q \to M(Y')$ is optimized at a geometric point $\overline{y}$ over y in an étale neighborhood Y' of $y \in Y$, and $P \to \mathcal{M}(X')$, is optimized at a geometric point $\overline{f(y)}$ over $f(y)$ in an étale neighborhood X' of X, with the induced map $Y' \to X'$.

optimization2 **Lemma 4.5.12.** *Let $Y \to X$ be a map of logarithmic varieties with rational log structures, and y be a point in Y. Then*

(1) *For any chart $(P \to M_X, Q \to M_Y, P \to Q)$ there exists an optimized chart $(P' \to M_X, Q' \to M_Y, P' \to Q')$.*
(2) *For any optimized chart $P \to \mathcal{M}_X$ there exist an optimized chart $(P \to M_X, Q \to M_Y, P \to Q)$.*

Proof. Any chart $P \to \mathcal{M}_X$ can be optimized by Lemma 4.5.8. The push-outs defined by the optimization map $P \to P'$, and $P \to Q$, determines the chart $P' \to Q'$ of $f : Y \to X$. It suffices to optimize $Q' \to M_Y$ by using Lemmas 4.5.8, and 4.5.12.

♣

4.6. **Normally log smooth morphisms.**

4.6.1. *Normal log smoothess.* The following definition extends the Kato Lemma 3.1.6 characterizing los smooth maps of log fs schemes[Kat89a] to the category of log schemes with rational structures. (see also [Niz], Lemma 2.8).

Kato22 **Definition 4.6.2.** A morphism $f : Y \to X$ of varieties with rational structures will be called *normally log smooth* (respectively *normally log étale*) iff there exists (étale locally on Y) a rational chart $(P \to \mathcal{M}_X, Q \to \mathcal{M}_Y, P \to Q)$ for the morphism f for which

1. The map $P \to Q$ induces a smooth (respectively étale) map $\mathrm{Spec}(\mathcal{O}_Y[Q^{gp}]) \to \mathrm{Spec}(\mathcal{O}_Y[P^{gp}])$
2. the induced map $X \to (Y \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[Q])^{\mathrm{nor}}$ (”n” stands for normalization) is étale.

A rational chart $(P_X \to \mathcal{M}_X, Q_Y \to \mathcal{M}_Y, P \to Q)$ for a normally log-smooth (respectively normally log étale) morphism $f : Y \to X$ will be called *log-smooth* (respectively log étale) if it satisfies conditions (1) and (2).

The definition is analogous to the Kato conditions for log smooth morphisms of fine and saturated logarithmic varieties, except it uses the normalized products and rational charts. The log-smooth morphism of normal fs varieties defines a normally log-smooth morphism of varieties with rational structures.

The conditions are equivalent to the standard definition in the case of morphisms of toroidal varieties (see Lemma 4.6.23) . In this case $Y \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[Q])$ is already normal.

We shall show that the basic functorial properties of the normally log smooth morphisms are similar to log smooth morphism of fs varities following the Kato line of arguments (see also [Niz]).

Kato-s1

4.6.3. *Kato's lemma on independence of charts.* The following proposition extends the analogous result by Kato for fs logarithmic structures to the normally log smooth varieties with rational structures.

Kato33 **Proposition 4.6.4.** *Let $Y \to X$ be a normally log smooth (respectively normally log étale) map of log schemes with rational structures. Then for any rational chart $P \to \mathcal{M}_Y$ there exist a log smooth (log étale) chart $(P \to M_X, Q \to M_Y, P \to Q)$ for f, extending $P \to \mathcal{M}_Y$.*

Before proving the proposition we need to gather some useful facts, many of which are due to Kato.

Kato2 **Proposition 4.6.5.** *Let $Y \to X$ be a log smooth (respectively log étale) map of normally log smooth varieties with rational structures. Let $P_1 \to \mathcal{M}(X)$ be a rational chart. Assume $(P_1 \to M_X, Q_1 \to M_Y, P_1 \to Q_1)$ is a log smooth (respectively log étale) chart for $X \to Y$*

(1) *If there are maps $P_1 \to P_2 \to \mathcal{M}(X)$ then the induced map $P_2 \to Q_2$ is log smooth (respectively log étale) chart for $Y \to X$ with Q_2 being the push ot of $Q_1 \leftarrow P_1 \to P_2$.*
(2) *If $P_1^{loc} \to Q$ is a log smooth (respectively log étale) chart for $Y \to X$ then $P_1 \to Q$ defines a log smooth (respectively log étale) chart for $Y \to X$.*
(3) *If $P_0 \to P_1$ is Kummer map (with the cokernel of $P_0^{gp} \to P_1^{gp}$ finite invertible on Y then $P_0 \to Q_1$ is a log smooth (respectively log étale) chart for $Y \to X$.*
(4) *If $H^* \subset P_1^*$ is a subgroup and $H^* \to G^*$ is Kummer then for the pushout P_2 of $P_1 \leftarrow H^* \to G^*$ the pushout Q_2 of $Q_1 \leftarrow P_1 \to P_2$. Then $Y_2 := (Y \times_{\mathrm{Spec}(\mathbb{Z}[P_1]} \mathrm{Spec}(\mathbb{Z}[P_2]))^{\mathrm{nor}} \to Y$ is étale, and $P_2 \to \mathcal{O}(Y_2)$, $P_2 \to Q_2$ is a log smooth (respectively log étale) chart for $Y \to X$.*

Proof. (1) Follows from the universal properties of the push out of monoids. (2) Follows from the fact that $\mathrm{Spec}(\mathbb{Z}[P^{loc}]$ is an open subset of $\mathrm{Spec}(\mathbb{Z}[P]$ so the both maps coincide.

(3) The maps $X \to \mathrm{Spec}(\mathbb{Z}[P_1]) \to \mathrm{Spec}(\mathbb{Z}[P_0])$ define the morphisms $X \to X \times_{\mathrm{Spec}(\mathbb{Z}[P_0]} \mathrm{Spec}(\mathbb{Z}[P_1]) \to X$, of the schemes of the same dimension. Passing to the normalizations we see that X is an (open) connected component of $(X \times_{\mathrm{Spec}(\mathbb{Z}[P_0]} \mathrm{Spec}(\mathbb{Z}[P_1])^{\mathrm{nor}}$. So taking pull back with $\mathrm{Spec}(\mathbb{Z}[Q_1]) \to \mathrm{Spec}(\mathbb{Z}[P_1])$ we see that $(X \times_{\mathrm{Spec}(\mathbb{Z}[P_1]} \mathrm{Spec}(\mathbb{Z}[Q_1])^{\mathrm{nor}}$ is open in $(X \times_{\mathrm{Spec}(\mathbb{Z}[P_0]} \mathrm{Spec}(\mathbb{Z}[Q_1])^{\mathrm{nor}}$ showing that $P_0 \to Q_1$ is a log smooth chart for $Y \to X$.

(4) $Y_2 = Y \times_{\mathrm{Spec}(\mathbb{Z}[P_1^*]} \mathrm{Spec}(\mathbb{Z}[H]) \to Y$ is induced by the étale map, so it is étale and thus Y is already normal: $(Y_2)^{\mathrm{nor}} = Y_2$.

♣

Kato-s2

4.6.6. *Proof of Kato's lemma.*

reduced11

Lemma 4.6.7. *Let $P \to \mathcal{M}(X)$ be a rational chart. Then:*

(1) *One can write P as $P = P^{\mathrm{red}} \oplus G$, where $G \subset P$ is the maximal finite subgroup of the order invertible on X, and P^{red} is a subgroup of P isomorphic to P/G*
(2) *The morphism $\phi : X \times_{\mathrm{Spec}(\mathbb{Z}[P]} \mathrm{Spec}(\mathbb{Z}[P/G]) \to X$ induced by the projection $P \to P/G$ is an open embedding.*

Proof. (1) The subgroup $P^* \subset P$ is saturated and contains a torsion part of P^{gp}. So $(P/P^*)^{gp}$ is torsion free and we can write $P = P/P^* \oplus P^*$. Then $P^* \simeq \mathbb{Z}^n \oplus \bigoplus_{p\in I} A_p$, where A_p are Sylow p-subgroups of the torision part $P^{\mathrm{tor}} = (P^*)^{\mathrm{tor}}$. Then $G = \bigoplus_{p \in I'} A_p$, where $I' \subset I$ corresponds to the primes invertible on X.

(2) The morphism ϕ is étale so open and a closed embedding. ♣

We shall call the chart $P \to \mathcal{M}(X)$ *reduced* if $P = P^{\mathrm{red}} = P/G$.

reduced22

Corollary 4.6.8. (1) *The map $P \to \mathcal{M}(X)$ is a chart iff $P^{\mathrm{red}} \to \mathcal{M}(X)$ is a chart.*
(2) *If a map of monoids $P \to Q$ is a log smooth rational chart of the morphism $f : Y \to X$. Then the reduced chart $P^{\mathrm{red}} \to Q$ is also log smooth. Moreover $P^{\mathrm{red}} \to Q$ is injective.*

Proof. (1) and the first part of (2) follow from Lemma 4.6.7. For the second part of (2) note that the kernel $P^{\rm red} \to Q$ is contained in $(P^{\rm red})^{\rm tor}$, and no nontivial elements in $(P^{\rm red})^{\rm tor}$ have order invertible on X.

♣

Kato2 **Lemma 4.6.9.** *(Kato)([Kat89a],[Niz]) Let $P \to P_1$ be an injective map of the monoids for which the induced map $P/P^* \to P_1/P_1^*$ is an isomorphism. Assume that $P_1 \to Q_1$ is an injective log smooth map of monoids.*

Then there exists maps of monoids $P \to Q$, and $P_1' \to Q_1'$, such that

(1) $P_1 \to P_1'$ is a certain étale extension
(2) $P_1' \to Q_1'$ is induced by the push-out $P_1' \leftarrow P_1 \to Q_1$.
(3) $P_1' \to Q_1'$ is induced by the push-out $P_1' \leftarrow P \to Q$.

Proof. (see Kato and Niziol) The problem reduces to maps of the groups $P^{gp} \to P_1^{gp}$, $P_1^{gp} \to Q_1^{gp}$. Let

$$G := P_1^{gp}/P^{gp} = P_1^*/P^*, \quad W := Q_1^{gp}/P_1^{gp}.$$

Denote by $\pi_1 : P_1^{gp} \to W$, and $\pi_2 : P_1^{gp} \to G$ the standard projections.

Then the push out of the exact sequence

$$0 \to P_1^{gp} \to Q_1^{gp} \to W \to 0$$

by the map $P_1^{gp} \to G$, induces the exact sequence

$$0 \to G \to T \to W \to 0.$$

We need to modify P_1 so that $f : T \to W$ has a section. Write the decoposition for

$$W = \bigoplus_{j=1}^{k} \mathbb{Z}\bar{e}_j \oplus \bigoplus_{i=1}^{s} (\mathbb{Z}/n_i\mathbb{Z})e_i.$$

Let $a = e_i$ be a torsion element in W and b its preimage in T. Then $n_i a = 0$ and $n_i b \in G = P_1^{gp}/P^{gp} = P_1^*/P^*$ is represented by $c \in P_1^*$. It suffices to extend P_1 to the push out P_1' of $P_1 \leftarrow P_1^* \to H$, where H is generated by n-th roots of P_1, with $n := n_1 \cdot \ldots \cdot n_s$. Let Q_1' be the push-out of $P_1' \leftarrow P_1 \to Q_1$. Note that such a modification leaves W intact (since $P_1^{gp} \to W$ is a zero map). Now $c \in P_1^* \subset (P_1')^*$, can be written as $c = n_i d$ for $d \in (P_1')^*$. The image $[d]$ of d in T' determines the element such that $n_i[d] = n_i b$, and $f[d] = 1$. Then $n_i(b - [d]) = 0$, and $f(b - [d]) = a$. Set new $b_i := b - [d] \in T'$. Since $n_i b_i = 0$, and $f(b_i) = e_i$ putting $s(e_i) = b_i$ defines a section on the torsion part of W, which extends to a section $s : W \to T$.

Let $s : W \to T$ be a section of $W \to T$. Consider the map $\alpha : Q_1^{gp} \to T$, $\alpha = \pi_2 - s\pi_1$. Its image is in in the kernel of $T \to W$. So it defines a (surjective) connecting $\alpha : Q_1^{gp} \to G$ which is the lift of $P_1^{gp} \to G$. Then we define Q^{gp} to be the kernel of α, and the monoid $Q := Q_1 \cap Q^{gp}$. By the definition Q_1^{gp} is the push out of $P_1^{gp} \leftarrow P^{gp} \to Q^{gp}$. Indeed such a push-out is a subgroup of Q_1^{gp} containing P_1^{gp} which surjects onto G. Consequently Q_1 is the push out of $P_1 \leftarrow P \to Q$. Indeed, the push-out is a submonoid Q_1' of Q_1, which is generated by Q and P_1. Note that by the universal property of the induced diagram $Q_1'/(Q_1')^* = Q/Q^*$, and since $P_1^*/P^* = Q_1'^*/Q_1'^* = G$, we get that $Q_1'^* \subset Q_1'$ is the push-out of $P_1^* \leftarrow P^* \to Q^*$, and is generated by P_1^*, and Q^*.

♣

In order to prove Proposition 4.6.4 we follow here some ideas of Kato. The problem is very similar to the case of fs charts of the log smoooth morphisms. Let $P \subset \mathcal{M}(Y)$ be any rational chart, and $(P_1 \subset \mathcal{M}(Y), Q_1 \subset \mathcal{M}(X), P_1 \to Q_1)$ be a log smooth chart of $X \to Y$.

Step 1. Construct a chart $P_1' \to \mathcal{M}(Y)$ 'containing' both P and P_1. For instance, consider $\phi : P_1^{gp} \oplus P^{gp} \to \mathcal{M}(Y)^{gp}$, with natural coordinate embeddings $P_1 \to P_1^{gp} \oplus P^{gp}$, $P \to P_1^{gp} \oplus P^{gp}$.

Define the monoid $P_1' := \phi^{-1}(\mathcal{M}(Y))$. Replace P_1 with P_1', and Q_1 with the induced push-out Q_1'. This induces the injective maps $P \to P_1'$, and $P_1 \to P_1'$.

By Proposition 4.6.9(1) we may replace $P_1 \to \mathcal{M}(Y)$ with $P_1' \to \mathcal{M}(Y)$, and Q_1 with the push out Q_1' of $P_1 \leftarrow P_1' \to Q_1$, such that there exists an *injective* map $P \to P_1 \subset \mathcal{M}(Y)$, where new $P_1 \to Q_1$ is also log smooth.

Step 2. Replace P and P_1 with their localizations $P^{loc}_{f(x)}$, and $(P_{1f(x)})^{loc}$ (by Proposition 4.6.9(1)(2)). Then $i : P/P^* \to P_1/P_1^*$ is Kummer with the cokernel invertible on Y.

Step 3. Consider the map $\psi : P_1 \to P_1/P_1^*$, and set $P_1' := \psi^{-1}(i(P/P^*))$. Then $P_1' \subset P_1$ is Kummer, so by Proposition 4.6.9(3) we can replace P_1 with P_1' and assume that $P_1 = PP_1^*$, with $P_1/P_1^* = P/P^*$.

Step 4. By Lemma 4.6.8, we can replace $P \to P_1$ with their reductions $P^{\rm red} \to P_1^{\rm red}$. This will make the map $P_1 \to Q_1$ injective.

Step 5. By Lemma 4.6.9, there is a map $P \to Q$ such that (after étale extension) $P_1 \to Q_1$ is induced by $P \to Q$. Thus $P \to Q$ is a log smooth map.

4.6.10. *Optimization of normally log smooth maps.*

optimization3 **Lemma 4.6.11.** *Let $Y \to X$ be a normally log smooth (log étale) morphism, logarithmic varieties with rational log structures, and y be a point in Y. Then for any optimized chart $P \to \mathcal{M}_Y$ there exist a log smooth (log étale) optimized chart $(P \to M_X, Q \to M_Y, P \to Q)$ for $f : Y \to X$.*

Proof. The same as for the proof of Lemma 4.5.12. ♣

Kummer11

4.6.12. *Kummer maps of log schemes with rational log structures.* One can extend the Kummer maps to the category of varieties with rational log structures. As before we have:

Definition 4.6.13. A map of log schemes with rational log structures $f : Y \to X$ is of *Kummer type* if for any $y \in Y$, the induced homomorphism of monoids $(\mathcal{M}/\mathcal{O}^*)_{X,\overline{f(y)}} \to (\mathcal{M}/\mathcal{O}^*)_{X,\overline{y}}$ is of Kummer type. The Kummer log étale map $f : Y \to X$ is a normally log étale map which is of Kummer type.

Lemma 4.6.14. *Let $Y \to X$ be a normally log étale map of varieties with rational log structures. Then $f : Y \to X$ is Kummer log étale iff for any $y \in Y$, and $x = f(y) \in X$, and any optimized at x log étale chart $(P \to M_X, Q \to M_Y, P \to Q)$, the map $P \to Q$ is Kummer log étale.*

Proof. Assume $f : Y \to X$ is Kummer log étale, and $(P \to M_X, Q \to M_Y, P \to Q)$ is an optimized log étale chart at $x \in X$.

This implies that the map $P/P^* \to Q/Q^*$ is Kummer as it is induced by the Kummer map $(\mathcal{M}/\mathcal{O}^*)_{X,\overline{f(y)}} \to (\mathcal{M}/\mathcal{O}^*)_{Y,\overline{y}}$. By the assumption $i : P^{gp} \to Q^{gp}$

and $j : P^{gp}/P^* \to Q^{gp}/Q^*$ are both Kummer. First let us show that $i : P^* \to Q^*$ is Kummer. The map $i : P^* \to Q^*$ is injective and for any $b \in Q^*$ there is $a \in P^{gp}$, such that $i(a) = b^n$. Since $P^{gp}/P^* \to Q^{gp}/Q^*$ is injective, for the classes $[a] \in P^{gp}/P^*$ of x , and $[b]$ in Q^{gp}/Q^* of y, we have $j([a]) = [b^n] = 1$, and $[a] = 1$. So $a \in P^*$, and thus $P^* \subset Q^*$ is Kummer. Moreover we showed that if $a \in P^{gp}$, and $j([a]) = 1$, then $i(a) \in Q^*$, which implies that $i(P^*) = Q^* \cap i(P^{gp})$.

To show that $P \to Q$ is also Kummer consider $y \in Q \subset Q^{gp}$. There is $a \in P^{gp}$ such that $i(a) = b^n \in Q$. Note that $a \in P' := i^{-1}(Q) \cap P^{gp}$. This implies that $P' \subset Q$ is Kummer. Moreover $P' \subset P^{gp}$ contains P, and $P/P^* \subset P'/P^* \subset Q/Q^*$ are also Kummer. So if $a \in P'$, then $a^n = bc$, with $b \in P$ and $i(c) \in Q^* \cap i(P^{gp}) = i(P^*)$. So $b, c \in P$ and $a^n \in P$. But $a \in P' \subset P^{gp}$, and since P is saturated in P^{gp} we get that $a \in P$. Consequently $P = P' \to Q$ is Kummer. ♣

Kummer21

4.6.15. *Fiber products and compositions.*

Definition 4.6.16. The fiber product $Y \times_X Z$ of the maps of the log schemes X, Y, Z with rational structures is the saturated normalized fiber square product $((Y \times_X Z)^{\mathrm{nor}})^{\mathrm{sat}}$ with the induced structure

$$\mathcal{M} := (((\pi_X^{-1}(\mathcal{M}_X) \otimes_{\pi_X^{-1}(\mathcal{M}_Z)} \pi_X^{-1}(\mathcal{M}_Y))^a)^{\mathrm{sat}}.$$

Let $P \to \mathcal{M}(Y \times_X Z)^{\mathrm{nor}}$ be the chart defined by the (nonsaturated) push-out of the charts for $\mathcal{M}(Y)$, $\mathcal{M}(Z)$ over $\mathcal{M}(X)$. Then, by definition, $\mathrm{Spec}(Z[P^{\mathrm{sat}}])$ is the disjoint union of some components in $\mathrm{Spec}(Z[P])^{\mathrm{nor}}$. On the other hand the scheme

$$(Y \times_X Z)^{\mathrm{nor}} \times_{\mathrm{Spec}(Z[P])} \mathrm{Spec}(Z[P])^{\mathrm{nor}}$$

contains $(Y \times_X Z)^{\mathrm{nor}}$ as its closed subscheme i.e the union of the irreducible components (As there is natural map $(Y \times_X Z)^{\mathrm{nor}} \to (Y \times_X Z)^{\mathrm{nor}} \times_{\mathrm{Spec}(Z[P])} \mathrm{Spec}(Z[P])^{\mathrm{nor}}$ which is the right inverse of the natural projection).

Then

$$((Y \times_X Z)^{\mathrm{nor}})^{\mathrm{sat}} = (Y \times_X Z)^{\mathrm{nor}} \times_{\mathrm{Spec}(Z[P]} \mathrm{Spec}(Z[P^{\mathrm{sat}}])$$

is an open and closed subset in $(Y \times_X Z)^{\mathrm{nor}} \times_{\mathrm{Spec}(Z[P]} \mathrm{Spec}(Z[P])^{\mathrm{nor}}$, hence the union of its some disjoint components.

Then $((Y \times_X Z)^{\mathrm{nor}})^{\mathrm{sat}}$ is an open and closed subset of $(Y \times_X Z)^{\mathrm{nor}}$, and is, in particular, normal.

Lemma 4.6.17. *Let $f : Y \to X$, and $g : Z \to X$ be two map of the log schemes with rational structures, defined locally by charts $(P \to \mathcal{M}_X, Q \to \mathcal{M}_Y, P \to Q)$ and $(P \to \mathcal{M}_X, R \to \mathcal{M}_Z, P \to R)$. Assume furthermore then the charts are optimized respectively at the points $y \in Y$, $z \in Z$ $f(y) = g(z) = x \in X$.*

Then the fiber product of $W = Y \times_X Z$ in the category of log schemes with rational structures admits étale locally the rational chart $T \to \mathcal{M}_W$, where T is the push out of $Q \leftarrow P \to R$ (in the category of fs monoids). Moreover T is optimized at the point $w \in W$ over x, y, z.

Proof. Since the chart Q, P, R generate the log -structure at y, x, and a point $z \in Z$ over x, the map $T \to \mathcal{M}_W$ generates the logarithmic structure at the point w over x, y, z, and thus it is optimized at w. We need to show that $T \to \mathcal{M}_W$ is a rational chart. By localizing and then optimizing charts $P \to \mathcal{M}_X$, $Q \to \mathcal{M}_Y$, $R \to \mathcal{M}_Z$ at respectively the points $x_1 = f(y_1), y_1, z_1$, as in Lemmas 4.5.6, 4.5.8, 4.5.12, 4.6.11 we construct the charts $P_1 \to \mathcal{M}_X, R_1 \to \mathcal{M}_Z$, $Q_1 \to \mathcal{M}_Y$, optimized for $x_1 = f(y_1), y_1, z_1$, which are Kummer over the localizations $P^{loc}_{x_1} \to \mathcal{M}_X$, $Q^{loc}_{y_1} \to \mathcal{M}_Y$, and $R^{loc}_{z_1} \to \mathcal{M}_Z$. Then the induced push-out T_1 of $R_1 \leftarrow P_1 \to Q_1$ is an optimized chart at w_1. Moreover T_1 is Kummer over the push out $\overline{T}^{loc}$ of the localizations $R^{loc}_{z_1} \leftarrow P^{loc}_{x_1} \to Q^{loc}_{y_1}$. But $\overline{T}^{loc}$ is contained naturally in the localization $T^{loc}_{w_1}$ so we get inclusions $\overline{T}^{loc} \subseteq T^{loc}_{w_1} \subset T_1$, which implies that $T^{loc}_{w_1} \subset T_1$ is Kummer. Consequently $T^{loc}_{w_1}/(T^{loc}_{w_1})^* \to T_1/T_1^* \simeq (\mathcal{M}/\mathcal{O}^*)_{w_1,W}$ is Kummer which shows that $T \to \mathcal{M}_W$ is rational. ♣

We can deduce from the proof the following result

product

Corollary 4.6.18. *The (saturated normalized) product of log-schemes with rational structure is a log scheme with rational structure. The chart of the product is defined by the push-out of the of the charts.*

Lemma 4.6.19. *Normally log étale maps, log smooth, and Kummer log étale maps are stable under base changes.*

Proof. Let $y \in Y$ be a point. We consider the case of normally log étale maps. The log-smooth case is exactly the same. Let $f : Y \to X$ be a log étale map of the log schemes with rational structures, defined locally by the optimized log étale chart $(P \to M_X, Q \to M_Y, P \to Q)$. Let $(P \to M_X, R \to M_Z, P \to R)$ be the optimized chart. Let $W \to Z$ be the (saturated normalized) pull-back of $Y \to X$ by $Z \to X$ defined by the optimized chart $(R \to M_Z, T \to M_W, R \to T)$ (as in the Lemma above).

Since $P \to Q$ is log étale (log smooth) the induced morphism $Y \to (X \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[Q]))^{\mathrm{nor}}$ is étale.

Taking pull back with $Z \to X$ we get that the induced morphism

$$(Y \times_X Z)^{\mathrm{nor}} \to (Z \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[Q]))^{\mathrm{nor}}$$

is étale.

Moreover by the universal property

$$(Z \times_{\mathrm{Spec}(\mathbb{Z}[R])} \mathrm{Spec}(\mathbb{Z}[T]))^{\mathrm{nor}} \to (Z \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[Q]))^{\mathrm{nor}}$$

is an open embedding since

$$\mathrm{Spec}(\mathbb{Z}[T]) = (\mathrm{Spec}(\mathbb{Z}[R] \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[Q]))^{\mathrm{sat}} \to (\mathrm{Spec}(\mathbb{Z}[R] \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[Q]))^{\mathrm{nor}}$$

is such.

Thus $W = ((Y \times_X Z)^{\mathrm{nor}})^{\mathrm{sat}} \to (Z \times_{\mathrm{Spec}(\mathbb{Z}[R])} \mathrm{Spec}(\mathbb{Z}[T]))^{\mathrm{nor}}$ is étale, and $W \to Z$ is normally log étale. Also, if $f : Y \to X$ is Kummer log étale then $P \to Q$ and $R \to T$ are Kummer, and since the charts are optimized by the previous Lemma the morphism $W \to Z$ is of Kummer type.

♣

Lemma 4.6.20. *Normally log étale maps, log smooth, and Kummer log étale maps are stable under composition.*

Proof. This reduces via optimized maps and Kato's Proposition 4.6.4 to the statements about the composition of monoids.

♣

4.6.21. *Normally log smooth maps between stably toroidal varieties.* The following result shows that the normalization in the log smooth maps of stably toroidal varieties is "locally toric".

stable map

Proposition 4.6.22. *A morphism $f : (Y, V) \to (X, U)$ of stably toroidal varieties is normally log smooth (respectively normally log étale) if iff for any toric chart $X \to X_{\sigma_1} \to X_{\delta_1}$, there exists (étale locally on X) a toric chart $Y \to X_{\sigma_2} \to X_{\delta_2}$, and the maps of cones with lattices $(\delta_2, N_2^0) \to (\delta_1, N_1^0)$, $(\sigma_1, N_1) \to (\sigma_2, N_2)$, commuting with face projections $\sigma_i \to \delta_i$ and yielding the extended normalized fiber square diagram:*

$$\begin{array}{ccccc} Y_2 \to (Y_1 \times_{X_{\delta_1}} X_{\delta_2})^{\mathrm{nor}} = Y \times_{X_{\sigma_1}} X_{\sigma_2} & \longrightarrow & (X_{\sigma_1} \times_{X_{\delta_1}} X_{\delta_2})^{\mathrm{nor}} = X_{\sigma_2} & \longrightarrow & X_{\delta_2} \\ \downarrow & & \downarrow & & \downarrow \\ Y_1 & \longrightarrow & X_{\sigma_1} & \longrightarrow & X_{\delta_1} \end{array}$$

such that the induced map $Y_2 \to (Y_1 \times_{X_{\delta_1}} X_{\delta_2})^{\mathrm{nor}}$ is étale, and the map $N_{\delta_2} \to N_{\delta_2}$ defines a smooth (respectively étale) map of tori.

Proof. Consider the toric chart $X_1 \to X_{\sigma_1} \to X_{\delta_1}$ from an étale neighborhood X_1 of X. By Proposition 4.6.2, there is étale locally a rational chart $(X_1 \to X_{\delta_1}, Y_1 \to X_{\delta_2}, \delta_2 \to \delta_1)$ of $f : Y \to X$, for which $Y_1 \to (X_1 \times_{X_{\delta_1}} X_{\delta_2})^{\mathrm{nor}}$ is étale.

Let σ_2 be the fiber product of the cones $\delta_2 \times_{\delta_1} \sigma_1$, and similarly for the lattices let $N_2 = N_1 \times_{N_1'} N_2'$. This allows to complete the diagram. Consequently the morphism $(Y \times_{X_{\delta_1}} X_{\delta_2})^{\mathrm{nor}} = (Y \times_{X_{\sigma_1}} X_{\sigma_2}) \to X_{\sigma_2}$ is étale. Since (X_{σ_1}, X_τ) is a toric doubleton it follows easily that the induced pull-back (X_{σ_2}, X_τ) is a toric doubleton again.

♣

toroidal2

Lemma 4.6.23. *Let $f : Y \to X$ be a normally log smooth map of the varieties with rational logarithmic structure where (X, U) is stably toroidal (toroidal). Then (Y, V) is stably toroidal (toroidal), where $V = f^{-1}(U)$. Moreover normally log smooth morphism of toroidal varieties is log smooth.*

Proof. Consider a toric chart $(X, U) \to (X_\sigma, X_\tau) \to X_\delta = \mathrm{Spec}(K[P])$, and a log-smooth chart $(P \to M_X, Q \to M_Y, P \to Q)$, for the morphism $f : Y \to X$ optimized for a point $y \in Y$.

Then the pullback of $X_\sigma \times_{\mathrm{Spec}(K[P]} \mathrm{Spec}(K[Q]$ defines the toric doubleton. Then there is an étale morphism to a toric doubleton

$$\phi : Y \to X \times_{\mathrm{Spec}(K[P]} \mathrm{Spec}(K[Q] \to X_\sigma \times_{\mathrm{Spec}(K[P]} \mathrm{Spec}(K[Q])$$

with $V = \phi^{-1}(X_\tau \times_{\mathrm{Spec}(K[P]} \mathrm{Spec}(K[Q]) = f^{-1}(U)$

Moreover the logarithmic structure at $y \in Y$ coincides with the one induced by the doubleton since the chart is optimized at y. Thus it coincides with the logaritmic structure of (Y, V).

This implies that Y is stably toroidal with the open subset $V = \phi^{-1}(X_\tau \times_{\mathrm{Spec}(K[P]} \mathrm{Spec}(K[Q]) = f^{-1}(U)$.

Similarly for toroidal varieties.

♣

Lemma 4.6.24. *The fiber product $X \times_Y Z \to X$ of the normally log smooth maps of stably toroidal varieties is a stably toroidal with the induced maps normally log smooth.*

Proof. By Lemma $X \times_Y Z \to Z$ is normally log-smooth, and $Z \to Y$ is normally log-smooth so $X \times_Y Z \to X$ is log-smooth. Then, by Lemma 4.6.23 $X \times_Y Z \to X$ is stably toroidal.

♣

4.6.25. *Local characterization of normally log-smooth morphisms of stably toroidal varieties.* The following proposition shows that the normally log smooth morphism of stably toroidal varieties is locally log smooth of the toroidal varieties with respect to a locally defined structure of the toroidal variety. To be more precise

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Proposition 4.6.26. *(1) Let (X_1, U_1) be a stably toroidal variety. Then for any point $y \in X_1$ there is an open neighborhood $X_1' \subset X_1$ with the induced $U_1' := U_1 \cap X_1'$, and an open subset $V_1 \subseteq U_1'$ such that (X_1', V_1), is a toroidal variety.*

(2) Let $f : (X_2, U_2) \to (X_1, U_1)$ be a normally log smooth morphism of stably toroidal varieties. Then the induced map $(X_2', V_2) \to (X_1', V_1)$ is log smooth morphism of toroidal varieties, where $X_2' := f^{-1}(X_1')$, $V_2 = f^{-1}(V_1)$.

Proof. (1) We define $X_1' \subset X_1$, $U_1' := U_1 \cap X_1'$ which admits an étale cover $(X_1'', U_1'') \to (X_1', U_1')$ and an étale chart $\alpha : X_1'' \to X_{\sigma_1}$.

(2) For $X_2' := f^{-1}(X_1')$, by Proposition 4.6.22 we find and an étale neighborhood $X_2'' \to X_2'$, with étale map $\beta : X_2'' \to X_{\sigma_2}$, $\sigma_2 \to \sigma_1$ defining a toric chart for $f' : X_2' \to X_1'$. We set $V_2 := \alpha^{-1}(T_2)$, $U_2 := \beta^{-1}(T_1) = f'^{-1}(V_1)$, where $T_2 \subset X_{\sigma_2}$, $T_1 \subset X_{\sigma_1}$ are the open tori. This defines map of the log smooth map of toroidal varieties, as in Proposition 4.6.23. ♣

Kummer top

4.7. **The Kummer étale site.**

4.7.1. *The Kummer étale site on stably toroidal varieties.* We shall equip the stably toroidal varieties with a Kummer topology which makes them (Kummer étale) locally isomorphic to the toroidal varieties. The construction is nearly identical to the one for toroidal varieties (and more generally f.s. logarithmic varieties). Similar construction can be done for the class of varieties with rational log structures.

Definition 4.7.2. Let (X, U) be a stably toroidal logarithmic variety. The *Kummer site* over X is the category $X_{\text{két}}$ of the normally Kummer étale maps from stably toroidal varieties $(Y, U_Y) \to (X, U)$. The *coverings* $\{\phi_i : V_i \to V\}$ are defined by the jointly surjective Kummer étale maps ϕ_i for which $\bigcup_i (V_i) = V$. The construction extends the Kummer topology to stably toroidal varieties.

The verification of the conditions for the covering in $X_{\text{két}}$ for the stably toroidal X reduces, by Lemma 4.6.26, to the toroidal case and is is streightforward:

Lemma 4.7.3. *(1) The identity $V \to V$ defines a covering.*

(2) If $\{\phi_i : V_i \to V\}$ is a covering, and $\{\phi_{ji} : V_{ji} \to V_i\}$ are covering then the induced family $\{V_{ij} \to V\}$ is a covering.

(3) If $V' \to V$ is a map and $\{\phi_i : V_i \to V\}$ is a covering then $V' \times_V V_i \to V'$ is a covering.

Proof. (1), and (2) follow from definition. To prove (3) We use the fact that the products of Kummer étale maps is Kummer étale . By Proposition 4.6.26 we can locally reduce the situation to the log-smooth maps of toroidal varieties. The surjectivity of the induced coverings follows from the analogous result of Nakayama [Nak97],[27, 2.2.2] for fine and saturated logarithmic structures, see also [Niz]. ♣

The lemma implies that the Kummer étale site on a stably toroidal variety defines the Kummer topology denoted as in the toroidal situation by $X_{\text{két}}$. The toroidal charts on X, so the Kummer étale maps $(Y, U_Y) \to (X, U)$ from toroidal varieties form a refinement X_{tor} of $X_{\text{két}}$ defining the same topology on X.

4.7.4. *Ideals on Kummer étale sites.* The presheaf $\mathcal{O}_{X_{\text{két}}}$ or $\mathcal{O}_{X_{\text{tor}}}$, is, in fact, a sheaf of rings. This can be verified locally, and it follows from the analogous result for the Kummer site over a toroidal variety. Indeed, locally, by Proposition 4.6.26, the stably toroidal varieties are toroidal with respect to the enlarged toroidal structure (replacing in étale charts the doubletons (X_σ, X_τ) by (X_σ, T)), and the normally log smooth maps of stably toroidal varieties can be viewed now as log smooth maps of toroidal varieties. (See Section 3.8.1)

We call a coherent finitely generated sheaves of ideals $\mathcal{I}$ of $\mathcal{O}_{X_{\text{két}}}$ *Kummer ideals* of X. Kummer-locally these are usual coherent ideals.

As before, we can consider ordinary ideals which are generated by an ideal $\mathcal{I} \subseteq \mathcal{O}_X$ and defined by assigning $\mathcal{I}_{X'} := \mathcal{I}\mathcal{O}_{X'}$ to objects X' of $X_{\text{két}}$.

4.7.5. *Kummer domains of the Kummer ideals.* As in the toroidal case for any Kummer ideal $\mathcal{I}$ on $X_{\text{két}}$ we shall consider the site $X_{\text{tor}}(\mathcal{I})$ where the local generators of the Kummer ideal sheaf $\mathcal{I}$ on $X_{\text{két}}$ are defined, and thus the ideals on the charts are compatible for the maps between the charts. (see Section 3.8.5).

5. Torification and Kummer covers

5.1. **Kummer covers.** Kummer covers is a particulary useful class of Kummer étale maps. They come equipped with the natural group actions of finite diagonalizable groups.

Kummer covers

5.1.1. *Groups of characters and Cartier dual.* Let K be a base field. Denote by $\overline{K}^*$ its algebraic closure. Let G be a finitely generated abelian group such that the order of the torsion part is relatively prime to $char(K)$. Recall that the group of $\overline{K}^*$-characters $Hom(G, \overline{K}^*)$ can be identified with the set of the closed points on the algebaric group $\operatorname{Spec} \overline{K}[G]$ and is known as Cartier dual.

Observe that $\overline{K}^*$ contains the group μ of all the roots of unity, and its subgroup μ^0 of the m-roots of units, with m relatively prime to $char(K)$. Te group μ^0 can be naturally embedded as a subgroup of the injective group $\mathbb{Q}/\mathbb{Z}$ in the category of abelian groups), and such that $Hom(G, \overline{K}^*) = Hom(G, \mu^0) = Hom(G, \mathbb{Q}/\mathbb{Z})$ for any finite group. In general, the subgroup $\mu^0 \subset \overline{K}^*$ is Zariski dense , and $Hom(G, \mu^0) \subset Hom(G, \overline{K}^*)$ is dense in $\operatorname{Spec} \overline{K}[G]$. On the other hand if G is finite of order n then $Hom(G, \mu) = Hom(G, \mu_n)$. Similarly, when passing to the relevant separable extension K' of K generated by μ_n we have that $Hom(G, \mu) = Hom(G, \mu_n) = Hom(G, K')$.

Thus, any the sequence

$$0 \to G_0 \to G_1 \to G_2 \to 0,$$

of finitely generated abelian groups such that the torsion part of G_2 is of order relatively prime to $char(K)$, defines the the exact sequence

$$0 \to Hom(G_2, \mu^0) \to Hom(G_1, \mu^0) \to Hom(G_0, \mu^0) \to 0$$

which determnies the sequences of the group of $\overline{K}^*$ characters. In fact, let $\phi : G_0 \to \mu^0$ be any group homomorphism, and $\psi : G_1 \to G_2$ is a given surjection. Write

$$G_2 = \bigoplus_{i=1,\ldots,k} \mathbb{Z}_{n_i} \cdot e_i \oplus \bigoplus_{i=k+1,\ldots,s} \mathbb{Z} \cdot e_i,$$

and choose $b_i \in G_1$ such that $\psi(b_i) = e_i$. We define the extension $\overline{\phi} : G_1 \to \mu^0$ of ϕ, by letting $\overline{\phi}(b_i) = (1/n_i) \cdot \phi(n_i b_i)$ for $i \leq k$, and $n_i b_i \in G_0$, and $\overline{\phi}(b_i) = 0$ for $i \geq k+1$.

$$0 \to Hom(G_2, \overline{K}^*) \to Hom(G_1, \overline{K}^*) \to Hom(G_0, \overline{K}^*) \to 0 \quad \text{and}$$

$$0 \to \operatorname{Spec} \overline{K}[G_2] \to \operatorname{Spec} \overline{K}[G_1]) \to \operatorname{Spec} \overline{K}[G_0]) \to 0$$

This shows that $Hom(G_1, \mu^0) \to Hom(G_0, \mu^0)$ is surjective.

Since all the homomorphisms are defined over K we get the exact sequence in the category of the group schemes over K:

$$0 \to \operatorname{Spec} K[G_2] \to \operatorname{Spec} K[G_1]) \to \operatorname{Spec} K[G_0]) \to 0$$

For a finitely generated abelian group G by the Cartier dual we mean the group scheme

$$G^\vee_{K^*} := \operatorname{Spec}(K[G]).$$

In principle, if G is finite then passing to a separable field extension K' of K containing the group $\mu_{|G|}$, the group scheme $G^\vee_{K^*}$ becomes a finite algebraic group which can be identified with the group of K^*-characters $G^\vee_{K^*} := Hom(G, K^*) = Hom(G, \mu_G)$, which are exactly the K-points of $\operatorname{Spec}(K[G])$. In particular $G^\vee_{\overline{K}^*} := Hom(G, \overline{K}^*) = G^\vee_{K^*}$ for G finite and the relavent extension of K.

Note that the constructions of the Kummer covers and torific ideals can be done directly over the field K if we consider the co-action of the relevant group schemes. We prefer to avoid the language of co-action and stick to the actions of finite groups when passing to the relevant separable field extensions. Note that the latter defines an étale cover so won't affect our reasoning.

5.1.2. *Kummer covers.*

Definition 5.1.3. Let Y be an affine stably toroidal variety, with a rational chart $P_1 \to \mathcal{M}(Y)$ and a Kummer inclusion $P_1 \subset P_2$ of the fine and saturated monoids.We define the *normal Kummer cover* $Y[P_2 : P_1]^n$ to be the normalization of

$$Y[P_2 : P_1] := Y \times_{\operatorname{Spec} K[P_1]} K[P_2],$$

Remark 5.1.4. There are analogous notation of this in [TV14] and [Ols03].

5.1.5. *Group actions on Kummer covers.* In the sequel we shall assume that all the Kummer maps of the groupifications of the monoids associated with a log scheme Y, have cokernel which is inverible on Y.

Lemma 5.1.6. *(1) For any fs monoid P there is a natural group action of the diagonalizable group: $(P^{gp})^\vee_{\overline{K}^*} = Hom(P^{gp}, \overline{K^*})$ on $\mathrm{Spec}(\overline{K}[P])$: For $\alpha \in (P^{gp})^\vee$, and $m \in P$ we consider the multiplication by the character $\alpha(m) \cdot m$.*

(2) If $P_1 \subset P$ is a Kummer map of the fs monoids then (by duality) after passing to a finite separable extension of K the finite group $(P^{gp}/P_1^{gp})^\vee_{\overline{K}^*} = (P^{gp}/P_1^{gp})^\vee_{K^*}$ is a subgroup of $(P^{gp})^\vee_{\overline{K}^*}$, acting on $\mathrm{Spec}(\overline{K}[P])$ and on $\mathrm{Spec}(K[P])$, and the natural map

$$\mathrm{Spec}(K[P]) \to \mathrm{Spec}(K[P_1])$$

is the geometric quotient by the group $(P^{gp}/P_1^{gp})^\vee_{K^*}$, where $P_1 = P_1^{gp} \cap P$.

In the sequel we simplify notation and write $G^\vee$ for $G^\vee_{K^*}$.

Proof. (1) Follows from definition. (2) $(P^{gp}/P^{gp})^\vee_{\overline{K}^*}$ consists of the homomorphisms $P^{gp})^\vee_{\overline{K}^*} := Hom(P^{gp}, \overline{K}^*)$ which are trivial on P_1^{gp}. Consequently $K[P]^{(P^{gp}/P^{gp})^\vee} = K[P_1]$.

♣

Definition 5.1.7. Let Y be a normal log smooth scheme. We shall call an fs monoid P a *Kummer chart* if there is a rational chart $P_0 \subset \mathcal{M}(Y)$, and $P_0 \subset P$ a Kummer map. Then, by $P(Y)$ we mean the maximal submonoid of P, extending to a rational chart $P(Y) \to \mathcal{M}(Y)$.

`reducible` **Lemma 5.1.8.** *Assume Y is a normal log scheme with a rational chart $P_1 \to \mathcal{M}(Y)$, and let $P_1 \subseteq P_2$ be a Kummer inclusion. There is a natural action of*

$$\Gamma_{P_2:P_1} := (P_2^{gp}/(P_1^{gp}))^\vee \qquad on$$

$$Y[P_2 : P_1] := Y \times_{\mathrm{Spec}(K[P_1]} \mathrm{Spec}(K[P_2]$$

and on its normalization $Y[P_2 : P_1]^n$. Moreover,

1. *$Y[P_2 : P_1]^n$ is irreducible iff $P_1 = P_2(Y)$.*
2. *$Y[P_2 : P_1]^n$ contains $|P_2(Y)^{gp} : P_1^{gp}|$ irreducible components isomorphic to $Y[P_2]^{\mathrm{nor}}$.*
3. *The subgroup $\Gamma_{P_2:P_2(Y)} = (P_2^{gp}/P_2(Y)^{gp})^\vee$ of $\Gamma_{P_2:P_1}$ acts on each irreducible component.*
4. *The quotient group $\Gamma_{P_2(Y):P_1} = \Gamma_{P_2):P_1}/\Gamma_{P_2:P_2(Y)}$ permutes components of $Y[P_2 : P_1]^n$*

Proof. The group $\Gamma_{P_2:P_1}$ acts naturally on $\mathrm{Spec}(K[P_2])$ and trivially on Y and $\mathrm{Spec}(K[P_1])$. This induces the action on $Y[P_2 : P_1]$ and its normalization. (1) Suppose $Y[P_2 : P_1]^n$ is reducible. Then $\Gamma_{P_2:P_1}$ permutes the set of the components, and let $\Gamma_0 \subset \Gamma_{P_2:P_1}$, be the kernel of this action. Let P_0 be the corresponding submonoid of P_2 containing P_1 and fixed by Γ_0. Then the group $\Gamma_{P_0:P_1} \simeq \Gamma_{P_2:P_1}/\Gamma_{P_2:P_0}$ acts on $Y[P_0]^n = Y[P_2]^{\mathrm{nor}}/\Gamma_0$ permuting the components, and $Y[P_0]^n \to Y[P_1]^n$ is a group quotient by the action of $\Gamma_{P_0:P_1}$, with isomorphic components which can be identified with Y. Thus elements of $P_0 \supseteq P_2$ are already regular on Y, so

$P_0 \subseteq P_2(Y)$. On the othe hand $P_0 \supsetneq P_1$, since $Y[P_1]^{\rm nor}$ is irreducible and $Y[P_0]^{\rm nor}$ is not. Thus $P_1 \neq P_2(Y)$. Conversely, if $P_1 \subsetneq P_0 := P_2(Y)$ then $Y[P_2 : P_1]^n \to Y$ factors through a not isomorphism $Y[P_0 : P_1]^n \to Y$. The latter admits the right inverse $Y \to Y[P_0 : P_1]^n$ since there is a map $P_0 \subset \mathcal{O}(Y)$ and thus contains a component isomorphic to Y, while being not isomorphic to Y.

(2), (3), (4) follow from the above. ♣

pure2 **Lemma 5.1.9.** *Let $P_1 \to \mathcal{M}(Y)$ be a rational chart. For any Kummer inclusions of monoids $P_1 \subseteq P_2 \subseteq P_3$.*

(1) $P_i \to \mathcal{M}(Y[P_i : P_1]^{\rm nor})$, are rational charts for $i = 2, 3$.
(2) $(Y[P_2 : P_1]^n)[P_3 : P_2]^n = Y[P_3 : P_1]^n$.
(3) If $P_3(Y) = P_2(Y) = P_1$ we have then $P_2 = P_3(Y[P_2]^{\rm nor})$.

Proof. (1), and (2) follow from definition and Corollary 4.6.18. (3) Let $P_2' := P_3(Y[P_2]^{\rm nor})$. Then $Y[P_2' : P_1]^n = (Y[P_2 : P_1]^n)[P_2' : P_2]^n$ is irreducible by the previous lemma. This implies that $P_2' = P_2$.

♣

5.1.10. *Kummer cover defined by monoids.*

Definition 5.1.11. Let Y be a normal log smooth scheme, and $P \supset P(Y) \to \mathcal{M}(Y)$ be a Kummer chart. Then by the *Kummer cover defined by P* we mean $Y[P]^{\rm nor} := Y[P : P(Y)]^{\rm nor}$,. Then the associated group acting on $Y[P]^{\rm nor}$ will be denoted by $\Gamma_P := \Gamma_{P:P(Y)}$.

Lemma 5.1.12. *(1) $Y[P]^{\rm nor}$ is irreducible*
(2) $Y = Y[P]/\Gamma_P = Y[P]^{\rm nor}/\Gamma_P = Y$.

Proof. (1) Follows from Lemma 5.1.8. The action of Γ_P defines the quotients $Y = Y[P]/\Gamma_P$ and its normalization $Y[P_2]^n/\Gamma_P$. Since Y is normal we get the equality. ♣

5.1.13. *Sharp and étale Kummer covers.*

Definition 5.1.14. We say that $Y[P]^{\rm nor}$ is a *sharp Kummer cover* of Y if $P^* = P(Y)^*$. We say that $Y[P]^{\rm nor}$ is an *étale Kummer cover* of Y if $P = P(Y)P^*$.

sharp **Lemma 5.1.15.** *(1) The sharp Kummer covers are exactly those which can be generated by the sharp monoids.*
(2) The étale covers $Y[P]^{\rm nor} = Y[P^*] \to Y$ are defined by the finite étale morphisms.
(3) Any Kummer cover $Y[P]^{\rm nor}$ can be written noncanonically as $Y[P]^{\rm nor} \to Y[P/P^*]^{\rm nor} \to Y$, where $Y[P]^{\rm nor} \to Y[P/P^*]^{\rm nor}$ is étale and $Y[P]^{\rm nor} = (Y[P^*])[P/P^*]^{\rm nor}$.

Proof. (1) We can write noncanonincally

$$P = \overline{P} \oplus P^* = \overline{P} \oplus P(Y)^*,$$

and thus it can be represented as $Y[P] = Y[\overline{P}]$, for the sharp monoid $\overline{P} = P/P^*$.

(2) Since P is the push-out of $P^* \leftarrow P_Y^* \subset P_Y$, we get $Y[P] = Y[P^*] \to Y$ is étale as it is generated bythe étale map $\mathrm{Spec}(K[P^*]) \to \mathrm{Spec}(K[P_Y^*])$. So $Y[P] = Y[P^*] = Y[P]^{\rm nor}$ is already normal.

(3) Using (1) the map $Y[P]^{\rm nor} \to Y[P/P^*]^{\rm nor} = Y[\overline{P} \oplus P_Y^*]^{\rm nor}$ is étale cover corresponding to the Kummer étale extension $\overline{P} \oplus P_Y^* \subset P = \overline{P} \oplus P^*$. This implies the second part. ♣

5.1.16. *Optimized charts.*

Definition 5.1.17. Let Y be a log scheme with rational structure. A rational chart $i : P_Y \to \mathcal{M}(Y)$ is *saturated* in $\mathcal{M}(Y)$ if it is injective and whenever $a^n \in i(P_Y)$ with $a \in \mathcal{M}(Y)$, and $n \in \mathbb{N}$ then $a \in i(P_Y)$. It is is *localized* in $\mathcal{M}(Y)$ if $P_Y^* = i^{-1}(\mathcal{O}(Y)^*)$. By the *localization* of P we mean the induced map from $P_Y^{loc} := P(i^{-1}(\mathcal{O}(Y)^*))^{-1}$.

A localized and saturated rational chart $P_Y \to \mathcal{M}(Y)$ is called *optimized* on Y if there exists a point $y \in Y$

$$P_Y/P_Y^* \simeq (\mathcal{M}/\mathcal{O}^*)_{Y,y} = \mathcal{M}(Y)/\mathcal{O}(Y)^*$$

The point $y \in Y$ will be called *an optimization point* for P_Y. If $x \in X$ then an étale neighborhood Y of x, togeteher with a rational chart $P_Y \to \mathcal{M}(Y)$ optimized at a point $y \in Y$ over $x \in X$ will be called an *optimized neighborhood.*

We shall call an a Kummer chart P *optimized* if there exists an optimized chart $P_Y \to \mathcal{M}(Y)$ such that $P_Y \subset P$ is Kummer.

Lemma 5.1.18. *Let $P_Y \to \mathcal{M}(Y)$ be a saturated chart. Then for any Kummer extension $P_Y \subset P$ we have $P_Y = P(Y)$.*

Proof. By definition $P_Y \subseteq P(Y)$ is Kummer and saturated so $P_Y = P(Y)$. ♣

5.1.19. *Canonical groups actions.* The actions of groups Γ_P naturally occur when considering the Kummer covers $Y[P]^n$. It is convenient to introduce the *sharp subgroup* corresponding to the sharp extension part in Lemma 5.1.15

$$G_P := (P^{gp}/(P^* P^{gp}(Y)))^\vee \subseteq \Gamma_P.$$

Remark 5.1.20. If $Y[P]^{\rm nor}$ is a sharp Kummer cover then $G_P = \Gamma_P$. It is easy to see that if Y is a stably toroidal variety and $P(Y)$ is an optimized toric chart on Y then G_P is the stabilizer $(\Gamma_P)_y$ at the optimization point $y \in Y$.

pure **Lemma 5.1.21.** *Any Kummer cover $Y[P]^{\rm nor} \to Y$ can be represented as the composition of a sharp Kummer cover and an étale Kummer cover. The sharp subgroup G_P corresponds to the sharp cover in this presentation.*

$$Y[P]^{\rm nor} \to Y_1 := Y[P]^{\rm nor}/G_P \to Y[P]^{\rm nor}/\Gamma_P = Y,$$

with the second morphism $Y_1 \to Y$ is an étale Kummer cover, and

$$Y[P]^{\rm nor} = Y_1[P]^{\rm nor}$$

is a sharp extension over Y_1 , with $\Gamma_P = G_P$.

Proof. Note that $Y_1 = Y[P]^{\rm nor}/G_P = Y[P^*P(Y)]^{\rm nor}$. The map $Y[P^*P(Y)] = Y[P^*] \to Y$ is an étale cover. So, since Y is normal $Y[P^*P(Y)]$ is normal as well and thus equal to $Y_1 = Y[P]^{\rm nor}/G_P = Y[P^*P(Y)]^{\rm nor}$. By Lemma 5.1.9, $Y[P]^{\rm nor} = Y_1[P]^{\rm nor}$. Moreover there exists a natural G_P-equivariant morphism

$$Y[P]^{\rm nor} = Y_1[P]^{\rm nor} = (Y[P]^{\rm nor}/G_P) \times_{{\rm Spec}(K[P^{G_P}])} {\rm Spec}(K[P]) \to Y_1,$$

defined by the sharp extension $Y_1[P]^{\rm nor}$. ♣

Lemma 5.1.22. *Let $Z = Y[P]^{\rm nor} \to Y$ be a Kummer cover. Then the groups G_P, and Γ_P and their actions on Z are canonically defined on $Y[P]^{\rm nor}$ for the optimized P. In particular, they are independent of the choice of the (optimized) P.*

Proof. We can assume that $P_1(Y) \subset P_2(Y)$ are optimized charts defined for two optimized Kummer charts P_1 , and P_2 such that $Y[P_1]^{\rm nor} = Y[P_2]^{\rm nor}$. Then comparing at the optimized point $y \in Y$ for $P_1(Y)$ we see that $P_1(Y)/P_1(Y)^* = P_2(Y)/P_2(Y)^* = (\mathcal{M}/\mathcal{O}^*)_{y,Y}$ and consequently $P_1(Y)P_2(Y)^* = P_2(Y)$. Then consider $P_2' := P_1P_2(Y)^* \subset P_2$, which is the push out of $P_1 \leftarrow P_1(Y) \to P_1(Y)P_2(Y)^* = P_2(Y)$. So $P_2'(Y) \supseteq P_1(Y)P_2(Y)^* = P_2(Y)$, which implies $P_2'(Y) = P_2(Y)$. By the push out argument and Lemma 5.1.8, $Y[P_1]^{\rm nor} = Y[P_2']^{\rm nor} = Y[P_2]^{\rm nor}$ is irreducible and $P_2' \subset P_2$ Kummer so $P_2 = P_2' = P_1P_2(Y)^*$, and $P_2^* = P_2(Y)^*P_1^*$. Consequently

$$\Gamma^\vee_{P_2} = P_2/P_2(Y) = (P_1P_2(Y)^*)/(P_1(Y)P_2(Y)^*) =$$

$$(P_1/(P_1 \cap (P_1(Y)P_2(Y)^*)) = P_1/P_1(Y) = \Gamma^\vee_{P_1}.$$

$$G^\vee_{P_2} = P_2/(P_2(Y)P_2^*) = (P_1P_2(Y)^*)/(P_1(Y)P_1^*P_2(Y)^*) =$$

$$P_1/(P_1 \cap (P_1(Y)P_1^*P_2(Y)^*) = P_1/(P_1(Y)P_1^*) = G^\vee_{P_1}.$$

♣

5.1.23. *Comparison of two optimized charts.*

comparison **Lemma 5.1.24.** *Let $P_{Y_1} \to \mathcal{M}(Y)$, $P_{Y_2} \to \mathcal{M}(Y)$ be two sharp optimized charts. Consider the natural isomorphisms $P_{Y_1} = P_{Y_1}/P^*_{Y_1} \simeq P_{Y_2}/P^*_{Y_2} = P_{Y_2}$.*

Let $P_Y \to \mathcal{M}(Y)$ be an optimized chart containing both P_{Y_1}, P_{Y_2}. Let $P_{Y_1} \subseteq P_1$ be a Kummer extension, and P_2 be the push-out of $P_1 \supset P_{Y_1} \simeq P_{Y_2}/P^*_{Y_2} \to P_{Y_2}$. Then there exists a Kummer cover $P \supseteq P_Y$, containing P_1, and P_2 such that

1. *There are natural isomorphisms $i : P_1 = P_1/P_1^* \to P/P^* \leftarrow P_2/P_2^*$.*
2. *There are natural group isomorphisms $G_{P_1} \leftarrow G_P \to G_{P_2}$.*
3. *$Y[P]^{\rm nor} = (Y[P_1]^{\rm nor})[P^*] = (Y[P_2]^{\rm nor})[P^*]$.*
4. *$Y[P]^{\rm nor} \to Y[P_1]^{\rm nor}$, and $Y[P]^{\rm nor} \to Y[P_2]^{\rm nor}$ are strongly G_P-étale.*
5. *The group $H := (P^*/P^*_Y)^\vee$ admits two injections $\phi_i : H \to \Gamma_P$, $i = 1, 2$ such that $Y[P_1]^{\rm nor} = Y[P]^{\rm nor}/H$, $Y[P_2]^{\rm nor} = Y[P]^{\rm nor}/H$, and $Y[P]^{\rm nor} \to Y[P_i]^{\rm nor}$ are étale.*

Proof. By definition, $P_{Y_i}/P^*_{Y_i} \to P_Y/P^*_Y$ are isomorphisms so $P_Y = P^*_YP_{Y_1} = P^*_YP_{Y_2}$. The maps $\phi_i : P_{Y_i} \to P_Y/P^*_Y$ defined by the inclusions coincide under the identification $\alpha : P_{Y_1} \to P_{Y_2}$, and thus define the map $\psi : P_{Y_1} \to P^*_Y$, $a \mapsto \phi_1(a) \cdot (\phi_2\alpha(a))^{-1} \in P^*_Y$. Let P^* be the push-out of $P_1 \leftarrow P_{Y_1} \to P^*_Y$. Consider the induced maps $\overline{\psi} : P_1 \to P^* \subset P_1P^*$, $\overline{\phi}_i : P_i \subseteq P_1P^*$, $\overline{\alpha} : P_1 \to P_2$, with $\overline{\phi}_1 \cdot \overline{\psi}^{-1} = \overline{\phi}_2\overline{\alpha} : P_1 \to P_1P^*$, and $\overline{\alpha}(P_1) = P_2$. Thus $\overline{\phi}_2 : P_2 \subset P_1P^*$ is a natural inclusion which is a lifting $P_{Y_2} \to P_Y$, so $P_1P^* = P_2P^*$.

One can represent $Y[P_i]^{\rm nor}$ as $Y[P_i]^{\rm nor} = Y[P_i']^{\rm nor}$, where $P_i' = P_i(P_Y)^*$ is the push out of $P_Y \leftarrow P_{Y_i} \to P_i$. Let $P := P_1'P_2' = P_1P^* = P_2P^*$ be the monoid generated by P_1', and P_2'. Then $P_i/P_i^* \to P/P^*$ is an isomorphism and both Kummer extensions $P_i' \subset P$ are étale with $Y[P]^{\rm nor} = (Y[P_i']^{\rm nor})[P^*]^{\rm nor} = (Y[P_i']^{\rm nor})[P^*]$, since $(Y[P_i']^{\rm nor})[P^*] \to Y[P_i']^{\rm nor}$ is étale and $Y[P_i']^{\rm nor})[P^*]$ is already normal. The group G_P is the same for $Y[P_i]^{\rm nor} = Y[P_i']^{\rm nor}$ for $i = 1, 2$, and $Y[P]^{\rm nor}$, and it acts trivially on P^*. Consequently the natural maps $Y[P]^{\rm nor} \to Y[P_i']^{\rm nor} = Y[P_i]^{\rm nor}$ are strongly étale G_P-equivariant, as they are induced by Kummer extensions generated by G_P-invariant functions P_i^*.

For the remaining part the two inclusions:

$$H = (P^*/P_Y^*)^\vee \simeq (P^* P_i^{gp}/(P_Y^* P_i^{gp}))^\vee = (P^{gp}/P_i^{gp} P_Y^*)^\vee \hookrightarrow (P^{gp}/P_Y^*)^\vee = \Gamma_P$$

correspond to two Kummer étale covers

$$Y[P] = (Y[P_i])[P] = (Y[P_i P_Y^*])[P]^{\mathrm{nor}} = Y[P : P_i P_Y^*]^{\mathrm{nor}}.$$

♣

5.2. Torific ideals and descent. [6]

←6

5.2.1. *Toroidal actions on log-smooth varieties.*

Definition 5.2.2. We shall call (Y, U) stably toroidal variety *torically optimized* if it has a toric rational optimized chart $P_Y \to \mathcal{M}(Y)$.

Definition 5.2.3. Let (Y, U) be a stably toroidal variety with an optimized rational chart $P_Y \to \mathcal{M}(Y)$, and assume a Kummer cover $Y[P]^{\mathrm{nor}} \in X_{\mathrm{tor}}$ of Y is toroidal with the (ordinary) chart $P \to \mathcal{M}(Y[P]^{\mathrm{nor}})$. We say that that the action of a subgroup $G \subseteq G_P$ on $Y[P]^{\mathrm{nor}}$ is *toroidal* if the action of G on $Y[P]^{\mathrm{nor}}$ is induced by the action G on $\mathrm{Spec}(K[P])$ via the morphism $Y[P]^{\mathrm{nor}} \to \mathrm{Spec}(K[P])$ that is $Y[P]^{\mathrm{nor}} = (Y[P]^{\mathrm{nor}}/G)[P]$ (without normalization!).

Remark 5.2.4. Such an action is automatically *relatively affine*, faithful and simple, and thus it coincides with the standard definition.

comparison2

Lemma 5.2.5. *Let P_{Y_1}, P_{Y_2}, P_Y, P_1, P_2, P be as in Lemma 5.1.24. Then $Y[P_1]^{\mathrm{nor}}$ is toroidal with the ordinary chart $P_1 \to \mathcal{M}(Y[P_1]^{\mathrm{nor}})$ if and only if $Y[P_2]^{\mathrm{nor}}$ is toroidal with the ordinary chart $P_2 \to \mathcal{M}(Y[P_2]^{\mathrm{nor}})$.*

Moreover, a subgroup $G \subset G_P$ acts toroidally on toroidal $Y[P_1]^{\mathrm{nor}}$ iff it acts toroidally on $Y[P_2]^{\mathrm{nor}}$ via the isomorphism $G_{P_1} \simeq G_{P_2} \simeq G_P$.

Proof. The first part follows from Lemma 5.1.24 (3) and (5). Assume $Y[P_1]^{\mathrm{nor}}$ is toroidal. By definition $G \subset G_P$ acts toroidally on $Y[P_1]^{\mathrm{nor}}$ iff $(Y[P_1^G]^{\mathrm{nor}})[P_1] = Y[P_1]^{\mathrm{nor}}$.

$$Y[P^G]^{\mathrm{nor}} = Y[(P_i \cdot P^*)^G]^{\mathrm{nor}} = Y[P_i^G \cdot P^*]^{\mathrm{nor}} = (Y[P_i^G]^{\mathrm{nor}})[P^*]$$

This implies that

$$(Y[P^G]^{\mathrm{nor}})[P] = (Y[P_1^G P^*]^{\mathrm{nor}})[P^* P_1][P] = (Y[P_1^G]^{\mathrm{nor}})[P_1][P^*] = Y[P_1]^{\mathrm{nor}})[P^*] = Y[P]^{\mathrm{nor}}$$

Since $H := (P^{gp}/P_i P_Y^*)^\vee$ acts trivially on $Y[P_i]^{\mathrm{nor}}$ we obtain

$$Y[P^G]^{\mathrm{nor}}/H = (Y[P_i^G]^{\mathrm{nor}})[P^*]/H = Y[P_i^G]^{\mathrm{nor}}$$

So we get

$$(Y[P^G])^{\mathrm{nor}})[P]/H = (Y[P_i^G]^{\mathrm{nor}}))[P_i][P^*]/H = (Y[P_i^G]^{\mathrm{nor}}))[P_i]$$

The above is equal to

$$Y[P]^{\mathrm{nor}}/H = Y[P_2 P^*]^{\mathrm{nor}}/H = (Y[P_2]^{\mathrm{nor}})[P^*]/H = Y[P_2]^{\mathrm{nor}}$$

♣

[6] (Dan) Comparison with torification paper

toroid **Lemma 5.2.6.** *Let Y be torically optimized. Assume $Y[P]^{\rm nor}$ is toroidal with the ordinary chart $P \to \mathcal{M}(Y[P]^{\rm nor})$. Then Y is toroidal iff G_P acts toroidally on $Y[P]^{\rm nor}$. In general, a subgroup $G \subset G_P$ acts toroidally on $Y[P]^{\rm nor}$ iff $Y[P]^{\rm nor}/G$ is toroidal.*

Proof. By Lemmas 5.2.5, 5.1.24 we can reduce the resoning to the case $P_Y = P^t_Y \subset \mathcal{M}(Y)$ is a sharp toric optimized chart, and $P = P^t$. This reduces the situation to a toric doubleton $Y = X_\sigma$, and $P_Y := P_{\delta,N'_\delta}$ so we can use Lemma notation from Lemma 4.2.4, and Lemma 4.3.1. Then the Kummer cover $Y[P]^{\rm nor}$ corresponds to the Kummer inclusion $N^1 \subset N^0_\delta$, and $Y[P]^{\rm nor} = Y[P_{\delta,N_1}]^{\rm nor}$ is toroidal if it is a regular doubleton $Y[P]^{\rm nor} = X_{\delta,N_1} \times X_\tau$. The action of $G \subset G_P$ is toroidal on $Y[P]^{\rm nor}$ if it is induced on $X_{\delta,N_1} = \operatorname{Spec}(K[P_{\delta,N_1}])$. This induces the isomorphism

$$(Y[P]^{\rm nor}/G = (X_{\delta,N_1} \times X_{\tau,N_\tau})/G \simeq X_{\delta,N_1}/G \times X_{\tau,N_\tau},$$

and shows that $Y[P]^{\rm nor}/G$ is toroidal (a regualar doubleton). Conversely if $Y[P]^{\rm nor}/G = (X_{\delta,N_1} \times X_{\tau,N_\tau})/G = Y[P^G]^{\rm nor}$ is toroial then it is a regular doubleton of the form $X_{\delta,N_2} \times X_{\tau,N_\tau}$, where $N_1 \subset N_2 \subset N^0_\delta$ and $X_{\delta,N_2} = X_\delta/G$.

Consequently $Y = Y[P]^{\rm nor}/G_P$ is toroidal G_P acts toroidally on $Y[P]^{\rm nor}$.

♣

As an immediate consequence of the prevous reasoning we obtain:

Corollary 5.2.7. *Let (Y,U) be a stably toroidal variety with a sharp toric rational chart $P^t_Y \to \mathcal{M}(Y)$, and assume a Kummer cover $Y[P^t]^{\rm nor}$ of Y is toroidal with the toric chart $P^t \to \mathcal{M}(Y[P^t]^{\rm nor})$. The action of a subgroup $G \subseteq G_{P^t}$ on $Y[P^t]^{\rm nor}$ is toroidal if the action of G on $Y[P^t]^{\rm nor}$ is induced by the action G on* $\operatorname{Spec}(K[P^t])$ *via the strongly G-smooth morphism $Y[P^t]^{\rm nor} \to$* $\operatorname{Spec}(K[P^t])$.

5.2.8. *Minimal Kummer covers of toric doubletons.*

minimal doubleton **Lemma 5.2.9.** *Let $(Y,U) = (X_\sigma, X_\tau) \to (X_{\delta,N^0_\delta}, T)$ be a toric doubleton with the induced chart $P_\delta = P_Y \subset \mathcal{M}(Y)$. Then there is a unique minimal Kummer toroidal cover $Y[P]^{\rm nor}$ which is defined by the lattice extension $N_\delta \subseteq N^0_\delta$.*

Proof. The Kummer extension of the lattices $N_\delta \subseteq N^0_\delta$ induces the Kummer map $N_\delta \times N_\tau = N_\sigma \times_{N^0_\delta} N_\delta \to N_\sigma$ corresponding to a regular doubleton. Converesely let $N^1_\delta \subseteq N^0_\delta$ be a Kummer extension which defines a regular doubleton for $N^1_\sigma = N_\sigma \times_{N^0_\delta} N^1_\delta$. Thus we get $N^1_\sigma = N^1_\delta \oplus N_\tau$, so $N^1_\delta \subset N^1_\sigma$. Consequently the inclusion $N^1_\sigma \subseteq N_\sigma$ determines its restriction $N^1_\delta \subseteq N_\delta \subset N^0_\delta$. Dualizing we see that P_δ defines the unique minimal toroidal Kummer cover.

♣

5.2.10. *Minimal toroidal Kummer covers.*

Definition 5.2.11. By a minimal toroidal Kummer cover $Y[P]^{\rm nor}$ we mean a toroidal Kummer cover $Y[P]^{\rm nor}$ with no nontrivial subgroup G of G_P acting toroidally

minimal2 **Corollary 5.2.12.** *(1) Given a sharp rational optimized chart $P_Y \subset \mathcal{M}(Y)$ on a torically optimized stably toroidal Y, there exists a unique sharp minimal toroidal Kummer cover $Y[P_{min}]^{\rm nor} \in X_{\rm tor}$, with the minimal Kummer extension $P_{min} \supseteq P_Y$.*

(2) *For any minimal toroidal Kummer cover* $Y[P]^{\rm nor}$ *there is an étale* G_P*-equivariant map to a minimal sharp toroidal Kummer cover* $Y[P]^{\rm nor} \to Y[P_{\rm min}]^{\rm nor}$ *with* $G_P = G_{P_{\rm min}}$

(3) *For any two rational optimized charts* $P_{1Y}, P_{2Y} \to \mathcal{M}(Y)$*, there exists two étale finite maps* $f_i : Y' \to Y$ *such that* $Y'[P_{1min}] = Y'[P_{2min}]$ *and thus there is a natural isomorphism of* $G^\vee_{P_{2min}} \to G^\vee_{P_{2min}}$.

Proof. First, consider an optimized toric chart $P^t_Y \subset \mathcal{M}(Y)$. It follows from the previous Lemma that there is a unique minimal (sharp) Kummer cover $Y[P^t_{min}]^{\rm nor}$. By Lemma 5.2.5 it corresponds to a unique sharp Kummer extension $Y[P_{\rm min}]^{\rm nor}$ with $P_Y \subseteq P_{\rm min}$ Kummer and sharp. The second part follows from Lemma 5.1.15(3). To prove the third part we replace both charts with the sharp ones by Lemma 5.1.15(3). Then we use Lemma 5.1.24. General case follows by taking the relevant étale extensions, determined by factorization as in Lemma 5.1.21. ♣

Corollary 5.2.13. *Assume* $Y[P]^{\rm nor}$ *is toroidal with the ordinary chart* $P \to \mathcal{M}(Y)$*. Then there is a maximal subgroup* $G \subset G_P$*, acting toroidally on* P.

Proof. It is a direct consequence of Lemmas 5.2.6, 5.2.12

♣

5.2.14. *Torification functor.*

Definition 5.2.15. We shall call such a Kummer cover $Y[P_{\rm min}]^{\rm nor}$ a *minimal toroidal Kummer cover* for Y, and the Kummer extension $P_Y \subset P$ a minimal toroidal Kummer extension. We shall denote the canonical group associated with torically optimized étale neighborhood $Y \to X$ by $G_Y := G_{P_{\rm min}}$, and the canonical group of characters:

$$M^{Tor}(Y) := \frac{P^{gp}_{min}}{P^*_{min} P^{gp}_Y} = G^\vee_Y,$$

and the associated order $m(Y) := |M^{Tor}|$.

5.2.16. *Torification functor on toric doubletons.* Recall that for any cone $\sigma \in N_Q \supseteq N$, the lattice $N_\sigma = {\rm span}(\sigma) \cap N$, and the lattices $M = Hom(N, \mathbb{Z})$, $M_\sigma = Hom(N_\sigma, \mathbb{Z})$ are their dual. Denote by

$$M^0(\sigma) = \{m \in M \mid m_{|\sigma} = 0\}$$

the subgroup of the monomials invertible on the orbit $O(\sigma)$, (invertible on X_σ). Then

$$M_\sigma = M/M^0(\sigma).$$

induced **Lemma 5.2.17.** *Let* (X_σ, X_τ) *be a toric doubleton in* $N = N_\sigma \times N'$ *with the dual lattice* $M = M_\sigma \times M'$*. Let* $(X_\delta \times X_\tau, T \times X_\tau)$ *be the induced minimal regular doubleton over* (X_σ, X_τ)*, with the lattice* $N^1 := N_\delta \times N_\tau \times N' \subset N = N_\sigma \times N'$*, and the dual lattice* $M^1 := Hom(N^1, \mathbb{Z})$*,*

$$M^1 = M_\delta \times M_\tau \times M' \supseteq M = M_\sigma \times M'.$$

Then $M^{Tor}(X_\sigma, X_\tau) = M^1/M \simeq (M_\delta \times M_\tau)/M_\sigma$*. Moreover for any face* σ' *of* σ *the open subset* $X_{\sigma'}$ *is an optimized , neighborhood of any* $p \in O_{\sigma'}$*, with an optimized chart* $X_{\sigma'} \to X_{\delta'}$.

(1) *The face inclusion* $(\sigma', N) \subset (\sigma, N)$ *determines the inclusions* $N_{\delta'} \subset N_\delta$, *and* $N_{\tau'} \subset N_\tau$ *such that* $N_{\delta'} \times N_{\tau'} \subset N_\delta \times N_\tau$. *The latter induces the injective homomorphism:*
$$\Psi : N_{\sigma'}/N_{\delta'} \times N_{\tau'} \to N_\sigma/(N_\delta \times N_\tau)$$
(2) *Its dual is the surjective homomorphism*
$$f^{Tor} : M^{Tor}(X_\sigma, X_\tau) \to M^{Tor}(X_{\sigma'}, X_{\tau'})$$
(3) *Any element in* $Ker(f^{Tor})$ *is represented by an element* m *from* $\sigma^\vee \cap M^1 = (\delta \times \tau)^\vee \cap M^1$ *which is invertible on* $X_{\delta' \times \tau'}$.

(4) *Then* $(X_{(\delta' \times \tau', N^1)}, X_{(\tau', N^1)})$ *is a minimal toroidal Kummer cover (usually not sharp) for* $(X_{\sigma', N}, X_{\tau', N})$.

Proof. (1),(2) follow from definition.

(3) By the duality the elements $m \in M^1$ representing $[m] \in Ker(f^{Tor}) \subset M^1/M$ are exactly those which satisfy the condition $m_{|\sigma'} = 0$. Take $\overline{m} \in \sigma^\vee \cap M$, such that $\overline{m}_{|\sigma \setminus \sigma'} > 0$. Then $\overline{m}^n$, where $n = |M^{Tor}(X_\sigma, X_\tau)|$ defines a trivial character in both $M^{Tor}(X_\sigma, X_\tau)$ and $M^{Tor}(X_{\sigma'}, X_{\tau'})$. Consequently the character $[m]$ is represented also by $m_1 := m\overline{m}^{nk}$, for sufficiently large k so that $m_1 \in \sigma^\vee \cap M^1$, and $[m] = [m_1] \in Ker(f^{Tor})$.

(4) as in Lemma 5.2.9, the minimal Kummer covers are defined by the monoids $P_{\delta'}$. ♣

5.2.18. *α-Torific ideals.*

Definition 5.2.19. Consider a toroidal Kummer site $X_{\rm tor}$ on a stably toroidal X. Let $Y[P]^{\rm nor} \in X_{\rm tor}$, be a minimal toroidal cover of an optimized neighborhood $Y \in X_{\rm \acute{e}t}$. For any character $\alpha \in M^{\rm Tor}(Y)$ of the group G_P define the *α-torific ideal* $\mathcal{I}^{Tor}_{Y[P]^{\rm nor},\alpha}$ to be generated by the G_P-semi-invariant functions on $Y[P]^{\rm nor}$ where the action is given by the character α.

induced2 **Lemma 5.2.20.** *Let X be a stably toroidal variety. Let $f : Y_1 \to Y_2$ be étale morphism over X of $Y_1, Y_2 \in X_{\rm \acute{e}t}$ and let Y_2 be torically optimized, and $y \in Y_1$ be any point. Let $P_{Y_2} \to \mathcal{M}(Y_2)$ be an optimized toric chart. Then there exists a Zariski open neighborhood $Y'_1 \subset Y_1$ of $y \in Y_1$ with torically optimized chart for $P_{Y'_1} \to \mathcal{M}(Y'_1)$, with the induced map $P_{Y_2} \to P_{Y'_1}$. Moreover there is the induced étale morphism of the minimal toroidal Kummer covers $\overline{f} : Y'_1[P_1]^{\rm nor} \to Y_2[P_2]^{\rm nor}$ such that*

(1) *P_1 is the push out of $P_1 \leftarrow P_{Y_2} \to P_{Y'_1}$*

(2) *there is an injective homomorphism $G_{P_1} \to G_{P_2}$.*

(3) *The étale morphism $f : Y'_1[P_1] \to Y_2[P_2]$ is G_{P_1}-equivariant.*

(4) *Let $f^{\rm Tor} : M^{\rm Tor}(Y_2) \to M^{\rm Tor}(Y'_1)$ be the induced epimorphism of the groups of the characters. For any $\alpha \in M^{\rm Tor}(Y_2)$,and $\beta = f^{\rm Tor}(\alpha) \in M^{\rm Tor}(Y'_1)$, we have*
$$f^*(I^{Tor}_{Y[P_2]^{\rm nor},\alpha}) = I^{Tor}_{Y'_1[P_1]^{\rm nor},\beta}.$$

Proof. We can assume that P_{Y_2} is a sharp monoid. Let an optimized toric chart $P_{Y_2} \to \mathcal{M}(Y_2)$ be defined by an étale morphism $g : Y_2 \to X_\sigma$ followed by the projection morhism $h : X_\sigma \to X_{\delta, N^0_\delta} = {\rm Spec}(K[P_{Y_2}])$. Consider the toric chart induced chart $\phi := hgf : Y_1 \to X_\sigma \to X_\delta$. Denote by σ' the smallest face of σ for which $X_{\sigma'}$ contains $\phi(y)$. Then there exists a Zariski open neighborhood $Y_1' \subset Y_1$ of $y \in Y$ such that the restriction map $\phi_1 := \phi_{Y_1'} : Y_1' \to X_{(\sigma', N_\sigma)} \to X_{(\delta', N_\delta^0)} = \mathrm{Spec}(K[P_{Y_1'}])$ is a toric optimized chart for Y_1'. This defines the chart $P_{Y_1'}$. Note the both charts $P_{Y_1'}$, and P_{Y_2} have the same groupification which is dual to the lattice N_δ^0.

By Lemma 5.2.12 the minimal Kummer extension $Y_2[P_2]^{\mathrm{nor}}$ corresponds to the induced toric chart $Y_2[P_2]^{\mathrm{nor}} \to X_\delta \times X_\tau \to X_{\delta, N_\delta}$, and by Lemma 5.2.17 (4), the induced extension $Y_1'[P_1]^{\mathrm{nor}}$ is minimal corresponding the the (non sharp) toric chart:

$$Y_1'[P_1]^{\mathrm{nor}} \to X_{\delta' \times \tau', N^1} \to X_{(\delta', N_\delta)}.$$

This induces, by the previous lemma, the surjective homomorphism

$$f^{\mathrm{Tor}} : M^{\mathrm{Tor}}(Y_2) \to M^{\mathrm{Tor}}(Y_1'),$$

and an injective homomorphism $G_{P_1} \to G_{P_2}$ and the étale G_{P_1}-equivariant map $\overline{f} : Y_1'[P_1]^{\mathrm{nor}} \to Y_2[P_2]^{\mathrm{nor}}$. The latter is the composition of the open embedding $Y_1'[P_2]^{\mathrm{nor}} \to Y_2[P_2]^{\mathrm{nor}}$, and the isomorphism $Y_1'[P_1]^{\mathrm{nor}} \to Y_1'[P_2]^{\mathrm{nor}}$. (Since P_1 is generated by P_2 and some elements in $P_{Y_1'}$).

The étale morphism $g : Y_2[P_2]^{\mathrm{nor}} \to X_\delta \times X_\tau = \mathrm{Spec}(K[x_1, \ldots, x_k, P_2]) = \mathrm{Spec}(K[\widetilde{P}]$, where $\widetilde{P}_2 := P_2 \cdot x^\alpha$, determines a G_{P_2}-equivariant étale map, with G_{P_2}-semiinvariant parameters x_i. The ideal $\mathcal{I}_{Y[P_2]^{\mathrm{nor}},\alpha}$ is defined by the semiinvariant monomials $x^c \cdot m$ of weight α, where $m \in P_2$.

Composing g with $\overline{f}$ defines a G_{P_1}-equivariant map $g\overline{f} : Y_1[P_1]^{\mathrm{nor}} \to X_\delta \times X_\tau = \mathrm{Spec}(K[x_1, \ldots, x_k, P_2]) = \mathrm{Spec}(K[\widetilde{P}]$. The latter factors through

$$Y_1'[P_1]^{\mathrm{nor}} \to X_{\delta'} \times X_{\tau'} = \mathrm{Spec}(K[x_1, \ldots, x_r, x_{r+1}, x_{r+1}^{-1}, \ldots, x_k, x_k^{-1}, P_1]$$

, which defines the toric chart $Y_1'[P_1]^{\mathrm{nor}} \to X_{\delta' \times \tau'} \to X_{\delta'} = \mathrm{Spec}(K[P_1])$.

Let $\alpha \in M^{\mathrm{Tor}}(Y_2)$, and $\beta = f^{\mathrm{Tor}}(\alpha) \in M^{\mathrm{Tor}}(Y_1')$. The ideal $I_{Y_1[P_1]^{\mathrm{nor}},\beta}$ is generated by the pullbacks of the monomials having G_{P_1}-weight β. If such a pullback $f^*(x^a m)$ has weight β, then the monomial $x^a m$ is of weight α', with $f^{\mathrm{Tor}}(\alpha') = \beta$. Hence $\alpha - \alpha'$ is in $\ker(f^{\mathrm{Tor}})$, and , by Lemma 5.2.17 (3), there exists a monomial $x^{c_1} m_1 \in \mathcal{O}(X_{\delta \times \tau})$ of the weight $\alpha - \alpha'$, such that its restriction to $Y_1'[P_1]$ is invertible. Thus the ideals

$$(f^*(x^c m)) = f^*(x^{c_1} m_1) \cdot f^*(x^c m) = (f^*(x^c m \cdot x^{c_1} m_1)$$

are equal, where $x^c m \cdot x^{c_1} m_1$ is of weight α. This implies $f^*(\mathcal{I}^{Tor}_{Y[P_2]^{\mathrm{nor}},\alpha}) = \mathcal{I}^{Tor}_{Y_1[P_1]^{\mathrm{nor}},\beta}$. ♣

5.2.21. *Torific ideals.*

Definition 5.2.22. Let $Y[P]^{\mathrm{nor}}$ be a minimal Kummer cover of $Y \in X_{\text{ét}}$. By the *torific ideal* on $Y[P]^{\mathrm{nor}}$ we mean

$$\mathcal{I}^{Tor}_{Y[P]^{\mathrm{nor}}} := \prod_{\alpha \in M^{Tor}_Y(Y)} \mathcal{I}^{Tor}_{Y[P]^{\mathrm{nor}},\alpha}.$$

Lemma 5.2.23. *If m is divisible by all $|M^{Tor}_Y(Y)|$ then the m power $(\mathcal{I}^{Tor}_{Y[P]^{\mathrm{nor}}})^m$ of the ideals define a unique ordinary ideal $(\mathcal{I}^{Tor})^m_X$ on X, which restricts on the each minimal Kummer toroidal cover to $(\mathcal{I}^{Tor}_{Y[P]^{\mathrm{nor}}})^m$.*

Proof. The ideals $(\mathcal{I}^{Tor}_{Y[P]^{\rm nor}})^m$ are generated by G_P-invariant elements, if $|G_P|$ divides m, and thus it descends to the ideal $(\mathcal{I}^{Tor})^m_Y$ on Y. The uniqueness of the descent (to Y) easily follows from Corollary 5.2.12.

Let Y_1, Y_2 be torically optimized étale neighborhoods be $f : Y_1 \to Y_2$ be an étale map over X. We will show that $f^*(\mathcal{I}^{Tor})^m_{Y_2} = (\mathcal{I}^{Tor})^m_{Y_1}$ at any point $y \in Y_1$. By Lemma 5.2.20 applied simulataneously to two étale maps $f : Y_1 \to Y_2$, and $\mathrm{id} : Y_1 \to Y_1$ and a point $y \in Y_1$, there exists a Zariski neighborhood $Y_1' \subseteq Y_1$ torically optimized for $y \in Y$, and the induced étale maps $Y_1'[P_1]^{\rm nor} \to Y[P_2]^{\rm nor}$, and $Y_1'[P_1']^{\rm nor} \to Y[P_1]^{\rm nor}$ of the relevant minimal Kummer covers. This shows that the both pull-backs of $(\mathcal{I}^{Tor}_{Y[P_i]^{\rm nor}})^m$, for $i = 1, 2$ define the torific ideals on $Y'[P]^{\rm nor}$, and by the uniquness descend to the same ideal on Y_1'. Hence the pull-backs of ordinary torific ideals $(\mathcal{I}^{Tor}_{Y_2})^m$ on Y_2 and $(\mathcal{I}^{Tor}_{Y_1})^m$ on Y_1 coincide on a neigborhood Y_1'. This means that we have equality $f^*(\mathcal{I}^{Tor})^m_{Y_2} = (\mathcal{I}^{Tor})^m_{Y_1}$ on any neigborhood Y_1' of any $y \in Y_1$.

Thus the torific ideal is compatible on the étale site so it descends to X. ♣

5.2.24. *Torifying properties of torific ideal.*

torific **Proposition 5.2.25.** *Let (X, U) be stably toroidal variety. The normalized blow-up of $\sigma : X' \to X$ of the torific ideal $(\mathcal{I}^{Tor})^m$ on X, for sufficiently divisible m, transforms X into a toroidal variety (X', U'), where $U' = \sigma^{-1}(U) \smallsetminus E$, where $E = V(\sigma^{-1}(\mathcal{I}^{Tor})^m)$ is the exceptional divisor.*

Proof. The normalized blow-up of $(\mathcal{I}^{Tor}_{X\alpha})^m$ defines the normalized blow-up $Y'[P]^{\rm nor} \to Y[P]^{\rm nor}$ which coincides with the normalized blow-up of $\mathcal{I}^{Tor}_{Y[P]^{\rm nor},\alpha}$ on any mimimal toroidal Kummer cover $Y[P]^{\rm nor}$.

Assume $Y \in X_{\rm ét}$ is torically optimized. This reduces the considerations to the case where $(Y, U_Y) = (X_\sigma, X_\tau)$ is a toric doubleton , and $Z := Y[P]^{\rm nor} = X_\delta \times X_\tau$ is a regular toric doubleton (a toric variety) and $\mathcal{J} := (\mathcal{I}^{Tor}) = \mathcal{J}_{\alpha_1} \cdot \ldots \cdot \mathcal{J}_{\alpha_k}$ is a toric torific ideal on Z. The blow-up of $\mathcal{J}$ coincides with the composition of the consecutive blow-ups of $\mathcal{J}_{\alpha_i}$ and their pull-backs in any order.

Let $z_1, \ldots, z_k$ be regular nonmonomial coordinates on $Z := Y[P]^{\rm nor} = X_\tau \times X_\delta$ with $U := T_\delta \times X_\tau$ defining the canonical toroidal structure on Z, where $T_\delta \subset X_\delta$ be the maximal torus. Let $U_i := T_\delta \times X_{\rho_i} = \{x \in X_\tau \times T_\delta \mid z_j \neq 0, j \neq i\}$. Then each couple (Z, U_i) defines a toroidal variety , with the difference

$$U \smallsetminus (\bigcup_{i=1,\ldots,k} U_i) = (T_\delta \times X_\tau) \smallsetminus (\bigcup_{i=1,\ldots,k} X_{\rho_i} \times T_\delta)$$

of codimension ≥ 2. Let α_i be the character of z_i. The ideal $\mathcal{J}_{\alpha_i} = (z_i, m_1, \ldots, m_s)$ is smooth-monomial with respect to (Z, U_i). The effect of its blow-up $\sigma_i : Z' \to Z$ is one of the following:

(1) in the chart z_i, the only free parameter z_i describes the exceptional divisor, and becomes monomial. Consequenty the pair (Z', U_i') becomes toroidal with no free coordiantes with respect to the enriched canonical structure. The action of G_P becomes toroidal, as there are no free parameters. The further blow-ups of the pull-backs of other J_α are monomial and they won't change these properties.
(2) in the chart m_j, the new modified uknown $z_i' := z_i/m_j$ admits the trivial action and is a free parameter. The action of G is again toroidal. The

generators of the pull-backs of other ideals J_α can be modified in a way that they do not contain z_i, as $z_i = z_i' \cdot m_j$ can be replaced replaced by m_j in the presentation of the generators of these pull backs, and this will not change the J_α . Normalization won't change this property. Thus the pull-back of other ideals J_α are necessarily monomial with respect to the logarithmic structure. Then (Z', U') is toroidal. The further blow-ups will not affect the toroidal action of G on toroidal (Z', U_i').

The effect of the normalized blow-up of $\mathcal{J}$ on Z, is a toric variety S, with the open toric subsets V, and V_i, such that each (S, V_i) is toroidal and admits toroidal action of G_P. Moreover locally (S, V_i) is of the form (X_σ, T), in the first case or $(X_\sigma, X_{\rho_i} \times T)$ in the second case, with $\sigma = \rho_i \times \delta_i$ and $X_\sigma = X_{\rho_i} \times X_{\delta_i}$ with X_{ρ_i} regular and G_P acting trivially on X_{ρ_i}. So the rays ρ_i are indendent of the other rays in σ. This implies that the face τ, generated by ρ_i is regular in X_σ, and $X_\sigma = X_\tau \times X_\delta$ with the trivial action of G_P on X_τ. So $(X_\sigma, X_\tau \times T) = (X_\tau \times X_\delta, X_\tau \times T)$ is toroidal. The open subset $X_\tau \times T$ is a minimal open affine subvariety containing $\bigcup X_{\rho_i} \times T$, and coincides with the union $\bigcup X_{\rho_i} \times T$ up to codimension two. On the other hand all the created divisorial components are removed from the toroidal subsets so V and V_i differ from U and U_i respectively in codimension two. So again $V \setminus \bigcup V_i$ is of codimension two. Moreover V is locally open affine, so corresponds to a cone and thus it coincides locally with $X_\tau \times T$, as all the rays of τ generate the cone. Thus (S, V) locally coincides with $(X_\sigma, X_\tau \times T)$, and is toroidal, with toroidal action of G_P. This shows that $Y = Y[P]^{\rm nor}/G_P$ is transformed into a toroidal variety S/G_P.

♣

projections

5.2.26. *Functoriality of torification with respect to isotropical morphisms.* First, let us introduce the maps of monoids which preserve the torific ideals of monoids. Consider the chart $P \to Q$ optimized for $f : Y \to X$ at $y \in Y$. Each Kummer extension $P \subset P_1$ defines the group action of $(P_1^{gp}/P^{gp})^\vee$ on the monomials in P_1 which is the isotropy group at $x = f(y)$. The push-out Q_1 of $Q \leftarrow P \to P_1$ determines the Kummer extension $Q \subset Q_1$. The resulting group homomorphism shall be an isomorphism of the isotropy groups $P_1^{gp}/(P^{gp}P_1^*) \to Q_1^{gp}/(Q^{gp}Q^*)$. Morover the torific ideal P_1^α shall genarate the monoid ideal Q_1^α generated by the elements with weight α. This motivates the following definition.

Definition 5.2.27. A map of monoids $i : P \to Q$ is *isotropical* if for any Kummer map $P \to P_1$, and the induced map $i_1 : P_1 \to Q_1$ defined by the push-out Q_1 $P_1 \leftarrow P \to Q$, we have that

(1) $Q_1 = i_1(P_1)Q$.
(2) $P_1^{gp}/(P^{gp}P_1^*) \to Q_1^{gp}/(Q^{gp}Q^*)$ is an isomorphism.

The map $P \to Q$ is *universally isotropical* if for any map of sf monoids $P \to P'$ the induced push-out map $P' \to Q'$ is isotropical.

Definition 5.2.28. A morphism $X \to Y$ of log schemes is isotropical (resp. universally isotropical) if for any $x \in X$ the map od monoids $(\mathcal{M}/\mathcal{O}^*)_{\overline{f(x)},X} \to (\mathcal{M}/\mathcal{O}^*)_{\overline{x},X}$ is isotropical (respectively *universally isotropical*).

Lemma 5.2.29. *Composition of universally isotropical (respectively isotropical) morphisms of fs log varieties (or varieties with rational log structures) is universally* *isotropical (respectively isotropical). The pull back of the universally isotropical maps is universally isotropical.*

Proof. We consider optimized charts for the morphisms by Lemma 4.5.8. This will reduce the problem to the maps of monoids where it is apparent.

♣

tori **Lemma 5.2.30.** *Let $i : P \to Q$ is isotropical , $P \to P_1$ be a Kummer extension, and $P_1 \to Q_1$ the push out map. Then for any characters $\alpha \in P_1^{gp}/P^{gp}Q_1^* = Q_1^{gp}/Q^{gp}Q_1^*$ the monoid ideals are equal:*

$$Q_1^\alpha = P_1^\alpha Q.$$

Proof. It follows from $Q_1 = i(P_1)Q$ that any $n \in Q_1$ with weight α has a form $n = i(m)n'$, where $m \in P_\alpha$, and $n \in Q$ is of weight zero.

♣

Proposition 5.2.31. *Let X be a stably toroidal variety. The ideal $(\mathcal{I}^{Tor})^m$ on X is defined functorially with respect to universallly isotropical normally log smooth morphisms.*

Proof. Let $f : Y \to X$ be a isotropical morphism of stably toroidal varieties. Then étale locally at $x \in X$ and $f(x) \in Y$ it is induced by the torically optimized chart $P_X \to P_Y$ so one can reduce the situation to toric doubletons. Consider the corresponding map of cones with lattices $(\delta_Y, N_Y) \to (\delta_X, N_X)$. The projection $\sigma_X \to \delta_X$, $\sigma_Y \to \delta_Y$ defined by the doubletons detemine the map $(\sigma_Y, N^0_Y) \to (\sigma_X, N^0_X)$, and the dual maps of monoids $P'_X \to P'_Y$ which is the push out of $P_X \to P_Y$.

By the assumption $P_X \to P_Y$ is univerally isotropical hence $P'_X \to P'_Y$ is isotropical. Consider the minimal Kummer extension $P_X \subset P_{1X}$, and its push-out $P_Y \to P_{1Y}$. Since $X[P_{1X}]^{\rm nor}$ is toroidal we conclude, by Lemma 4.6.23 that $Y[P_{1Y}]^{\rm nor}$ is also toroidal, as the map $Y[P_{1Y}]^{\rm nor} \to X[P_{1X}]^{\rm nor}$ is log smooth, induced by $f : Y \to X$. If $Y[P_{1Y}]^{\rm nor}$ was not minimal toroidal Kummer cover then there was a subgroup of $G \subset G_{P_{1X}} = G_{P_{1Y}}$ acting toroidally on $Y[P_{1Y}]^{\rm nor}$. In such a case consider two actions of G on $X[P_{1X}]^{\rm nor}$: the toroidal action on $X[P_{1X}]^{\rm nor}$ and the natural action defined by G. The pull-backs of both actions would coincide on $Y[P_{1Y}]^{\rm nor}$, so they should coincides on $X[P_{1X}]^{\rm nor}$. Hence G acts toroidally on $X[P_{1X}]^{\rm nor}$ which, by Lemma 5.2.6, contradicts to the minimality of Kummer cover $X[P_{1X}]^{\rm nor}$. It suffices to apply Lemma 5.2.30 to $P'_X \to P'_Y$ and its push out $P'_{1X} \subset P'_{1Y}$ to get the equality $f^*(\mathcal{I}^{Tor}_{Y[P_{1Y}]^{\rm nor}}) = \mathcal{I}^{Tor}_{Y[P_{1X}]^{\rm nor}}$ of the ideals on the minimal covers which implies the equality of the ordinary ideals $f^*(\mathcal{I}^{Tor}_Y)^m = (\mathcal{I}^{Tor}_X)^m$

♣

To complete the proof of the functoriality of the resolution with universally isotropical morphisms it suffices to show the lemma.

Lemma 5.2.32. *If $Y \to X$ is universally isotropical map of stably toroidal varieties then. Let $X' \to X$ is a blow-up of Kummer ideal on $X_{\rm két}$, and $Y' \to Y$ is the induced blow-up then $Y' \to X'$ is universally isotropical.*

Proof. By the formulas in 3.2.3 the blow-ups on X at the monomial and the smooth-monomial centers and the induced blow-ups on Y define locally the map of monoids (charts) $P \to P'$, and $Q \to Q'$, where Q' is the push out of $Q \leftarrow P \to P'$. By the universal property $P' \to Q'$ is universally isotropical.

♣

6. Stably toroidal varieties as ambient schemes

[7]

←7

6.1. **Smooth monomial centers on stably toroidal varieties.**

6.1.1. *Blow-ups of smooth Kummer centers.* .

Lemma 6.1.2. *Let $\mathcal{I}$ be a coherent ideal on $X_{\rm tor}$. Then for sufficiently divisible $m \in \mathbb{N}$, the restriction of the power $\mathcal{I}^m$ to $X^{\rm Zar}$ induces the ordinary ideal $\mathcal{I}^{[m]}$ on $X_{\rm tor}$, such that the saturations $(\mathcal{I}^{[m]})^{\rm sat}$, and $(\mathcal{I}^m)^{\rm sat}$ are the same.*

Proof. It suffices to show that these saturations coincide on the Kummer covers $Y[P]^{\rm nor}$. Let $\mathcal{I}_{Y[P]^{\rm nor}}$ be the ideal defined on $Y[P]^{\rm nor}$, and for any $\alpha \in G_P^\vee$ by $\mathcal{I}_{Y[P]^{\rm nor},\alpha}$ denote the ideal generated by the G_P-semi-invariant functions $f \in \mathcal{I}_{|Y[P]^{\rm nor}}$ on $Y[P]^{\rm nor}$. Then

$$\mathcal{I}_{Y[P]^{\rm nor}} = \sum_{\alpha \in G_P^\vee} \mathcal{I}_{Y[P]^{\rm nor},\alpha}$$

For $m = |G_P|$, each ideal $\mathcal{I}^m_{Y[P]^{\rm nor},\alpha}$ is generated by the G_P-invariant elements. Thus $\mathcal{I}^{[m]}$ on $Y[P]^{\rm nor}$ contains $\sum_{\alpha \in G_P^\vee} \mathcal{I}^m_{Y[P]^{\rm nor},\alpha}$. The latter has the same saturation as $\mathcal{I}^m_{Y[P],\alpha}$.

♣

Lemma 6.1.3. *Let X be a stably toroidal variety. Let $\mathcal{J}$ be an ideal which is defined on $X_{\rm tor}(\mathcal{J})$. Then the normalized blow-up $X' = bl_{\mathcal{J}^{[m]}}(X)$ of the ideal $\mathcal{J}^{[m]}$ on X, defines a map $f : X' \to X$, which induces the blow-up of $\mathcal{J}_Z^{[m]}$ on each $Z \in X_{\rm tor}(\mathcal{J})$. The latter coincides with the blow-up of $\mathcal{J}^{[m]}$ $\mathcal{J}$ on each $Z \in X_{\rm tor}(\mathcal{J})$*

Proof. This is a consequenece of the previous lemma, and the fact that the normalized blow-ups depend only on the saturations of the centers. ♣

Def:Kummer

Definition 6.1.4. Let X be stably toroidal variety. A *smooth-monomial center* on $X_{\rm tor}$ is a Kummer ideal $\mathcal{J}$, which is locally at any $p \in X_{\rm tor}(\mathcal{J}) \supseteq X_{\rm tor}(m_1, \ldots, m_r)$ of the form

$$\mathcal{J} = (x_1, \ldots, x_k, m_1, \ldots, m_r),$$

where m_i are the monomials and $x_i \in \mathcal{O}_X$ restrict to local parameters on the stratum s_p. We say that the center is *monomial* (respectively *smooth*) if $k = 0$ (respectively $r = 0$) at all the points of its vanishing locus.

Note that the monomials $m_1, \ldots, m_r$ can be (informally) represented on the étale site $X_{\rm ét}$ by fractional powers.

[7](Dan) Cross references to orbifold paper from here to the end of this paper

Blow-up2

Lemma 6.1.5. *Let (X, U) be a stably toroidal variety. Let $\mathcal{J}$ be a smooth-monomial Kummer center on $X_{\rm tor}(\mathcal{J})$. Then $\mathcal{J}^{[m]}$ is an ordinary ideal on X for sufiiciently divisible m, and the normalized blow-up $X' = bl_{\mathcal{J}^{[m]}}(X)$ of X, defines a map $f : X' \to X$, such that:*

1. *X' is a stably toroidal variety, with respect to the log structure (X', U'), where $U' = f^{-1}(U) \smallsetminus E$, where E is the exceptional divisor.*
2. *$\mathcal{J}\mathcal{O}_{X'}$ is invertible (in the Kummer topology).*
3. *If $\mathcal{J}$ is a monomial ideal on X, that is it can be written as $\mathcal{J} = (m_1, \ldots, m_s)$ on Z in $X(m_1, \ldots, m_s)$ then the normalized blowing up $\pi : X' \to X$ of $\mathcal{J}$ is a normally log smooth morphism.*
4. *If $X_1 \to X$ is normally log smooth then $\mathcal{J}_{X_1} = \mathcal{J} \cdot \mathcal{O}^{\rm tor}_{X_1}$ is a Kummer center on $X_1^{\rm tor}$, and $X_1' \to X_1$ is the normalized blow-up of $\mathcal{J}^{[m]}_{X_1}$ then $X_1' = (X' \times_X X_1)^{\rm sat}$ is the (saturated) product taken in the category of stably toroidal logarithmic schemes. It is given by the closure of $U \times_X X_1$ in the normalization of the usual pullback $X' \times_X X_1$.*

Proof. (1) The normalized blow-ups $X' \to X$ of the ordinary ideal $\mathcal{J}^{[m]}$ induces a local blow-up $\sigma_Y : Y' \to Y$ of some étale neighborhoods, and defines the normalized blow-up $\sigma_{Y[P]^{\rm nor}} : Y'[P]^{\rm nor} \to Y[P]^{\rm nor}$ of $\mathcal{J}$ on a sharp Kummer cover $Y[P]^{\rm nor} \in X_{\rm \acute{e}t}(\mathcal{J})$ of Y, with $Y'[P]^{\rm nor} \to Y'$ a Kummer cover. One can replace Y locally with the toric doubleton (X_σ, X_τ). Then the blow-up Y' of $\mathcal{J}^{[m]}$ is toric, as well as its lifting to $Y[P]^{\rm nor}$. By Proposition 3.2.4, $Y'[P]^{\rm nor}$ is toroidal (and toric) with respect to the new logarithmic structure $(Y'[P]^{\rm nor}, U'[P]^{\rm nor})$, so it locally a regular doubleton $(X_{\delta'} \times X_{\tau'}, X_{\tau'})$. Then the induced ilogarithmic variety (Y', U') is the quotient $Y'[P]^{\rm nor}/G_P = (Y' \times_Y Y[P]^{\rm nor})/G_P = Y'$ of $(Y'[P]^{\rm nor}, U'[P]^{\rm nor})$ by a finite toric group action G_P. Thus, locally it is the quotient of $(X_{\delta'} \times X_{\tau'}, X_{\tau'})$, and has a form $(X_{\sigma'}, X_{\tau'})$, where τ' is a regular face. This shows that $\sigma' = \tau' + \delta' \simeq \tau' \times \delta'$, where τ', δ' are faces of σ', and $\tau' \subset \sigma'$ is regular. Hence $(X_{\sigma'}, X_{\tau'})$ is a toric doubleton and X' is stably toroidal with respect to the new structure (X', U').

(2) is obvious. (3) The ideal $\mathcal{J}^{[m]}$ is monomial, and induces the normally log smooth map on étale covers, by Proposition 3.2.4, and thus on X.

(4) We can reduce the situation to the toric maps of doubletons $X_1 = (X_{1\sigma}, X_{1\tau}) \to X = (X_\sigma, X_\tau)$, induced by $X_{1\delta} \to X_\delta$ and corresponding to the logarithmic map of étale neighborhoods of $X_1 \to X$.

The ideal $\mathcal{J}$ on the (sharp) toroidal Kummer cover $Y[P]^{\rm nor}$ (regular doubleton) is smooth-monomial. So its transform $\mathcal{J}_1$ on $X_1[P]^{\rm nor}$ is also smooth-monomial. Denote by $X'[P]^{\rm nor} \to X[P]^{\rm nor}$ the blow-up of $\mathcal{J}$, where $Y' \to Y$ be the blow-up of $\mathcal{J}[m]$ (as in (1)). Set $X_1' = X' \times_X X_1$. By the universal property of the normalized blow-up the induced map $X_1'[P^n] = X_1[P]^{\rm nor} \times_{X[P]^{\rm nor}} X'[P]^{\rm nor} \to X_1[P]^{\rm nor}$ (with the saturated normalized product) is the blow-up of $\mathcal{J}_1$.

Passing to the (toric) quotient by G_P we obtain that $X_1' \to X'$ is the blow-up of $\mathcal{J}^{[m]} = \mathcal{J}^{G_P}$. Moreover we have the factorization $X_1' \to X_1 \times_X X' \to X'$, where the induced map $X_1' \to X_1 \times_X X'$ is projective birational and finite of normal varieties so it is an isomorphism.

Finally the the normalization of the usual pullback $X' \times_X X_1$ is the union of disjoint components. The component dominating X is given by the closure of $U \times_X X_1$ in normalization of the usual pullback $X' \times_X X_1$. The other components map into $X \smallsetminus U$, and are eliminated in the process of taking saturated closure as in Proposition 3.2.4.

♣

6.2. **Logarithmic differential operators on stably toroidal varieties.** Associating to objects X' of $X_{\rm tor}$ the sheaf $\mathcal{D}^{\leq i}_{X'}$ one obtains presheaves $\mathcal{D}^{\leq i}_{X_{\rm tor}}$. It follows from Lemmas 3.1.7 and 3.1.12 that each $\mathcal{D}^{\leq i}_{X_{\rm tor}}$ is a locally free sheaf (in the Kummer étale topology). If X is already logarithmically smooth (toroidal) then, $\mathcal{D}^{1}_{X_{\rm k\acute{e}t}}$ is locally free already for the étale topology. Finally, if X is a strict logarithmically smooth then it is locally free on Zariski topology of X. For any Kummer ideal $\mathcal{I}$, applying $\mathcal{D}^{\leq i}_{X_{\rm tor}}$ as a presheaf and sheafifying one obtains a Kummer ideal $\mathcal{D}^{\leq i}_{X_{\rm tor}}(\mathcal{I})$.

tangentsheaf2 **Lemma 6.2.1.** *Assume that $X_{\rm tor}$ is a stably toroidal variety equipped with the normal Kummer topology. Then the logarithmic tangent sheaf $\mathcal{D}^1_X$ is Kummer locally free of rank* $\dim(X)$. *In general, $\mathcal{D}^{\leq i}_{X_{\rm tor}}$ are free in the normal Kummer étale topology.*

Proof. Follows from Lemmas 3.1.7, and 3.1.12 ♣

6.3. **Clean and balanced ideals on stably smooth variety.** We are now in position to rewrite basic notions introduced in the previous Chapter in the lanaguage of stably toroidal varieties with the Kummer topology.

One can extend the following definition to stably toroidal varieties.

Definition 6.3.1. Let X be a stably toroidal variety and $\mathcal{I}$ an ideal sheaf on $X_{\rm tor}$. Define the *monomial saturation of* $\mathcal{I}$ to be

$$\mathcal{M}(\mathcal{I}) := \bigcap_{\substack{\tilde{\mathcal{I}} \supseteq \mathcal{I} \\ \tilde{\mathcal{I}} \text{ monomial}}} \tilde{\mathcal{I}}.$$

As before we have:

Th:monomial-part2 **Theorem 6.3.2.** *Let X be a stably toroidal variety and $\mathcal{I}$ an ideal sheaf on $X_{\rm tor}$.*

(1) $\mathcal{I}$ is monomial if and only if $\mathcal{D}^{(\leq 1)}_X \mathcal{I} = \mathcal{I}$.
(2) $\mathcal{D}^{\infty}_X \mathcal{I} = \mathcal{M}(\mathcal{I})$
(3) If $Y \to X$ is logarithmically smooth then $\mathcal{M}(\mathcal{I}\mathcal{O}_Y) = \mathcal{M}(\mathcal{I})\mathcal{O}_Y$.

Proof. It is an immediate consequence of Theorem 3.4.2.

♣

The following is an extension of Defintion 3.6.1.

Def:classify-ideals2 **Definition 6.3.3.** A nowhere zero ideal $\mathcal{I}$ on stably smooth variety X is

- *balanced* if the monomial ideal $\mathcal{M}(\mathcal{I})$ is invertible in Kummer topology.
- *clean* if $\mathcal{M}(\mathcal{I}) = 1$.

Given a balanced ideal $\mathcal{I}$ on we define its *clean part*

$$\mathcal{I}^{cln} := (\mathcal{M}(\mathcal{I}))^{-1}\mathcal{I}.$$

As before, $\mathcal{I}$ factors as $\mathcal{I}^{cln} \cdot \mathcal{M}(\mathcal{I})$ and this is compatible with differentiation:

commute2 **Lemma 6.3.4.** *Let $\mathcal{M}$ be a monomial ideal and $\mathcal{I}$ an arbitrary ideal. Then*

$$\mathcal{D}_X^{(\leq i)}(\mathcal{M}\mathcal{I}) = \mathcal{M}\mathcal{D}_X^{(\leq i)}(\mathcal{I}) = \mathcal{M}(\mathcal{I}).$$

Proof. It is a consequence of Lemma 3.6.2.

♣

The following extens Proposition 3.6.5.

Absolute2 **Proposition 6.3.5.** *Let $\sigma : X' \to X$ be the monomial modification associated to an ideal $\mathcal{I}$, that is the blow-up of $\mathcal{M}(\mathcal{I})$. Then the resulting ideal $\mathcal{I}\mathcal{O}_{X'}$ is balanced. The blow-up is functorial with respect to logarithmically smooth maps.*

6.4. **Working with stably toroidal varieties.** In the remaining part of the paper we shall work with ambient schemes which are stably toroidal varieties. In practice it means that the sufficiently small neighborhoods $Z \in X_{\rm tor}$ in the Kummer topology, are toroidal. The maps between different charts are Kummer étale maps. In order to run the canonical algorithm on such a stable toroidal variety all the transformations and constructions defined on the toroidal site shall be functorial with respect to Kummer étale maps. In fact they are functorial with respect to the logarithmic maps.

7. Marked ideals on stably toroidal varieties

Sec:marked-ideals

7.1. **Marked ideals.** Inspired by Hironaka [Hir77], and following Villamayor [Vil89] and Bierstone–Milman [BM91], it is convenient to consider *marked ideals* $(\mathcal{I}, a)$, that is coherent ideals $\mathcal{I}$ on stably toroidal varieties X with associated integer weights a which keep track of the singular locus of an ideal. In Section 7.2.6 below the marking a dictates the transformation of a marked ideal by an admissible blowing up.

Definition 7.1.1. Let X be a stably toroidal variety equipped with the Kummer toroidal site $X_{\rm tor}$. Let $\mathcal{I}$ be an coherent ideal on X. The *logarithmic cosupport* or simply *cosupport* $\mathrm{supp}(\mathcal{I}, a)$ of a marked ideal $(\mathcal{I}, a)$ is a functor on the Kummer toroidal site $X_{\rm tor}$

$$Z \mapsto \mathrm{supp}(Z, \mathcal{I}, a) := \{p \in Z \mid \mathrm{logord}_p(\mathcal{I}) \geq a\} \subset Z.$$

Remark 7.1.2. The fact that $\mathrm{supp}(\mathcal{I}, a)$ is a functor follows from the property that the logaritmic order of an ideal $\mathcal{I}$ is preserved by normally log smooth maps. Moreover by compatibility the functor naturally extends to $X_{\text{két}}$. Also, It defines a vanishing ideal on $X_{\text{két}}$.

Definition 7.1.3. A marked ideal $(\mathcal{I}, a)$ is *of maximal order* if $\mathrm{logord}_p(\mathcal{I}) = a$ for every $p \in \mathrm{supp}(Z, \mathcal{I}, a)$, $Z \in X_{\rm tor}$.

Note that if $(\mathcal{I}, a)$ is of maximal order then necessarily $\mathcal{I}$ is clean (Definition 3.6.1). This includes the case when $\mathrm{supp}(\mathcal{I}, a) = \emptyset$.

As an example, a principal marked ideal $((x), 1)$ is of maximal order if and only if x is a regular parameter along $V(x)$.

Note that $\mathcal{I}_{|s(p)} = 0$ if and only if $\mathrm{logord}_p(\mathcal{I}) = \infty$. Therefore, an ideal $\mathcal{I}$ on X is clean if and only if $\mathrm{logord}_p(\mathcal{I}) < \infty$ for every $p \in V(\mathcal{I})$.

Lem:supports2

Lemma 7.1.4. *If $\mathcal{J} \subseteq \mathcal{I}$ and $(\mathcal{J}, a)$ is of maximal order then $(\mathcal{I}, a)$ is of maximal order.*

Proof. Assume $(\mathcal{J}, a)$ is of maximal order. Then $\mathcal{J}$ is clean and, by Lemma 3.6.3(2), $\mathcal{D}_X^{(\leq a)}(\mathcal{J}) = 1$, thus $\mathcal{D}_X^{(\leq a)}(\mathcal{I}) = 1$. ♣

The following lemma is the analogue of [Wło05, Lemma 3.2.2] and [Wło05, Lemma 3.3.3], which in turn refer to [Vil89]:

Lem:supports

Lemma 7.1.5. *Let $(\mathcal{I}, a)$ be a marked ideal.*

(1) $\operatorname{supp}(\mathcal{D}_X^{(\leq i)}(\mathcal{I}), a - i) = \operatorname{supp}(\mathcal{I}, a)$
(2) $\operatorname{supp}(\mathcal{I}, a) = V(\mathcal{D}_X^{(\leq a-1)}(\mathcal{I}))$ *is closed.*
(3) If $(\mathcal{I}, a)$ is of maximal order then for $i < a$ the marked ideal $(\mathcal{D}_X^{(\leq i)}(\mathcal{I}), a-i)$ is also of maximal order.

Proof. Thanks to Lemma 3.1.14, the statements can be checked in the structure sheaves of strata, where the result follows from the non-logarithmic situation. ♣

To simplify notation we will use in the sequel the following

Def:derivative-marked

Definition 7.1.6. Given a marked ideal $(\mathcal{I}, a)$ and $i \geq 0$ we set

$$\mathcal{D}^{(\leq i)}(\mathcal{I}, a) = (\mathcal{D}^{(\leq i)}(\mathcal{I}), a - i).$$

Sec:admissible-marked

7.2. Admissible modifications for marked ideals.

7.2.1. *Integral closure and monomial saturation.* Note that, unlike the smooth centers in the classical situation, the smooth-monomial centers need not to be integrally closed. And even if we assume that the centers are integarally closed still their powers need not be such. This creates some minor technical problem explained in Section 7.4. To resolve it we need to consider the integral closure of the powers of the smooth-monomial centers. Note that the powers are necessarily locally monomials with respect to the coordinates and logarithmic structure so the integral closure is very simple and can be seen as, merely, the monomial saturation, and canonically expressed as

$$(\mathcal{J}^a)^{\text{sat}} = \{f \in \mathcal{O} \mid f^m \in (\mathcal{J}^a)^m.\}$$

It is immediate from this definition that

Lemma 7.2.2. *(1)* $D_X^i(\mathcal{J}^a)^{\text{sat}} \subseteq (\mathcal{J}^{a-i})^{\text{sat}}$.
(2) If x is a coordinate then $((\mathcal{J}^a)^{\text{sat}})_{|V(x)} = (\mathcal{J}^a_{|V(x)})^{\text{sat}}$.

7.2.3. *Admissibility.*

Def:admissible

Definition 7.2.4. Let $(\mathcal{I}, a)$ be a marked ideal on X and $\sigma\colon X' \to X$ a Kummer blowing up with a center $\mathcal{J}$. If $\mathcal{I} \subseteq (\mathcal{J}^a)^{\text{sat}}$ we say that $\mathcal{J}$ is *admissible* for the marked ideal $(\mathcal{I}, a)$ and σ is an $(\mathcal{I}, a)$-*admissible Kummer blowing up.*

Our notion of admissibility is equiavalent to Hironaka's original definition [Hir64, I.2, Definition 6]: $\mathcal{I} \subseteq \mathcal{J}^a$, once we drop the logarithmic structure. When the logarithmic structure is trivial, the definition means that C is regular and the order of $\mathcal{I}$ along C is at least a.

3 **Lemma 7.2.5.** *Let $(\mathcal{I}, a)$ be a marked ideal, let $\sigma : X' \to X$ be the $(\mathcal{I}, a)$-admissible Kummer blowing up at $\mathcal{J}$, and let $I_E = \sigma^{-1}(\mathcal{J})$. Then*

(1) $V(\mathcal{J}) = V((\mathcal{J})^{\mathrm{sat}}) = \mathrm{supp}((\mathcal{J})^{\mathrm{sat}}, 1) \subseteq \mathrm{supp}(\mathcal{I}, a)$.
(2) $\mathcal{I}\mathcal{O}_{X'} \subseteq \mathcal{I}_E^a$.

Proof. (1) Applying the Leibniz rule to the Kummer ideal $(\mathcal{J}^a)^{\mathrm{sat}}$ we obtain that

$$\mathcal{D}_X^{(\leq a-1)}(\mathcal{I}) \subseteq \mathcal{D}_X^{(\leq a-1)}((\mathcal{J}^a)^{\mathrm{sat}}) \subseteq \mathcal{J}^{\mathrm{sat}}.$$

Therefore on the level of supports we get:

$$\mathrm{supp}((\mathcal{J})^{\mathrm{sat}}, 1) \subseteq V(\mathcal{D}_X^{\leq a-1}(\mathcal{I})) = \mathrm{supp}(\mathcal{I}, a).$$

(2) We have $\mathcal{I}\mathcal{O}_{X'} \subseteq (\mathcal{J}^a)^{\mathrm{sat}}\mathcal{O}_{X'} = \mathcal{I}_E^a$. ♣

Sec:controlled-transform

7.2.6. *The controlled transform of a marked ideal.*

Def:controlled-transform **Definition 7.2.7.** By the *controlled transform* of the marked ideal $(\mathcal{I}, a)$ under the admissible Kummer blowing up σ we mean the ideal

$$\sigma^c(\mathcal{I}, a) := \mathcal{I}_E^{-a}(\mathcal{I}\mathcal{O}_{X'}).$$

The *marked controlled transform* is $(\sigma^c(\mathcal{I}), a) := (\sigma^c(\mathcal{I}, a), a)$. The logarithmic structure on X' is enhanced by the exceptional divisor E, see [ATW16b, §4.1.1].

We extend this definition to a sequence of admissible modifications

Eq:admissible-sequence
$$X' =: X_n \xrightarrow{\sigma_n} X_{n-1} \xrightarrow{\sigma_{n-1}} \dots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 := X. \tag{1}$$

with centers $\mathcal{J}_i$ on X_i admissible for the controlled transform $\mathcal{I}_i = \sigma_i^c(\mathcal{I}_{i-1}, a)$, see Definition 7.3.1. The logarithmic structure on X_i is enhanced by the exceptional divisor E_i of σ_i.

Writing $\sigma : X' \to X$ for the composite map, we denote $\sigma^c(\mathcal{I}, a) := \mathcal{I}_n$. This can be unwound as follows. Write $\mathcal{I}_E = \prod \mathcal{I}_{E_i}\mathcal{O}_{X'}$, and then in general

$$\sigma^c(\mathcal{I}, a) = \mathcal{I}_E^{-a}(\mathcal{I}\mathcal{O}_{X'}).$$

7.2.8. *Derivatives and transformed ideals.* Lemma 3.3.1 implies the following

4 **Corollary 7.2.9.** $\mathcal{I}_E^i \cdot (\mathcal{D}_X^{(\leq i)}(\mathcal{I})\mathcal{O}_{X'}) \subseteq \mathcal{D}_{X'}^{(\leq i)}(\mathcal{I}\mathcal{O}_{X'})$.

Proof. Let y be the local equation of the divisor E. Observe that $\mathcal{I}_E^i \cdot (\mathcal{D}_X^{(\leq i)}(\mathcal{I})\mathcal{O}_{X'})$ is locally generated by the functions $y^i\sigma^*(\nabla(f))$, where $\mathcal{I}_E = (y), \nabla \in \mathcal{D}_X^{(\leq i)}$ and $f \in I$. Lemma 3.3.1 gives that $y^i\sigma^*(\nabla) \in \mathcal{D}_{X'}^{(\leq i)}$ so $y^i\sigma^*(\nabla(f)) = y^i\sigma^*(\nabla)(\sigma^*(f)) \in \mathcal{D}_{X'}^{(\leq i)}(\mathcal{I}\mathcal{O}_{X'})$. ♣

Note that marked ideals $(\mathcal{I}, a)$ with weight a are transformed by the rule $\sigma^c(\mathcal{I}) = \mathcal{I}_E^{-a}(\mathcal{I}\mathcal{O}_{X'})$. From this perspective the derivatives of order i transform as "marked objects" of weight $-i$, see Notation 3.3.3. This is not so surprising since the dual objects, the cotangent sheaves, are transformed as marked ideals with weight 1.

44 **Lemma 7.2.10.** *Let $(\mathcal{I}, a)$ be a marked ideal and $\sigma : X' \to X$ be an $(\mathcal{I}, a)$-admissible Kummer blowing up with exceptional divisor E. Then* $\sigma^c(\mathcal{D}_X^{(\leq i)}(\mathcal{I}, a)) \subseteq \mathcal{D}_{X'}^{(\leq i)}(\sigma^c(\mathcal{I}, a))$.

Proof. Recall from Definition 7.1.6 that $\mathcal{D}_X^{(\leq i)}(\mathcal{I},a) = (\mathcal{D}_X^{(\leq i)}(\mathcal{I}), a-i)$. Also $\mathcal{I}_E$ is monomial. Using Corollary 7.2.9 and Lemma 3.6.2 we have

$$\begin{aligned}\mathcal{I}_E^a\sigma^c\left(\mathcal{D}_X^{(\leq i)}(\mathcal{I}), a-i\right) &= \mathcal{I}_E^i\left(\mathcal{D}_X^{(\leq i)}(\mathcal{I})\mathcal{O}_{X'}\right)\\ &\subseteq \mathcal{D}_{X'}^{(\leq i)}(\mathcal{I}\mathcal{O}_{X'}) = \mathcal{D}_{X'}^{(\leq i)}(\mathcal{I}_E^a\sigma^c(\mathcal{I},a))\\ &= \mathcal{I}_E^a\mathcal{D}_{X'}^{(\leq i)}(\sigma^c(\mathcal{I},a))\end{aligned}$$

So $\sigma^c(\mathcal{D}_X^{(\leq i)}(\mathcal{I}), a-i) \subseteq \mathcal{D}_{X'}^{(\leq i)}(\sigma^c(\mathcal{I},a))$, as needed

♣

7.2.11. *The order does not increase!*

5 **Proposition 7.2.12.** *Let $(\mathcal{I},a)$ be a marked ideal of maximal order, and $\sigma: X' \to X$ be the $(\mathcal{I},a)$-admissible Kummer blowing up at center $\mathcal{J}$. Then*

(1) $\mathcal{D}_{X'}^{(\leq a)}(\sigma^c(\mathcal{I},a)) = (1)$.
(2) $(\sigma^c(\mathcal{I}),a)$ is a marked ideal of maximal order.
(3) For any $p' \in X'$, and $p = \sigma(p') \in X$, we have $\mathrm{logord}_{p'}(\sigma^c(\mathcal{I},a)) \leq \mathrm{logord}_p(\mathcal{I})$.

Proof. By assumption $(\mathcal{D}_X^{(\leq a)}(\mathcal{I}))\mathcal{O}_{X'} = (1)$. By Lemma 7.2.10 we have

$$(1) = (\mathcal{D}_X^{(\leq a)}(\mathcal{I}))\mathcal{O}_{X'} = \sigma^c(\mathcal{D}_X^{(\leq a)}(\mathcal{I}),0) \subseteq \mathcal{D}^{(\leq a)}(\sigma^c(\mathcal{I},a)).$$

This proves (1), which implies (2).

For (3) we may assume that $p \in C \subseteq \mathrm{supp}(\mathcal{I},a)$. By (1),

$$\mathrm{logord}_{p'}(\sigma^c(\mathcal{I})) = \mathcal{D}\mathrm{ord}_{p'}(\sigma^c(\mathcal{I})) \leq a = \mathrm{logord}_p(\mathcal{I}).$$

♣

Rem:efficiency **Remark 7.2.13.** The fact that clean ideals remain clean under blowing , may be somewhat surprising and it is absolutely crucial for our method. This makes algorithm much faster and efficient removing the unnecessary notorious "cleaning" procedure in the traditional algorithm at the multiple step procedure.

Part (2) of Proposition 7.2.12 allows us to define:

Def:order-reduction **Definition 7.2.14.** By an *order reduction* of a marked ideal $(\mathcal{I},a)$ of maximal order we mean an $(\mathcal{I},a)$-admissible sequence of Kummer blowings up (1) such that

$$\mathrm{supp}(\sigma^c(\mathcal{I},a)) = \emptyset.$$

If one uses the same centers but eliminates a smaller power of the exceptional divisor then the obtained ideal will still be balanced:

admissibility reduced **Corollary 7.2.15.** *Let $(\mathcal{I},a)$ be a balanced ideal, not necessarily of maximal order. Let b be the order of the clean factor $\mathcal{I}^{cln}$ of $\mathcal{I}$, and let $(\mathcal{I}^{cln},b)$ be the associated ideal of maximal order. Assume $b \geq a$, and let $\sigma: X' \to X$ be the $(\mathcal{I}^{cln},b)$–admissible Kummer blowing up at center $\mathcal{J}$. Then*

(1) $\mathcal{J}$ is admissible for $(\mathcal{I},a)$
(2) $\sigma^c(\mathcal{I},a)$ is balanced.

Proof. (1) $(\mathcal{J}^a)^{\rm sat} \supseteq (\mathcal{J}^b)^{\rm sat} \supseteq \mathcal{I}^{cln} \supseteq \mathcal{I}$.

(2) We have that

$$\begin{aligned}\sigma^c(\mathcal{I},a) &= \sigma^c(\mathcal{I}^{cln}\cdot\mathcal{M}(\mathcal{I}),a) = \mathcal{I}_E^{-a}(\mathcal{I}^{cln}\mathcal{O}_{X'})\cdot(\mathcal{M}(\mathcal{I})\mathcal{O}_{X'})\\ &= \left(\mathcal{I}_E^{b-a}(\mathcal{M}(\mathcal{I})\mathcal{O}_{X'})\right)\ \sigma^c(\mathcal{I}^{cln},b).\end{aligned}$$

The ideal $\sigma^c(\mathcal{I}^{cln},b)$ is clean by Proposition 7.2.12(2), and $\mathcal{M}(\mathcal{I})\mathcal{O}_{X'}$ is invertible and monomial by the assumption, hence the ideal $\sigma^c(\mathcal{I},a)$ is balanced. ♣

derivative

Corollary 7.2.16. *Let $(\mathcal{I},a)$ be a marked ideal of maximal order, let $\sigma: X'\to X$ be the $(\mathcal{I},a)$-admissible Kummer blowing up at center $\mathcal{J}$, and let $0<i<a$. Then*

(1) $\mathcal{J}$ is admissible for $\mathcal{D}_X^{(\leq i)}(\mathcal{I},a)$ and
(2) $\sigma^c(\mathcal{D}_X^{(\leq i)}(\mathcal{I},a))$ is a marked ideal of maximal order.

Proof. (1) Recall that $\mathcal{D}_X^{(\leq i)}(\mathcal{I},a) = (\mathcal{D}_X^{(\leq i)}(\mathcal{I}),a-i)$. Now $\mathcal{I}\subseteq(\mathcal{J}^a)^{\rm sat}$ implies that $\mathcal{D}_X^{(\leq i)}(\mathcal{I})\subseteq\mathcal{D}_X^{(\leq i)}((\mathcal{J}^a)^{\rm sat})\subseteq(\mathcal{J}^{a-i})^{\rm sat}$.

(2) By Lemma 7.1.5(3), $(\mathcal{D}_X^{(\leq i)}(\mathcal{I}),a-i)$ is of maximal order, so $\sigma^c(\mathcal{D}_X^{(\leq i)}(\mathcal{I},a)) = \sigma^c(\mathcal{D}_X^{(\leq i)}(\mathcal{I}),a-i)$ is of maximal order by Proposition 7.2.12(2). ♣

We have seen that the ideal $\sigma^c(\mathcal{D}_X^{(\leq i)}(\mathcal{I},a))$ satisfies $\sigma^c(\mathcal{D}_X^{(\leq i)}(\mathcal{I},a))\subseteq\mathcal{D}_{X'}^{(\leq i)}(\sigma^c(\mathcal{I},a))$, So property (2) above is perhaps expected.

Sec:order-reduction

7.3. Order reduction.

Def:admissible-sequence

Definition 7.3.1 (Admissible Kummer sequence)**.** Let X be a stably toroidal variety, $\mathcal{I}$ a clean ideal with maximal logarithmic order a. An $(\mathcal{I},a)$-*admissible Kummer sequence* $(X_i,\mathcal{I}_i,\mathcal{J}_i)$ consists of

- a sequence of stably toroidal varieties

missible-Kummer-Sequence

$$X_n\to X_{n-1}\to\cdots\to X_1\to X_0=X, \tag{2}$$

- clean ideal sheaves $\mathcal{I}_i$ of maximal logarithmic order $\leq a$ for $i\leq n$,
- Kummer ideals $\mathcal{J}_i$ on X_i for $i<n$,

such that for all $i<n$

- $\mathcal{J}_i$ is $(\mathcal{I}_i,a)$-admissible,
- X_{i+1} is the $\mathcal{J}_i$-Kummer blowing up of X_i, with exceptional divisor E_{i+1},
- $\mathcal{I}_i\mathcal{O}_{X_{i+1}} = \mathcal{I}_{i+1}\mathcal{I}^a_{E_{i+1}}$,

As in the classical situation the resolution of singularities will be achieved by the resolution of marked ideals

Th:order-reduction2

Definition 7.3.2 (Order reduction)**.** Let $(\mathcal{I},a)$ be a marked ideal on a stably toroidal variety with $\mathcal{I}$ clean ideal and maximal logarithmic order a on the Kummer toroidal site. By *a resolution of* $(\mathcal{I},a)$ we mean an $(\mathcal{I},a)$-admissible Kummer sequence $(X_i,\mathcal{I}_i,\mathcal{J}_i)$ such that $\mathcal{I}_n$ has maximal logarithmic order $<a$ or alternatively $\mathrm{supp}(\mathcal{I}_n,a)=\emptyset$ on $(X_n)_{\rm tor}$.

Equivalence

7.4. **Equivalence and domination of marked ideals.** We adapt to our setup the notions of equivalence of marked ideals, sums and products of marked ideals, and homogenization of marked ideals.

In analogy with [Wło05], let us introduce the following domination and equivalence relations for marked ideals:

Definition 7.4.1. Let $(\mathcal{I}, a)$ and $(\mathcal{J}, b)$ be marked ideals on a stably toroidal variety X

(1) We say that $(\mathcal{I}, a)$ is *dominated* by $(\mathcal{J}, b)$ and write

$$(\mathcal{I}, a) \preccurlyeq (\mathcal{J}, b)$$

if any sequence of $(\mathcal{J}, b)$-admissible Kummer blowings up is also a sequence of $(\mathcal{I}, a)$-admissible Kummer blowings up.

(2) If both $(\mathcal{I}, a) \preccurlyeq (\mathcal{J}, b)$ and $(\mathcal{J}, b) \preccurlyeq (\mathcal{I}, a)$ then we say that the marked ideals are *equivalent* and write $(\mathcal{I}, a) \approx (\mathcal{J}, b)$

Lem:scaling

Lemma 7.4.2. *If $(\mathcal{I}, a)$ is a marked ideal and $k > 0$ is an integer then if $a \geq b$ then $(\mathcal{I}, a) \preccurlyeq (\mathcal{I}, b)$, (b) if $\mathcal{I} \subseteq \mathcal{J}$ then $(\mathcal{I}, a) \preccurlyeq (\mathcal{J}, a)$.*

(2) If $(\mathcal{I}, a) \preccurlyeq (\mathcal{J}, b)$ then automatically $\operatorname{supp}(\mathcal{I}, a) \supseteq \operatorname{supp}(\mathcal{J}, b)$.

The main reason for introducing of the saturation is that the conditions for the centers $\mathcal{I} \subseteq \mathcal{J}^a$, and $\mathcal{I}^k \subseteq \mathcal{J}^{ka}$ are not equivalent.

Since the power of the marked ideals plays quite marginal role in the constructions mostly in the coefficient ideals just to equalize weights in the logarithmic setting we rather consider the following *simple power* of the ideal:

$$\mathcal{I}^{(k)} := (f^k \mid f \in \mathcal{I})$$

Immediately from the definition. Then $\mathcal{I} \subset \mathcal{J}^{\mathrm{sat}}$ is equiavelent to $\mathcal{I}^{(k)} \subset (\mathcal{J}^{ka})^{\mathrm{sat}}$.

Power2

Lemma 7.4.3. *(1) $(\mathcal{I}, a) \approx (\mathcal{I}^{(k)}, ka)$.*

(2) $\sigma^c(\mathcal{I}^{(k)}, ka) = \sigma^c(\mathcal{I}, a)^{(k)}$ for any $(\mathcal{I}, a)$-admissible sequence of Kummer blowings up $\sigma\colon X' \to X$.

The following example of domination will be very useful.

diff

Lemma 7.4.4. *If $(\mathcal{I}, a)$ is of maximal order and $i \leq a$ then*

(1) $\mathcal{D}_X^{(\leq i)}(\mathcal{I}, a) \preccurlyeq (\mathcal{I}, a)$.

(2) $\sigma^c(\mathcal{D}_X^{(\leq i)}(\mathcal{I}, a)) \subseteq \mathcal{D}_{X'}^{(\leq i)}(\sigma^c(\mathcal{I}, a))$ for any $(\mathcal{I}, a)$-admissible sequence of Kummer blowings up $\sigma\colon X' \to X$.

Proof. The proof is analogous to the classical one and follows from Lemma 7.2.10. ♣

7.5. **Addition and multiplication of marked ideals.** Following [Wło05, BM08] define the following operations of addition and multiplication of marked ideals:

(1) $(\mathcal{I}_1, a_1) + \ldots + (\mathcal{I}_m, a_m)$

$$:= (\mathcal{I}_1^{(a_2 \cdot \ldots \cdot a_m)} + \mathcal{I}_2^{(a_1 a_3 \cdot \ldots \cdot a_m)} + \ldots + \mathcal{I}_m^{(a_1 \ldots a_{k-1})}, a_1 a_2 \ldots a_m).$$

(2) $(\mathcal{I}_1, a_1) \cdot \ldots \cdot (\mathcal{I}_m, a_m) := (\mathcal{I}_1 \cdot \ldots \cdot \mathcal{I}_m, a_1 + \ldots + a_m)$

The following is immediate:

Lem:operation-pullback **Lemma 7.5.1.** *Given a morphism $X' \to X$ we have*

$$((\mathcal{I}_1, a_1) + \ldots + (\mathcal{I}_m, a_m))\mathcal{O}_{X'} = (\mathcal{I}_1, a_1)\mathcal{O}_{X'} + \ldots + (\mathcal{I}_m, a_m)\mathcal{O}_{X'}$$

and

$$((\mathcal{I}_1, a_1) \cdot \ldots \cdot (\mathcal{I}_m, a_m))\mathcal{O}_{X'} = (\mathcal{I}_1, a_1)\mathcal{O}_{X'} \cdot \ldots \cdot (\mathcal{I}_m, a_m)\mathcal{O}_{X'}.$$

The following facts will not be used, but they clarify the definition.

Remark 7.5.2. The sum is compatible with equivalence. So, although the definition is not associative, it is associative up to equivalence.

Lem:sum-max-order **Lemma 7.5.3.** *If one of the marked ideals $(\mathcal{I}_1, a_1), \ldots, (\mathcal{I}_m, a_m)$ is of maximal order then $(\mathcal{I}_1, a_1) + \ldots + (\mathcal{I}_m, a_m)$ is of maximal order.*

Proof. By Lemma 7.4.2, one of the ideals $\mathcal{I}_1^{a_2 \cdot \ldots \cdot a_m}, \ldots, \mathcal{I}_m^{a_1 \ldots a_{k-1}}$ is clean of maximal order $a_1 a_2 \ldots a_m$. By Lemma 7.1.4 $(\mathcal{I}_1, a_1) + \ldots + (\mathcal{I}_m, a_m)$ is of maximal order, as needed. ♣

le: operations **Lemma 7.5.4.** *Suppose $(\mathcal{I}_1, a_1), \ldots, (\mathcal{I}_m, a_m)$ are of maximal order.*

(1) If $(\mathcal{I}_i, a_i) \preccurlyeq (\mathcal{I}_1, a_1)$ for $2 \leq i \leq m$ then $(\mathcal{I}_1, a_1) + \ldots + (\mathcal{I}_m, a_m) \approx (\mathcal{I}_1, a_1)$
(2) If $(\mathcal{I}_i, a_i) \preccurlyeq (\mathcal{I}_1, a_1)$ for $2 \leq i \leq m$ then $(\mathcal{I}_2, a_1) \cdot \ldots \cdot (\mathcal{I}_m, a_m) \preccurlyeq (\mathcal{I}_1, a_1)$.

Proof. This is proved in the same way as in the classical situation, so we omit the details. ♣

7.6. **Hypersurfaces of maximal contact.** To achieve Order Reduction we induct on the dimension using *hypersurfaces of maximal contact,* which are constructed locally near a point $p \in X$ where $\mathcal{I}$ has logarithmic order precisely a, as follows. A clean ideal $\mathcal{I}$ of logarithmic order a has the property that $\mathcal{D}_X^{(\leq a)}\mathcal{I} = (1)$, see Lemma 3.6.3. If $a > 0$ then there exists a local section x of $\mathcal{D}_X^{(\leq a-1)}\mathcal{I}$ which is a regular parameter, defining a hypersurface of maximal contact $H = \{x = 0\}$. Note that H defines locally, a stably toroidal variety which is a subvariety of X. Indeed, H defines the toroidal subvariety on the Kummer toroidal chart $Z \in X_{\mathrm{tor}}$, and its projects to a locally closed stably smooth variety H_X on X.

Definition 7.6.1. Let $(\mathcal{I}, a)$ be of maximal order. Consider its restriction to $Z \in X_{\mathrm{tor}}$. Write $\mathcal{T}(\mathcal{I}, a) := \mathcal{D}_X^{(\leq a-1)}(\mathcal{I})$. A *maximal contact element* of $\mathcal{I}$ at p is an element $x \in \mathcal{T}(\mathcal{I}, a)_p$ which is a local parameter at p. The locus $H = V(x)$ is called a *hypersurface of maximal contact* of $\mathcal{I}$ at p.

By Lemma 7.1.5(1) we have the following useful fact:

Lem:max-contact-supp **Lemma 7.6.2.** *Let $(\mathcal{I}, a)$ be of maximal order and $x \in \mathcal{T}(\mathcal{I}, a)_p$ a hypersurface of maximal contact at p. Then $\mathrm{supp}(\mathcal{I}, a)_p \subseteq V(x)_p$.*

Finally, Lemma 3.5.3 implies that various notions we have introduced in this section are compatible with logarithmically smooth morphisms:

functorlem **Lemma 7.6.3.** *Assume that $f \colon X' \to X$ is a logarithmically smooth morphism between logarithmically smooth varieties, $\mathcal{I}$ is an ideal on X with $\mathcal{I}' = \mathcal{I}\mathcal{O}_{X'}$, and $a > 0$. Then*

(1) $\mathrm{supp}(\mathcal{I}', a) = f^{-1}(\mathrm{supp}(\mathcal{I}, a))$.

(2) If f is surjective then $(\mathcal{I}',a)$ is of maximal order if and only if $(\mathcal{I},a)$ is of maximal order.

(2) If H is a hypersurface of maximal contact to $(\mathcal{I},a)$ then $H\times_X X'$ is a hypersurface of maximal contact to $(\mathcal{I}',a)$.

7.6.4. *Preservation of maximal contact in Kummer blowings up.*

333 **Proposition 7.6.5.** *Let $(\mathcal{I},a)$ be of maximal order and $x\in\mathcal{D}^{a-1}(\mathcal{I})_p$ be a maximal contact element of $\mathcal{I}$ at p. Let $\sigma: X'\to X$ be a sequence of Kummer blowings up at $(\mathcal{I},a)$-admissible centers, and $q\in\sigma^{-1}(p)$ be a point in $\operatorname{supp}(\sigma^c(\mathcal{I},a))$.*

Then

(1) $x'=\sigma^c(x,1)$ is a maximal contact element for $\sigma^c(\mathcal{I},a)$.
(2) there is a coordinate system $\bar{x}:=(x,x_1,\dots,x_n)$ and a monoidal chart $u\colon M:=\overline{M}_{q'}\hookrightarrow\mathcal{O}_{X,q'}$ at a point $q'\in X'$ over q on an étale neigborhood of X' of X_n such that

$$\frac{\partial^i}{\partial x^i}(\sigma^c(\mathcal{I}))\subseteq\sigma^c(\mathcal{D}_X^{(\leq i)}(\mathcal{I})).$$

Proof. Write $\sigma_n: X_n\to X_{n-1}$ and $\tau: X_{n-1}\to X$, so that $\sigma=\tau\circ\sigma_n$. By induction we may assume that the result holds for τ.

Since $x'=\tau^c(x,1)\ \in\ \tau^c(\mathcal{D}^{a-1}(\mathcal{I}),1)\ \subseteq\ \mathcal{D}^{a-1}(\tau^c(\mathcal{I},a))$,, and $\mathcal{J}$ is admissible for $\tau^c(\mathcal{I},a)$ it is admissible for $\tau^c(\mathcal{D}^{a-1}(\mathcal{I}),1)$. So we can write the center in a neighborhoof of q in the form $\mathcal{J}=(x'=x_1,\dots,x_k,m_1,\dots,m_k)$ for the coordinate system $\bar{x}:=(x'=x_1,\dots,x_n)$ and a monoidal chart $u\colon M:=\overline{M}_p\hookrightarrow\mathcal{O}_{X,p}$ at a point $p\in X$.

Thus $x''=\sigma_n^c(x',1)=\sigma^c(x,1)\in\sigma^c(\mathcal{D}^{a-1}(\mathcal{I}),1)$ is a parameter at its vanishing locus containig the support $\operatorname{supp}(\mathcal{I},a)\subseteq V(x'')$.

Now, for the second part, by the inductive hypothesis passing to an étale neihborhhod we can assume that there is a local system of parameters $\bar{x}':=(x'=x'_1,\dots,x'_n)$ and a monoidal chart $u\colon M:=\overline{M}_p\hookrightarrow\mathcal{O}_{X',q'}$ at a point $q'\in X'$, such that $\frac{\partial^i}{\partial x'^i}(\tau^c(\mathcal{I}))\subseteq\tau^c(\mathcal{D}_X^{(\leq i)}(\mathcal{I}))$..

Consider the Weierstrass division by x' with respect to the coordinates system (see proof below) we can write any function in an étale neighborhood of X'' of X , as $f=qx+r(x'_2,\dots,x'_k,m'_1,\dots,m'_s)$, with the monomials m'_i. Then we write the coordinates of the center $\mathcal{J}=(x'=x_1,\dots,x_n,m_1,\dots,m_k)$ as $x_i=q_ix'+x''_i$, for $j\geq 2$ and put $x''_1=x'$ so $\mathcal{J}=(x'=x''_1,\dots,x''_n,m_1,\dots,m_k)$. Then $\frac{\partial^i}{\partial x''^i}=\frac{\partial^i}{\partial x''^i}$. Moreover, by the chain rule as in 3.3.2 and Remark 3.3.2, we see that $\sigma_n^c\left(\frac{\partial}{\partial x''}\right)=\frac{\partial}{\partial x'}$. So

$$\frac{\partial^i}{\partial x''^i}(\sigma^c(\mathcal{I}))=\frac{\partial^i}{\partial x''^i}(\sigma_n^c\tau^c(\mathcal{I}))=\sigma_n^c\left(\frac{\partial^i}{\partial x''^i}\tau^c(\mathcal{I})\right)\subseteq\sigma_n^c\tau^c(\mathcal{D}_X^{(\leq i)}(\mathcal{I}))=\sigma^c(\mathcal{D}_X^{(\leq i)}(\mathcal{I})).$$

♣

7.6.6. *Weierstrass division and Taylor series on the toroidal varieties.*

Weierstrass **Lemma 7.6.7.** *Given a coordinate system $x:=(x,x_2,\dots,x_n)$ and $u\colon M:=\overline{M}_p\hookrightarrow\mathcal{O}_{X,p}$ at a point p of a toroidal variety X, and the hypersurface $V(x)\subset X$, there exists an étale neighborhood $h: X'\to X$ of $p\in X$ such that*

(1) *The inverse image* $h^{-1}(V(x))$ *is isomorphic to* $V(x)$*, so* $V(x)$ *can be identified with a closed subscheme of* X'*.*
(2) *There is a smooth morphism* $\pi : X' \to V(x)$ *which restricts to the identity on* $V(x) \subset X'$*.*
(3) *Any function* f *on* $V(x)$ *lifts to a unique function* $\tilde{f} := \pi^*(f)$ *on* X' *with the property* $\frac{\partial \tilde{f}}{(\partial x)} = 0$
(4) *(Weierstrass division) Any function* f *on* X' *can be written as* $f = qx + r$*, where* $r = f_{(V(x)}$ *is independent of* x*.*
(5) *(Taylor series expansion) Any function* f *on* X' *can be written as a truncated Taylor series*

$$f = \sum_{i=0}^{k-1} \tilde{c}_{i,f} x^a + c(x) \cdot x^k,$$

where $k \in \mathbb{N}$ *and* $\tilde{c}_{if}$ *is the lifting of* $c_{if} := \left(\frac{1}{i!}\frac{\partial^i f}{\partial x^i}\right)_{|V(x)}$.

Proof. (1) and (2) Consider the smooth projection

$$\pi_0 : U \to \mathbb{A}^{n-1} \times \operatorname{Spec} k[\overline{M}_p]$$

from an open neighborhood $U \subset X$ of p, which is given by $(x_2, \dots, x_n, u)$. Let

$$\phi := \pi_{0|V(x)} : V(x) \to \mathbb{A}^{n-1} \times \operatorname{Spec} k[\overline{M}_p]$$

be its restriction to $V(x)$. We can assume that ϕ is étale by shrinking U, if necessary. The fiber product

$$U' = U \times_{\mathbb{A}^{n-1} \times \operatorname{Spec} k[\overline{M}_p]} V(x)$$

is étale over U (and usually reducible). It contains a closed subset

$$V' = V(x) \times_{\mathbb{A}^{n-1} \times \operatorname{Spec} k[\overline{M}_p].} V(x),$$

which, in turn, contains diagonally embedded $V(x)$ as one of its connected components.

Let X' be obtained from U' by removing all irreducible components of V' different from the diagonally embedded copy of $V(x)$. We get the induced projection $\pi : X' \to V(x)$ which restricts to the identity on $V(x)$. (3) Any function f on $V(x)$ defines its extension $\pi_0^*(f)$ to X' whose restriction $\pi_0^*(f)_{|V(x)}$ to $V(x)$ is equal f. Note that the functions coming from $\pi_0^{-1}(\mathcal{O}(V(x)))$ are factually those vanishing with respect to $\frac{\partial}{\partial x}$. Indeed the morphisms π_0 and π are determined by the coordinates $(x_2, \dots, x_n, u)$. In particular it defines an inclusion on the completions of local rings for any $p' \in X'$:

$$\widehat{\pi}^*_{X',\pi(p')} : \widehat{\mathcal{O}}_{V(x),p'} = k[\![\overline{M}_p, x_2, \dots, x_n]\!] \to k[\![\overline{M}_p, x_1, x_2, \dots, x_n]\!] = \widehat{\mathcal{O}}_{X',p'}$$

(4) Follows from (3).

(5) In the étale neighborhood X' of X there is a Weierstrass division with respect to x. We can write any function f of the form $f = q \cdot x + r(f)$ with the remainder $r(f) = \tilde{f}_{|V(x)}$ independent on x. Using induction on k we can write f as a truncated power series, with coefficients independent of x. The formulas for the coefficient c_{if} follow from simple differentiation.

♣

Sec:homogenization

7.7. **Homogenization.** Hypersurfaces of maximal contact exist only locally and are not unique. To obtain a global algorithm we adapt from [Wło05] the concept of homogenization and a corresponding gluing lemma. This does not involve any essential modification. Let $(\mathcal{I}, a)$ be a marked ideal of maximal order, with $\mathcal{T}(\mathcal{I}) := \mathcal{D}^{a-1}\mathcal{I}$. By the corresponding *homogenized ideal* we mean the marked ideal

$$\begin{aligned}\mathcal{H}(\mathcal{I}, a) &:= (\mathcal{H}(\mathcal{I}), a)\\ &= (\mathcal{I} + \mathcal{D}\mathcal{I}\cdot\mathcal{T}(\mathcal{I}) + \ldots + \mathcal{D}^i\mathcal{I}\cdot\mathcal{T}(\mathcal{I})^i + \ldots + \mathcal{D}^{a-1}\mathcal{I}\cdot\mathcal{T}(\mathcal{I})^{a-1}, a).\end{aligned}$$

omogenization-equivalent **Lemma 7.7.1.** *Let $(\mathcal{I}, a)$ be of maximal order. Then $(\mathcal{I}, a) \approx (\mathcal{H}(\mathcal{I}), a)$.*

Proof. Follows from Lemmas 7.5.4 and 7.4.4. ♣

omogenization-functorial **Lemma 7.7.2.** *Let $(\mathcal{I}, a)$ be of maximal order and $f : Y \to X$ logarithmically smooth. Then $\mathcal{H}(\mathcal{I}\mathcal{O}_Y) = \mathcal{H}(\mathcal{I})\mathcal{O}_Y$.*

Proof. This follows from Lemmas 7.5.1 and 7.5.4 since $(\mathcal{D}_X^{(\leq i)}\mathcal{I})\mathcal{O}_Y = \mathcal{D}_Y^{(\leq i)}(\mathcal{I}\mathcal{O}_Y)$, by Corollary 3.1.13. ♣

The main point of the following lemma is that restrictions of the homogenized ideal onto hypersurfaces of maximal contact are isomorphic étale-locally.

le: homo **Lemma 7.7.3.** *(Gluing Lemma) Let $(\mathcal{I}, a)$ be a marked ideal of maximal order on a logarithmically smooth variety X, and let $x, y \in \mathcal{T}(\mathcal{I}, a)$ be maximal contact elements at $p \in \operatorname{supp}(\mathcal{I}, a)$. Then there exist étale neighborhoods $\phi_x, \phi_y : \overline{X} \to X$ of $p = \phi_x(\overline{p}) = \phi_y(\overline{p}) \in X$, where $\overline{p} \in \overline{X}$, and a marked ideal $(\overline{\mathcal{I}}, a)$ on $\overline{X}$, such that*

1. *$\phi_x^*(\mathcal{H}(\mathcal{I}))\mathcal{O}_{\overline{X}} = \phi_y^*(\mathcal{H}(\mathcal{I}))\mathcal{O}_{\overline{X}} = \overline{\mathcal{I}}$.*
2. *$\phi_x^*(x) = \phi_y^*(y) \in \mathcal{T}(\overline{\mathcal{I}}, a)$.*
3. *For any $\overline{q} \in \operatorname{supp}(\overline{\mathcal{I}}, a)$, $\phi_x(\overline{q}) = \phi_y(\overline{q})$.*
4. *For any $(\mathcal{I}, a)$-admissible Kummer sequence the induced modifications $\phi_x^*(X_i)$ and $\phi_y^*(X_i)$ of $\overline{X}$ coincide and define an $(\overline{\mathcal{I}}, a)$-admissible Kummer sequence $\overline{X}$.*

Proof. The proof is identical to that of [Wło05, Lemma 3.5.5], taking logarithmic structures into account. We focus on parts (1)-(3), part (4) being longer and not requiring changes. The key point is that $\mathcal{H}(\mathcal{I})$ is tuned to the Taylor expansion in terms of $h = (x - y)$, which is insensitive to the logarithmic structure.

Let $U \subseteq X$ be an open subset for which there exist parameters $x_2, \ldots, x_n$ for the logarithmic stratum s_p through p which are transversal to x and y on U, as well as a sharp toric chart $U \to \mathbb{A}_M := \mathbb{A}_{\overline{\mathcal{M}}_p}$ for the logarithmic structure of X at p. In particular $x, x_2, \ldots, x_n$ and $y, x_2, \ldots, x_n$ form two systems of local parameters on U. Let $\mathbf{A}^n$ be the affine space with coordinates $z_1, \ldots, z_n$.

We have étale morphisms $\phi_1, \phi_2 : U \to \mathbf{A}^n \times \mathbb{A}_M$ with

$$\phi_1^*(z_1) = x, \quad \phi_1^*(z_i) = x_i \quad \text{for } i > 1 \qquad \text{and} \quad \phi_2^*(z_1) = y, \quad \phi_2^*(x_i) = x_i \quad \text{for } i > 1.$$

and sending $u \in M$ to $\mathcal{O}_X$ via the given toric chart.

Consider the fiber product $U \times_{\mathbf{A}^n \times \mathbb{A}_M} U$ associated to the morphisms ϕ_1 and ϕ_2. Let ϕ_x, ϕ_y be the natural projections $\phi_x, \phi_y : U \times_{\mathbf{A}^n \times \mathbb{A}_M} U \to U$, so that

$\phi_1\phi_x = \phi_2\phi_y$. Define $\overline{X}$ to be the irreducible component of $U \times_{\mathbf{A}^n \times \mathbb{A}_M} U$ whose images $\phi_x(U)$ and $\phi_y(U)$ contain p.

$$w_1 := \phi_x^*(x) = (\phi_1\phi_x)^*(z_1) = (\phi_2\phi_y)^*(z_1) = \phi_y^*(y),$$
$$w_i := \phi_x^*(x_i) = \phi_y^*(x_i) \qquad \text{for } i \geq 2.$$

Then $w_1, \ldots, w_n$ form a system of local parameters on $\overline{X}$ for the relevant stratum $s'_{p'}$. The monoid structure $\overline{M}$ on $\overline{X}$ can be identified with M, and the étale morphisms $\phi_x, \phi_y : \overline{X} \to X$ send $x, x_2, \ldots, x_n$ and $y, x_2, \ldots, x_n$, respectively, to $w_1, \ldots, w_n$.

The rest of the proof is nearly identical as in the classical situation and is left to the reader. [Wło05, Lemma 3.5.5(4)].

♣

This guarantees that a *functorial* procedure for $\mathcal{H}(\mathcal{I})$ is independent of H_i, glues across patches, and applies to $\mathcal{I}$. In the language of [Kol07, 3.53], the ideal $\mathcal{H}(\mathcal{I})$ is MC-invariant.

We may now replace $\mathcal{I}$ by $\mathcal{H}(\mathcal{I}, a)$ and assume given a global hypersurface of maximal contact H.

It remains to prove order reduction for a clean ideal $\mathcal{I}$ with given hypersurface of maximal contact H, functorially *with respect to the data $(\mathcal{I}, H)$.*

7.8. **Coefficient ideals.** Homogenization is not sufficient. By induction on dimension one can principalize $\mathcal{I}\mathcal{O}_H$; However in general this does not imply that $\mathcal{I}$ itself is principalized, even in a neighborhood of H.

To correct this, we follow the principles of [Wło05, Section 3.6] and introduce the *coefficient ideal* $C(\mathcal{I}, a)$, a clean ideal of order $a!$. We stress that the treatment here, following [Wło05], differs from earlier treatments of [Vil89, Vil92] and [BM97], in that $C(\mathcal{I}, a)$ is an ideal on X and not on a hypersurface of maximal contact. First, we use the calculus of *marked ideals* (Section 7.1) to show:

Sec:coefficient

Let $(\mathcal{I}, a)$ be a marked ideal. We define the coefficient ideal as in [Wło05], see also [Vil89, BM08]:

$$C(\mathcal{I}, a) := \sum_{i=0}^{a-1} \mathcal{D}_X^{(\leq i)}(\mathcal{I}, a),$$

with associated weight $a!$. It follows from Lemma 7.1.4 that if $(\mathcal{I}, a)$ is of maximal order then $C(\mathcal{I}, a)$ is of maximal order. Moreover

Lem:C-equiv

Lemma 7.8.1. *If $(\mathcal{I}, a)$ is of maximal order then*

$$(\mathcal{I}, a) \approx C(\mathcal{I}, a)$$

Proof. By Lemma 7.4.4 we have $\mathcal{D}^i(\mathcal{I}, a) \preccurlyeq (\mathcal{I}, a)$, hence the claim follows from Lemma 7.5.4(1). ♣

m:coefficient-functorial

Lemma 7.8.2. *Let $(\mathcal{I}, a)$ be of maximal order and $f : Y \to X$ logarithmically smooth. Then $C(\mathcal{I}\mathcal{O}_Y, a) = C(\mathcal{I}, a)\mathcal{O}_Y$.*

Proof. Recall that $(\mathcal{D}_X^{(\leq i)}\mathcal{I})\mathcal{O}_Y = \mathcal{D}_Y^{(\leq i)}(\mathcal{I}\mathcal{O}_Y)$, by Corollary 3.1.13. The claim follows by Lemma 7.5.1. ♣

7.9. **Restriction to a maximal contact.**

Proposition 7.9.1. *Let $\mathcal{I}$ be a clean ideal of logarithmic order a.*

- (See Lemma 7.8.1.) *Any sequence of $(\mathcal{I}, a)$-admissible centers is also a sequence of $C(\mathcal{I}, a)$-admissible centers and vice versa.*
- (See Lemma 7.8.2.) *If $Y \to X$ is logarithmically smooth, then*
$$C(\mathcal{I}, a)\mathcal{O}_Y = C(\mathcal{I}\mathcal{O}_Y, a).$$

Next, we consider a hypersurface of maximal contact $H = \{x = 0\}$ and compare, as in the classical case, $C(\mathcal{I}, a)$-admissible centers with $C(\mathcal{I}, a)|_H$-admissible centers. This is the key property of coefficient ideals.

Proposition 7.9.2 (See Proposition 7.9.3)**.** *Let $\mathcal{I}$ be a clean ideal of logarithmic order a.*

- *If $H = \{x = 0\}$ is a hypersurface of maximal contact for $C(\mathcal{I}, a)$, any sequence of $C(\mathcal{I}, a)$-admissible centers is supported inside the corresponding proper transforms of H and forms a sequence of $(C(\mathcal{I}, a)|_H, a!)$-admissible centers.*
- *A sequence of admissible centers for $(C(\mathcal{I}, a)|_H, a!)$ is admissible for $(\mathcal{I}, a)$.*

2a **Proposition 7.9.3.** *Let $(\mathcal{I}, a)$ be of maximal order, and $\sigma : X' \to X$ be the composition of an $(\mathcal{I}, a)$-admissible Kummer sequence. Assume that $x \in \mathcal{D}_X^{(\leq a-1)}(\mathcal{I})$ is a maximal contact element and $x' = \sigma^c(x, 1)$. Then*

(1) $\sigma^c(C(\mathcal{I}, a))$ is of maximal order contained in $C(\sigma^c(\mathcal{I}), a)$.
(2) Let $\mathcal{J} = (x_2, \ldots, x_n, m_1, \ldots, m_s)$ be a Kummer center on $V(x')$, let $\tilde{x}_i$ be lifts of x_i, and let $\tilde{m}_i$ be monomial lifts of m_i. Consider the Kummer center
$$\widetilde{\mathcal{J}} = (x', \tilde{x}_2, \ldots, \tilde{x}_n, \tilde{m}_1, \ldots, \tilde{m}_s)$$
on X'. Then $\mathcal{J}$ is admissible for $\sigma^c(C(\mathcal{I}, a)_{|V(x)})$ if and only if $\widetilde{\mathcal{J}}$ is admissible for $\sigma^c(\mathcal{I}, a)$.
(3) $\operatorname{supp}(\sigma^c(\mathcal{I}), a) = \operatorname{supp}(\sigma^c(C(\mathcal{I}, a)_{|V(x)}))$.

Proof of Proposition 7.9.3. Part (1) follows from Corollary 7.2.16(2) and Lemma 7.2.10, and part (3) follows from (2), so it remains to prove (2).

Assume that the center $\mathcal{J}$ is admissible for $\sigma^c(C(\mathcal{I}, a))_{|V(x')}$. This implies that
$$\sigma^c(\mathcal{D}^i(\mathcal{I}, a))_{|V(x')} \subseteq (\mathcal{J}^{a-i})^{\mathrm{sat}},$$
for any $i = 0, \ldots, a-1$.

Passing to an étale neighborhood X' we can assume, by Lemmas 7.6.7(4), and 7.6.5, that there is a Weierstrass division by x, and that $\frac{\partial^i}{\partial x^i}(\sigma^c(\mathcal{D}^i(\mathcal{I})) \subseteq (\sigma^c(\mathcal{D}^i(\mathcal{I}))$.

By Lemma 7.6.7(5), for any function $f \in \sigma^c(\mathcal{I}, a)$ we have $f \equiv \sum \tilde{c}_{if}(x')^i$ mod $(x')^a$. Then $c_{if} := \left(\frac{1}{i!}\frac{\partial^i(f)}{\partial x^i}\right)_{|V(x)} \in \sigma^c(\mathcal{D}^i(\mathcal{I}))_{|V(x)} \subseteq (\mathcal{J}^{a-i})^{\mathrm{sat}}$.

Thus $\tilde{c}_{if} \in \widetilde{(\mathcal{J}^{a-i})}^{\mathrm{sat}}$ and $x' \in \tilde{\mathcal{J}}$ we have $f \in (\tilde{\mathcal{J}}^a)^{\mathrm{sat}}$, as needed. So $\sigma^c(\mathcal{I}, a) \subseteq \widetilde{(\mathcal{J}^a)}^{\mathrm{sat}}$, and $\tilde{\mathcal{J}}$ is $\sigma^c(\mathcal{I}, a)$-admissible, giving one direction on the étale neighborhoods and thus on X.

Now, if $\sigma^c(\mathcal{I}, a) \subseteq \widetilde{(\mathcal{J}^a)}^{\mathrm{sat}}$ then
$$\sigma^c(\mathcal{D}^i(\mathcal{I})) \subseteq \mathcal{D}^i(\sigma^c(\mathcal{I}))) \subseteq (\tilde{\mathcal{J}}^{a-i})^{\mathrm{sat}}.$$

Passing to the restrictions we get

$$\sigma^c(\mathcal{D}^i(\mathcal{I}))_{|V(x')} \subseteq \mathcal{D}^i(\sigma^c(\mathcal{I})))_{|V(x')} \subseteq (\mathcal{J}^{a-i})^{\mathrm{sat}}$$

and this implies that

$$\sigma^c(C(\mathcal{I},a)_{|V(x')}) \subseteq (\mathcal{J}^{a!})^{\mathrm{sat}},$$

and $\mathcal{J}$ is $\sigma^c(C(\mathcal{I},a)_{|V(x')})$-admissible, giving the other direction.

♣

SSecadmmpaxp

8. Resolution of marked ideals

8.1. **Classifying lifts of admissible centers.** In this section we classify the transformations obtained by lifting an admissible center from a maximal contact hypersurface $x = 0$. The results of this section play a central role in constructing the order reduction functor in §8.2.

A1

Theorem 8.1.1 (Admissible lifting of a zero center)**.** *Let $(\mathcal{I},a)$ be a clean ideal on a stably toroidal variety X, and let $x \in \mathcal{D}^{a-1}(\mathcal{I})$ be a maximal contact element. If $C(\mathcal{I},a)_{|x} = 0$ then $\mathcal{I} = (x^a)$ with $\mathrm{supp}(\mathcal{I},a) = V(x)$, and the $(\mathcal{I},a)$-admissible modification at $\mathcal{J} = (x)$ creates a new marked ideal $\sigma^c(\mathcal{I},a)$ with empty cosupport.*

Proof. It follows that $\mathcal{I}_{|V(x)} = 0$, and $D_X^{(\le i)}(\mathcal{I})_{|V(x)} = 0$ for all $i < a$ and thus $\mathcal{I} = (x^a)$, with $\mathrm{supp}(\mathcal{I},a) = V(x)$. The center $\mathcal{J} = (x)$ is admissible for $(\mathcal{I},a)$ since $\mathcal{I} \subseteq (x)^a$. The blowing up at $\mathcal{J}$ is the identity on the underlying variety, but makes the maximal contact $x = 0$ into the exceptional divisor, which becomes part of the logarithmic structure. The ideal $\mathcal{I} = (x^a)$ becomes monomial and $\sigma^c(\mathcal{I}) = (1)$. ♣

A2

Theorem 8.1.2 (Admissible lifting of a monomial center)**.** *Let $(\mathcal{I},a)$ be a marked ideal of maximal order on a stably toroidal variety X, and let $x \in D_X^{(\le a-1)}(\mathcal{I})$ be a maximal contact element for $(\mathcal{I},a)$. Let $C_x = C(\mathcal{I},a)_{|V(x)}$, let $\mathcal{J} = (m_1,\ldots,m_s) = \mathcal{M}(C_x)$ be its monomial part, and let $\tilde m_i$ be the monomial lifts of m_i. Then the Kummer center $\widetilde{\mathcal{J}} := (x, \tilde m_1^{1/a!},\ldots,\tilde m_s^{1/a!})$ (defined in relevant Kummer neighborhoods $Z \in X_{\mathrm{tor}}(m_1^{1/a!},\ldots,\tilde m_s^{1/a!})$) is $(\mathcal{I},a)$-admissible, and the Kummer blowing up $\sigma\colon X' \to X$ along $\widetilde{\mathcal{J}}$ creates a new marked ideal $\sigma^c(C(\mathcal{I},a))$ whose restriction $\sigma^c(C(\mathcal{I},a))_{|V(x')} = (\sigma_{|V(x)})^c(C_x,a!)$ to the transformed maximal contact $V(x')$ is clean of order bounded by*

$$\mathrm{logord}_{p'}(\sigma_{|V(x)})^c(C_x,a!) \le \mathcal{D}\mathrm{ord}_p(C_x)$$

for any $p \in V(x)$ and $p' \in X'$ above p.[8]

Proof.

mixed

Lemma 8.1.3. *Let $\mathcal{I}$ be an ideal on X and let $\mathcal{M} = \mathcal{M}(\mathcal{I})$ be its monomial part. Then the Kummer blowing up at $\mathcal{J} = \mathcal{M}^{1/a}$ is $(\mathcal{I},a)$-admissible (defined in the relevant Kummer neighborhoods $Z \in X^\top(\mathcal{M}^{1/a})$). Moreover if a is so large that $\mathcal{M} = \mathcal{D}_X^{(\le a)}(\mathcal{I})$ then $(\sigma^c(\mathcal{I},a),a)$ is of maximal order.* [9]

[8]See Footnote 9.

[9]We do not use the claim that $(\sigma^c(\mathcal{I},a),a)$ is of maximal order when $\mathcal{M} = \mathcal{D}_X^{(\le a)}(\mathcal{I})$, nor the logarithmic order inequality in Theorem 8.1.2. They might be useful in improving the algorithm.

Foot:speeding

Proof. The admissibility is clear since $\mathcal{I} \subseteq \mathcal{M} \subseteq (\mathcal{J}^a)^{\mathrm{sat}}$. To prove the second claim note that $\mathcal{M}(\mathcal{I}\mathcal{O}_{X'}) = \mathcal{M}\mathcal{O}_{X'} = \mathcal{I}_E^a$ by Corollary 3.1.13. Assuming $\mathcal{M} = \mathcal{D}_X^{(\leq a)}(\mathcal{I})$, Proposition 3.6.5(1) gives $\mathcal{D}_{X'}^{(\leq a)}(\mathcal{I}\mathcal{O}_{X'}) = \mathcal{M}\mathcal{O}_{X'}$. Note that

$$\sigma^c(\mathcal{I}, a) = \mathcal{I}_E^{-a}(\mathcal{I}\mathcal{O}_{X'}) = (\mathcal{M}\mathcal{O}_{X'})^{-1}(\mathcal{I}\mathcal{O}_{X'}).$$

Consequently

$$(\mathcal{M}\mathcal{O}_{X'})\mathcal{D}_{X'}^{(\leq a)}(\sigma^c(\mathcal{I}, a)) = \mathcal{D}_{X'}^{(\leq a)}(\mathcal{I}\mathcal{O}_{X'}) = \mathcal{D}_X^{(\leq a)}(\mathcal{I})\mathcal{O}_{X'} = \mathcal{M}\mathcal{O}_{X'},$$

Thus, $\mathcal{D}_{X'}^{(\leq a)}(\sigma^c(\mathcal{I}, a)) = 1$, as needed. ♣

By Lemma 8.1.3, the Kummer center $\mathcal{J}^{1/a!} = (m_1^{1/a!}, \dots, m_s^{1/a!})$ is admissible for $(C_x, a!)$ on a relevant Kummer neighborhood (where it is defined). By Proposition 7.9.3 the Kummer ideal $\widetilde{\mathcal{J}}$ is admissible for $(\mathcal{I}, a)$.

By Proposition 7.6.5 the Kummer blowing up σ of $\widetilde{\mathcal{J}}$ transforms the maximal contact x into a maximal contact x', and by [ATW16b, Lemma 5.4.16] its restriction $V(x') \to V(x)$ coincides with the Kummer blowing up of $\mathcal{J}^{1/a!}$. So, by Proposition 3.6.5, the ideal C_x becomes $C_x\mathcal{O}_{V(x')}$ with $\mathcal{M}(C_x\mathcal{O}_{V(x')})$ invertible.

Since $x \in \mathcal{D}^{a-1}(\mathcal{I})$ we get that $x^{a!} \in C(\mathcal{I}, a)$. Consider the logarithmic structure $(X, \mathcal{M}_X^+)$ associated to the open set $U_X \smallsetminus V(x)$. Note that $\mathcal{J}$ is monomial with respect to this structure, hence the Kummer blowing up σ is toroidal with respect to the enriched structure $(X, \mathcal{M}_X^+)$. Denote by $\mathcal{D}_{(x)}$ the sheaf of logarithmic derivations of $(X, \mathcal{M}_X^+)$, namely logarithmic derivations on $(X, \mathcal{M}_X)$ preserving the ideal (x). Then we have

Lem:enriched **Lemma 8.1.4.** *(1)* $(\mathcal{D}_{(x)})(\mathcal{I}')_{V(x)} = \mathcal{D}_{V(x)}(\mathcal{I}'_{|V(x)})$ *for any ideal sheaf* $\mathcal{I}'$*.*
(2) $\sigma^*(\mathcal{D}_{(x)})$ *are logarithmic derivations preserving* (x')*:* $\sigma^*(\mathcal{D}_{(x)}^\infty) \subseteq \mathcal{D}_{(x')}^\infty$*.*
(3) $\mathcal{D}_{(x)}(\mathcal{J}) = \mathcal{D}_{(x)}^\infty(\mathcal{J}) = \mathcal{J}$*.*
(4) $\mathcal{D}_{(x)}^\infty(\mathcal{I}) \subseteq \mathcal{J}^a$*.*
(5) $\mathcal{D}_{(x)}^\infty(C(\mathcal{I}, a)) \subseteq \mathcal{J}^{a!}$*.*

Proof. Let $x, x_1, \dots, x_r$ restrict to a system of regular parameters on a log stratum at a point $p \in V(x)$, and $M = \mathcal{M}_{X,p}$. The sheaf $\mathcal{D}_{(x)}$ is generated locally by $x\frac{\partial}{\partial x}$, $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_r}$ and the logarithmic derivations $u\frac{\partial}{\partial u}$ for $u \in \alpha_X(M)$. This implies (1) and (3).

Part (2) follows from Lemma 3.1.12.

Since $\mathcal{J}$ is admissible for $(\mathcal{I}, a)$ and $(C(\mathcal{I}, a), a!)$ we obtain that $\mathcal{D}_{(x)}^\infty(\mathcal{I}) \subseteq \mathcal{D}_{(x)}^\infty((\mathcal{J}^a)^{\mathrm{sat}}) = \mathcal{J}^a$, giving (4) and similarly $\mathcal{D}_{(x)}^\infty(C(\mathcal{I}, a)) \subseteq \mathcal{J}^{a!}$, giving (5). ♣

Consider the ideal $\mathcal{M}_{(x)} := \mathcal{D}_{(x)}^\infty(C(\mathcal{I}, a))$. By Lemma 8.1.4(1) its restriction to $V(x)$ coincides with $\mathcal{J} = \mathcal{M}(C_x)$. On the other hand by Lemma 8.1.4(3) $\mathcal{M}_{(x)}$ is $\mathcal{D}_{(x)}$-stable and thus, by Theorem 3.4.2(1), is generated by monomials $u^\alpha \cdot x^a$ of the logarithmic structure $\mathcal{M}_X^+$. Therefore the lifting monomials $\widetilde{m}_1, \dots, \widetilde{m}_s$ are in $\mathcal{M}_{(x)}$ and by Lemma 8.1.4(5) we have

$$(x^{a!}, \widetilde{m}_1, \dots, \widetilde{m}_s) \subseteq \mathcal{M}_{(x)} \subseteq \widetilde{\mathcal{J}}^{a!} = (x, (\widetilde{m}_1)^{1/a!}, \dots, (\widetilde{m}_s)^{1/a!})^{a!}.$$

Since the saturations of $\mathcal{J}$ and $(m_1^{1/a!}, \dots, m_s^{1/a!})^{a!}$ coincide, the integral closure of $(x^{a!}, \widetilde{m}_1, \dots, \widetilde{m}_s)$ on the left hand side coincides with the integral closure of $\widetilde{\mathcal{J}}^{a!}$.

Therefore we get equality of total transforms $(x^{a!}, \widetilde{m}_1, \ldots, \widetilde{m}_s)\mathcal{O}_{X'} = \widetilde{J}^{a!}\mathcal{O}_{X'}$, giving equalities throughout

$$\widetilde{J}^{a!}\mathcal{O}_{X'} = \mathcal{M}_{(x)}\mathcal{O}_{X'} = \widetilde{J}^{a!}\mathcal{O}_{X'}.$$

Considering controlled transforms we obtain

$$\sigma^c(\mathcal{M}_{(x)}, a!) = \sigma^c(\widetilde{J}^{a!}, a!) = (1).$$

By Lemma 7.2.10 and admissibility

$$\begin{aligned}(1) = \sigma^c(\mathcal{D}^{\infty}_{(x)}(C(\mathcal{I}, a))) \subseteq \mathcal{D}^{\infty}_{(x')}(\sigma^c(C(\mathcal{I}, a))) \subseteq \mathcal{D}^{\infty}_{(x')}(\sigma^c(\widetilde{J}^{a!}, a!)) \\ = \mathcal{D}^{\infty}_{(x')}(1) = (1),\end{aligned}$$

yielding $\mathcal{D}^{\infty}_{(x')}(\sigma^c(C(\mathcal{I}, a))) = (1)$. Thus

$$\begin{aligned}(1) &= (\mathcal{D}^{\infty}_{(x')}(\sigma^c(C(\mathcal{I}, a)))_{|V(x')} = (\mathcal{D}^{\infty}_{(x')})_{|V(x')}(\sigma^c(C(\mathcal{I}, a))_{|V(x')}) \\ &= (\mathcal{D}^{\infty}_{V(x')})(\sigma^c(C(\mathcal{I}, a))_{|V(x')}).\end{aligned}$$

In other words the restriction of $\sigma^c(C(\mathcal{I}, a))$ to $V(x')$ is clean. Finally, note that

$$\sigma^c(C(\mathcal{I}, a))_{|V(x')} = \tau^c(C_x),$$

where $\tau \colon V(x') = Bl_{\mathcal{J}}V(x) \to V(x)$ is the Kummer blowing up along $\mathcal{J}$. By Proposition 3.6.5 and Lemma 3.6.3 the differential logarithmic order of $\tau^c(C_x)$ does not increase. ♣

A3 **Theorem 8.1.5** (Admissibile lifting for balanced ideals)**.** *Let $(\mathcal{I}, a)$ be a clean ideal with a maximal contact x defining the hypersurface $H = V(x)$, such that the restriction $(C(\mathcal{I}, a)_{|H}, a!)$ is* balanced*. Let $\sigma_H : H' \to H$ be the composition of a sequence of Kummer blowings up at centers*

$$\mathcal{J}_i = (x_{i,1}, \ldots, x_{i,k_i}, m_{i,1}, \ldots, m_{i,l_i})$$

admissible for $((C(\mathcal{I})_{|H})^{cln}, b)$ with $b \geq a!$. Then

1. *The Kummer ideals $\mathcal{J}_i$ on the modifications of H determine an $(\mathcal{I}, a)$-admissible Kummer sequence, composing to $\sigma : X' \to X$, with centers $(x^{(i)}) + \widetilde{\mathcal{J}}_i$, where $\widetilde{\mathcal{J}}_i = (\tilde{x}_{i,1}, \ldots, \tilde{x}_{i,k_i}, \tilde{m}_{i,1}, \ldots, \tilde{m}_{i,l_i})$ is generated by liftings of the regular parameters and monomial generators of $\mathcal{J}_i$, and $V(x^{(i)}) = H_i$ are the strict transforms of H.*
2. *The ideal $\sigma^c(C(\mathcal{I}, a))_{|H'}$ is balanced, with*

$$(\sigma^c(C(\mathcal{I}, a))_{|H'})^{cln} = \sigma^c_H((C(\mathcal{I})_{|H})^{cln}, b).$$

Proof. Write $C = C(\mathcal{I}, a)$. By Corollary 7.2.15, the centers $\mathcal{J}_i$ are admissible for $(C_{|H}, a!)$. Proposition 7.9.3(2) implies that the centers $\widetilde{\mathcal{J}}_i$ are admissible for $(\mathcal{I}, a)$.

Write $\sigma = \tau \circ \pi$, where τ is the composition of the first $n-1$ blowings up. Denote the exceptional of π by E. Name the restrictions to the corresponding hypersurfaces of maximal contact $\sigma_H = \tau_H \circ \pi_H$. We assume by induction that

$$\tau^c(C, a!)_{|H_{n-1}} \quad = \quad (\tau^c(C, a!)_{|H_{n-1}})^{cln} \ \cdot \ \mathcal{M}(\tau^c(C, a!)_{|H_{n-1}}),$$

where $\mathcal{M}_{n-1} := \mathcal{M}(\tau^c(C, a!)_{|H_{n-1}})$ is an invertible monomial ideal and

$$(\tau^c(C, a!)_{|H_{n-1}})^{cln} = \tau^c_H(C_{|H}, b)$$

is clean. Applying controlled transforms along $\pi : X_n \to X_{n-1}$, we have

$$\begin{aligned}\sigma^c(C, a!)_{|H} &= \pi_H^* \tau_H^c(C_{|H}, b) \cdot \mathcal{I}_{E_{|H_n}}^{-a!} \cdot \pi_H^*(\mathcal{M}_{n-1}) \\ &= \sigma_H^c(C_{|H}, b) \cdot \mathcal{I}_{E_{|H_n}}^{b-a!} \cdot \pi_H^*(\mathcal{M}_{n-1})\end{aligned}$$

where the factor $\mathcal{I}_{E_{|H}}^{b-a!} \cdot \pi_H^*(\mathcal{M}_{n-1})$ is an invertible monomial ideal, and, by Proposition 7.2.12(1), $\sigma_H^c(C_{|H}, b) = \pi_H^c \tau_H^c(C_{|H}, b)$ is clean, as required. ♣

algsec

8.2. Resolution of marked ideals.

Th:order-reduction

Theorem 8.2.1 (Order reduction). *(1)* **Existence:** *Let X be a stably toroidal variety, $\mathcal{I}$ a clean ideal with maximal logarithmic order a. Then there is an $(\mathcal{I}, a)$-admissible Kummer sequence $(X_i, \mathcal{I}_i, \mathcal{J}_i)$ such that $\mathcal{I}_n$ has maximal logarithmic order $< a$.*

(2) **Functoriality:** *The procedure assigning to $(X, \mathcal{I})$ the sequence $(X_i, \mathcal{I}_i, \mathcal{J}_i)$ is functorial for logarithmically smooth morphisms: if $X' \to X$ is a logarithmically smooth morphism, with associated sequence $(X'_j, \mathcal{I}'_j, \mathcal{J}'_j), j = 1, \ldots, n'$, then there is a strictly increasing function $i(j)$ such that*

- $X'_j = X' \times_X X_{i(j)}$,
- $\mathcal{I}'_j = \mathcal{I}_{i(j)} \mathcal{O}_{X'_j}$, *and*
- $\mathcal{J}'_j = \mathcal{J}_{i(j)} \mathcal{O}_{X'_j}$,

while for the remaining i, those not in the image of $i(j)$, we have that $\mathcal{J}_i \mathcal{O}_{X_{n'}}$ is the unit ideal.

Proof. We apply induction on $\dim X$. If $\dim X = 0$ then there are no nontrivial clean ideals so the theorem is valid. Assume that $\dim X \geq 1$. Let x be a maximal contact. Let us replace $\mathcal{I}$ with its homogenized ideal $\mathcal{H}(\mathcal{I}, a)$. Consider the coefficient ideal $C(\mathcal{I}, a)$ and its restriction $\mathcal{I}_x := C(\mathcal{I}, a)_{|V(x)}$ to a hypersurface of maximal contact. The homogenization procedure ensures that the considered invariants and centers do not depend upon choice of maximal contact, and lift uniquely to the ambient stably toroidal variety

Step 0: *Reduction to the case when $\mathcal{I}_x$ is nowhere zero.*

Let H denote the set of points $p \in \mathrm{supp}(\mathcal{I})$ at which $\mathcal{I}_x = 0$, and assume that H is non-empty. By Lemma 8.1.1, H is a smooth hypersurface, which is of maximal contact to $\mathcal{I}$ at any point $p \in H$. Note that $X' = Bl_H(X) \to X$ is an isomorphism on the level of schemes while the toroidal structure of X' is obtained from that of X by adding H. In addition, and H is disjoint from the cosupport of the transform $\mathcal{I}' = \sigma^c(\mathcal{I})$ by Lemma 8.1.1. Thus, we did not change the situation over $X \smallsetminus H$ and we resolved $\mathcal{I}$ over H. In particular, $\mathcal{I}'_x$ is nowhere zero.

From here on we assume that $\mathcal{I}_x$ is nowhere zero.

Step 1: *Construction of σ with $\sigma^c(C(\mathcal{I}, a))_{|V_{x'}}$ a clean ideal.*

We apply to $H = V(x)$ the blowing up $\bar{\sigma} : H' \to H$ at the monomial ideal $\mathcal{J} = \mathcal{M}(\mathcal{I}_x)$. This creates an ideal $\mathcal{I}_x \mathcal{O}_{H'} = \mathcal{I}_{x'}^{cln} \mathcal{J}'$ which factors into the product of the clean part $\mathcal{I}_{x'}^{cln}$ and the invertible monomial ideal $\mathcal{J}' = \mathcal{J} \mathcal{O}_{H'}$. We call such ideals *balanced*. By Theorem 8.1.2, if $\mathcal{J} = (m_1, \ldots, m_k)$ on $V(x)$ then the Kummer center $\widetilde{\mathcal{J}} = (x) + (\tilde{m}_1, \ldots, \tilde{m}_k)^{1/a!}$,, on a relevant Kummer neighborhood, is admissible for $C(\mathcal{I}, a)$, resulting in a Kummer blowing up $\sigma : X' \to X$. The transformed restricted ideal $\sigma^c C(\mathcal{I}, a)_{|V_{x'}} = \mathcal{I}_{x'} = \mathcal{I}_{x'}^{cln}$ is now clean, of order b.

If $b < a!$ then this means that $\mathrm{maxord}(\sigma^c(\mathcal{I}, a)) < a$, so the order of the ideal $(\mathcal{I}, a)$ is reduced. If $b \geq a!$ then we proceed to Step 2, where we assume that $\sigma^c(C(\mathcal{I}, a))_{|V_{x'}}$ is a clean ideal.

Step 2: *Order reduction on H' and its extension.*

We are given an admissible $\sigma : X' \to X$ restricting to an admissible $\bar\sigma : H' \to H$ where $\sigma^c(C(\mathcal{I}, a))_{|V_{x'}}$ is a clean ideal of maximal order $b \geq a!$. By induction on $\dim X$, there exists a sequence of integers

$$b = b_0 > b_1 \ldots > b_{r-1} \geq a!,$$

a sequence of morphisms of stably toroidal varieties

$$H'' := H_r \xrightarrow{\bar\sigma_r} H_{r-1} \xrightarrow{\bar\sigma_{r-1}} \cdots \xrightarrow{\bar\sigma_1} H_1 \xrightarrow{\bar\sigma_0} H_0 = H',$$

and clean ideals $\mathcal{I}_{x'} = \mathcal{I}_{x',0}, \ldots, \mathcal{I}_{x',r}$ on $H_0, \ldots H_r$ such that

$$(\mathcal{I}_{x',0}, b_0), \ldots, (\mathcal{I}_{x',r-1}, b_{r-1})$$

are of maximal order, $\bar\sigma_i$ is the functorial order reduction of $(\mathcal{I}_{x',i}, b_i)$ with $\bar\sigma_i^c(\mathcal{I}_{x',i}, b_i) = \mathcal{I}_{x',i+1}$, and finally $\mathcal{I}_{x',r}$ has maximal order $< a!$.

By Corollary 7.2.15 all these transformations are $(\mathcal{I}_{x'}^{cln}, a!)$-admissible, and if $\bar\sigma' \colon H'' \to H$ denotes the composition then $\bar\sigma^c(\mathcal{I}_{x'}, a)$ is balanced with the clean part $\mathcal{I}_{x',r}$. By Theorem 8.1.2 the sequence of centers extends canonically to $(\mathcal{I}, a)$-admissible centers, giving rise to a functorial sequence

$$X'' := X_r \xrightarrow{\sigma_r} X_{r-1} \xrightarrow{\sigma_{r-1}} \cdots \xrightarrow{\sigma_1} X_1 \xrightarrow{\sigma_0} X_0 = X',$$

with composition $\sigma' : X'' \to X'$ such that ${\sigma'}^c\sigma^c(C(\mathcal{I}, a))_{H''}$ is balanced with clean part $({\sigma'}^c\sigma^c C(\mathcal{I}, a)_{|H''})^{cln} = \mathcal{I}_{x',r}$ of maximal order $< a!$.

Step 3: *Cleaning up the extension.* Let x'' be the transform of x', namely the local equation of H'', and let $\mathcal{J} = (m_1, \ldots, m_k)$ be the monomial part of the balanced ideal ${\sigma'}^c\sigma^c C(\mathcal{I}, a)_{|H''}$. Consider the Kummer center $\widetilde{\mathcal{J}} = (x'') + (m_1, \ldots, m_k)^{1/a!}$ on X''.

By Proposition 7.9.3(2) $\widetilde{\mathcal{J}}$ is ${\sigma'}^c\sigma^c(\mathcal{I}, a)$-admissible. Let $\tau : X''' \to X''$ be the resulting Kummer blowing up. We have that $\tau^c{\sigma'}^c\sigma^c(C(\mathcal{I}, a), a!)_{|H''} = \mathcal{I}_{x',r}$ is clean of logarithmic order $< a!$, hence $\max\mathrm{logord}(\tau^c{\sigma'}^c\sigma^c(\mathcal{I}, a)) < a$, as needed. ♣

8.3. **Principalization.**

Proof of the Principalization Theorem 1.3.2. Let $\mathcal{I}$ be any coherent ideal on a stably smooth X. First apply the blow-up of $\mathcal{M}(I)$ which bt Proposition 6.3.5 transforms $\mathcal{I}$ functorially into a balanced ideal $\mathcal{I} = \mathcal{I}^{cln}\mathcal{M}(\mathcal{I})$ on $\mathcal{X}'$, where $\mathcal{I}^{cln}$ is clean and $\mathcal{M}(\mathcal{I})$ an invertible monomial ideal.

Assume $\mathcal{I}^{cln}$ has maximal logarithmic order a. If $a = 0$ then $\mathcal{I}' = (1)$ and we are done. Assume by induction we can principalize an ideal with maximal logarithmic order $< a$.

By Order Reduction (Theorem 8.2.1) there is a logarithmic morphism $X'' \to X'$, the result of a functorial Kummer sequence, such that $\mathcal{I}'\mathcal{O}_{X''} = \mathcal{I}''\mathcal{P}$ with $\mathcal{P}$ an invertible monomial ideal and $\mathcal{I}''$ clean with maximal logarithmic order $< a$.

By the induction assumption there is a functorial logarithmic $X''' \to X''$ where $\mathcal{I}''\mathcal{O}_{X'''}$ is an invertible monomial ideal. The ideals $\mathcal{P}\mathcal{O}_{X'''}$ and $\mathcal{L}\mathcal{O}_{X''}$ are also invertible monomial ideals, since $\mathcal{P}$ and $\mathcal{L}$ are and since the morphisms are logarithmic. So the product $\mathcal{I}\mathcal{O}_{X'''}$ is an invertible monomial ideal, and the theorem follows.

♣

Sec:resolution

8.4. **Canonical resolution of the logarithmic varieties.**

Proof of Theorem 1.3.8.

Choice of embedding of constant codimension.

Let X be any logarithmic variety with fine and saturated structure. It suffices to construct such a desingularization functor $\mathcal{F}$ étale locally for étale cover, which locally in Zariski topology posseses the charts.

Indeed assume $X_0 \to X$ is such a local étale cover. Moreover assume there is a functorial desingularization $F : X_0' \to X_0$, which is obtained by a sequence of blow-ups on X_0. Then consider the normalized product$X_1 := X_0 \times_X X_0$, with the induced étale maps (the natural projections) $\pi_i : X_1 \to X_0$, for $i = 0, 1$. Then the resolution functor is compatible with both maps. The centers of blow-ups $\pi_i^*(\mathcal{J}_i)$ coincide on X_1, giving the same reolution on X_1. Thus by the compatibility the centers $\mathcal{J}_i$ descent to the ordinary ideals $\mathcal{J}_{Xi}$, and define the canonical resolution on X_0.

By the above we can assume that X is a logarithmic variety, which admits locally in Zariski toplogy local charts.

Then by Corollary 3.7.4 we can further assume that X possesses a strict closed immersion into a toroidal variety Y. Moreover, since X is locally equidimensional such a closed embedding $i\colon X \hookrightarrow Y$ can be constructed so that X is of a constant codimension d in Y. Indeed, if X_i and Y_j are the connected components of X and Y, then it suffices to take an embedding of the form $X = \coprod_i X_i \hookrightarrow \coprod_i Y_{j(i)} \times \mathbb{A}^{n_i}$.

Principalization of I_Y: blowing up Y is synchronous. Let $\mathcal{I}$ be the ideal of X in Y, let $Y_{n+1} \to \cdots \to Y_0 = Y$ be the principalization sequence from Theorem 1.3.2, and let $X_i \hookrightarrow Y_i$ be the sequence of strict transforms of X. We claim that all generic points of Z are blown up for the first time at the same stage i. Indeed, this is the stage when we restrict the ideal onto an iterative d-th maximal contact $H^{(d)}$. Moreover, since all generic point of X_i are of codimension d, at the generic points of $H^{(d)}$ are precisely the generic points of X_i. Therefore, $\mathcal{I}_{i|H^{(d)}} = 0$ and by Theorem 8.1.1 the blowing up of $H^{(d)}$ principalizes $\mathcal{I}_i$. This proves that $i = n$ and $X_n = H^{(d)}$ is a stably toroidal variety. In addition, the centers of $X_{i+1} \to Y_i$ with $i < n$ do not contain the generic points of X_i and hence the morphisms $X_{i+1} \to X_i$ with $i < n$ are modifications.

The process ic functorial with respect to the log smooth maps. **Indepedence of choices with different embedding dimension.** It remains to compare embeddings i and i' of constant codimensions d and d'. Since for a fixed codimension the resolution is independent of the embedding, it suffices to compare i and an embedding of the form $i'\colon X \hookrightarrow Y' = Y \times \mathbb{A}^n$, for which the Re-embedding Principle, Proposition **??**, applies.

Functoriality. Finally, we claim that the resolution is compatible with logarithmically smooth morphisms $h\colon X' \to X$. Again, it suffices to check this locally on X and then by Lemma 3.7.6 we can find extend h to a logarithmically smooth morphism $Y' \to Y$ such that $X' = X \times_Y Y'$. Then X' is given by the ideal

$\mathcal{I}' = f^*(\mathcal{I})$, principalizations of $\mathcal{I}$ and $\mathcal{I}'$ are compatible by Theorem 1.3.2 and hence the induced desingularizations of X and X' are compatible. ♣

8.4.1. *The logarithmically smooth locus is preserved.* First, étale functoriality means that it suffices to consider X logarithmically smooth. This means $X \to \operatorname{Spec} k$ is logarithmically smooth (toroidal). The resolution of $\operatorname{Spec} k$ is trivial, therefore by Functoriality so is the resolution of X.

Sec:to-varieties

8.4.2. *From stably toroidal to toroidal varieties.* Combining the above algorithm for stably toroidal desingularization of logarithmic varieties with the torification algorithm, one obtains a desingularization algorithm that outputs a toroidal variety. However, the torification process is compatible with isotropical logarithmically smooth morphisms. To achieve functoriality with respect to all logarithmically smooth morphisms one needs to resolve logarithmic varieties by stably toroidal varieties.

Sec:to-smooth

8.4.3. *From toroidal varieties to smooth varieties.* One can also *functorially* resolve the singularities of a toroidal variety. One way to do this is using non-embedded resolution in characteristic 0, see [Niz, Theorem 5.10], [IT14, Theorem 3.3.16], [GM15, Theorem 9.4.5]. This in particular preserves the normal crossings locus. Another is a simple combinatorial method provided in [ACMW14, Theorem 4.4.2] by combining barycentric subdivisions with the lattice reduction algorithm of [KKMSD73, Theorem 11*], but has the disadvantage that it modifies the normal crossings locus. A simple functorial combinatorial algorithm which does not modify the normal crossings locus is provided in [Wło17].

8.4.4. *The role of equidimensionality.* Next, let us explain where the equidimensionality assumption came from. The argument in the proof of Theorem 1.3.8 shows that if $X \hookrightarrow Y \hookrightarrow Y'$ are strict closed immersions, Y and Y' are toroidal and Y is of constant codimension in Y' then both $X \hookrightarrow Y$ and $X \hookrightarrow Y'$ induce the same desingularization of X. However, if the codimension of Y in Y' varies then the induced desingularizations may differ. For example, two components of X that were resolved at the same time in Y and have different codimension in Y' will be resolved at different times in Y' because it will take different number of passages to the maximal contact to get to them.

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